Proceedings of the Edinburgh Mathematical Society (2004) **47**, 257–270 ©
DOI:10.1017/S0013091502000767 Printed in the United Kingdom

THREE SOLUTIONS FOR A QUASILINEAR TWO-POINT BOUNDARY-VALUE PROBLEM INVOLVING THE ONE-DIMENSIONAL p-LAPLACIAN

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(Received 7 August 2002)

Abstract In this paper we prove the existence of at least three classical solutions for the problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(t,u)h(u'), \\ u(a) = u(b) = 0, \end{cases}$$

when λ lies in an explicitly determined open interval.

Our main tool is a very recent three-critical-points theorem stated in a paper by D. Averna and G. Bonanno (*Topolog. Meth. Nonlin. Analysis* **22** (2003), 93–103).

Keywords: critical points; three solutions; two-point boundary-value problem; one-dimensional p-Laplacian

2000 *Mathematics subject classification:* Primary 34B15

1. Introduction

The aim of this paper is to prove the existence of at least three classical solutions for the following quasilinear two-point boundary-value problem:

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(t,u)h(u'), \\ u(a) = u(b) = 0, \end{cases} \tag{P}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $h : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function such that $0 < \inf h$, $p > 1$, and λ is a positive parameter.

Several results are known about the existence of multiple solutions for problems involving the one-dimensional p-Laplacian (see, for example, [**1**, **4**, **5**] and the references cited therein), in which the right-hand side is independent of u'; principally they use methods of quadrature, lower and upper solutions, or fixed-points theorems.

Here, under suitable hypotheses, we prove that the problem (P) has at least three classical solutions when λ lies in an explicitly determined open interval.

Our approach is of variational type, and is based on the following recent three-critical-points theorem of [**2**].

Theorem 1.1 (Theorem B of [2]). *Let X be a reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(i) $\lim_{\|x\|\to+\infty}(\Phi(x) + \lambda\Psi(x)) = +\infty$ *for all* $\lambda \in [0, +\infty[$;

(ii) *there is $r \in \mathbb{R}$ such that*

$$\inf_X \Phi < r$$

and

$$\varphi_1(r) < \varphi_2(r),$$

where

$$\varphi_1(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x) - \inf_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi}{r - \Phi(x)},$$

$$\varphi_2(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \sup_{y\in\Phi^{-1}([r,+\infty[)} \frac{\Psi(x) - \Psi(y)}{\Phi(y) - \Phi(x)},$$

and $\overline{\Phi^{-1}(]-\infty,r[)}^w$ is the closure of $\Phi^{-1}(]-\infty,r[)$ in the weak topology.

Then, for each

$$\lambda \in \left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[,$$

the functional $\Phi + \lambda\Psi$ has at least three critical points in X.

However, $\varphi_1(r)$ in Theorem 1.1 could be 0. In this and similar cases, here and below, we agree to read $\frac{1}{0}$ as $+\infty$.

In § 2, the variational approach is justified and the regularity of an appropriate functional involved is proved. In § 3, we prove our main result (Theorem 3.1) and give some examples of applications. Here, by way of an example, we present a very particular case of Theorem 3.1.

Theorem 1.2. *Let $p > 1$ and assume that $\beta : \mathbb{R} \to \mathbb{R}$ is a non-negative continuous function such that*

$$\beta(d) > 0$$

https://doi.org/10.1017/S0013091502000767

for some $d > 0$,

$$\lim_{x \to 0^+} \frac{\beta(x)}{x^{p-1}} = 0,$$

and

$$\lim_{x \to +\infty} \frac{\beta(x)}{x^q} \in \mathbb{R}$$

for some $q \in]0, p-1[$.

Then, for every

$$b > \bar{b} := \left(\frac{2^{2p}}{p} \frac{d^p}{\int_0^d \beta(x)\,\mathrm{d}x} \right)^{1/p},$$

the problem

$$\begin{cases} -(|u'|^{p-2}u')' = \beta(u), \\ u(0) = u(b) = 0 \end{cases} \tag{Pb}$$

admits at least two non-trivial classical solutions.

For basic notation and definitions we refer to [**7**].

2. Preliminaries

Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and let $h : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function such that $0 < m := \inf h$. Let us put $M := \sup h$.

Let us consider the following two-point problem:

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(t, u) h(u'), \\ u(a) = u(b) = 0, \end{cases} \tag{P}$$

where λ is a positive parameter and $p > 1$.

We say that u is a classical solution to (P) if $u \in C^1([a, b])$, $|u'|^{p-2}u' \in C^1([a, b])$, $u(a) = u(b) = 0$, and

$$-(|u'(t)|^{p-2}u'(t))' = \lambda f(t, u(t)) h(u'(t))$$

for every $t \in [a, b]$.

For $p > 1$, define $\varphi : \mathbb{R} \to \mathbb{R}$ by putting, for every $x \in \mathbb{R}$,

$$\varphi(x) := \int_0^x \frac{(p-1)|s|^{p-2}}{h(s)}\,\mathrm{d}s.$$

Clearly, φ is an increasing homeomorphism on $\mathbb{R}$ and one has

$$|\varphi(x)| \leqslant \frac{|x|^{p-1}}{m}$$

for all $x \in \mathbb{R}$.

https://doi.org/10.1017/S0013091502000767

We say that u is a weak solution to (P) if $u \in W_0^{1,p}([a,b])$ and

$$\int_a^b \varphi(u'(t))v'(t)\,\mathrm{d}t = \lambda \int_a^b f(t,u(t))v(t)\,\mathrm{d}t$$

for every $v \in W_0^{1,p}([a,b])$.

The following straightforward lemma is the key tool for proving that classical and weak solutions to (P) coincide.

Lemma 2.1. *Let $x : [a,b] \to \mathbb{R}$ be continuous at $t_0 \in [a,b]$. If one among $t \mapsto \varphi(x(t))$ and $t \mapsto |x(t)|^{p-2}x(t)$, $t \in [a,b]$, is differentiable at t_0, then the other is also differentiable at t_0 and we have*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\varphi(x(t))\right)_{t=t_0} = \frac{(\mathrm{d}/\mathrm{d}t)|x(t)|^{p-2}x(t))_{t=t_0}}{h(x(t_0))}.$$

Proof. For every $t \neq t_0$, by the first mean-value theorem and by the continuity of x at t_0, we can write

$$\int_{x(t_0)}^{x(t)} \frac{(p-1)|s|^{p-2}}{h(s)}\,\mathrm{d}s = \frac{1}{h(\xi_t)} \int_{x(t_0)}^{x(t)} (p-1)|s|^{p-2}\,\mathrm{d}s,$$

where $\lim_{t \to t_0} \xi_t = x(t_0)$, from which the conclusion follows in an obvious way. □

Proposition 2.2. *Classical and weak solutions to* (P) *coincide.*

Proof. If u is a classical solution to (P), then $u \in W_0^{1,p}([a,b])$ and

$$-\frac{(|u'(t)|^{p-2}u'(t))'}{h(u'(t))} = \lambda f(t,u(t))$$

for every $t \in [a,b]$.

Multiplying by $v \in W_0^{1,p}([a,b])$, by integration by parts between a and b, and taking into account that, by the previous lemma, $\varphi(u'(t))$ is a primitive of

$$\frac{(|u'(t)|^{p-2}u'(t))'}{h(u'(t))},$$

we obtain that u is a weak solution to (P).

If u is a weak solution to (P), then by using usual methods, and taking into account that φ is a homeomorphism, we have that u and $\varphi \circ u'$ lie in $C^1([a,b])$, and $-(\varphi(u'(t)))' = \lambda f(t,u(t))$ for every $t \in [a,b]$, where $(\varphi(u'(t)))'$ is the usual derivative. Thus we obtain that u is a classical solution to (P) by virtue of Lemma 2.1. □

Remark 2.3. We explicitly observe that the continuity of f can be weakened if we ask for generalized solutions to problem (P). To this end, we recall that a function $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is said to be L^1-Carathéodory if

(a) $t \mapsto f(t,x)$ is measurable for every $x \in \mathbb{R}$;

(b) $x \mapsto f(t,x)$ is continuous for almost every $t \in [a,b]$;

(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([a,b])$ such that

$$\sup_{|x|\leqslant\rho} |f(t,x)| \leqslant l_\rho(t)$$

for almost every $t \in [a,b]$.

A function $u : [a,b] \to \mathbb{R}$ is said to be a generalized solution to (P) if $u \in C^1([a,b])$, $|u'|^{p-2}u' \in AC([a,b])$, $u(a) = u(b) = 0$, and $-(|u'(t)|^{p-2}u'(t))' = \lambda f(t,u(t))h(u'(t))$ for almost every $t \in [a,b]$.

Therefore, arguing as in Proposition 2.2 and taking into account that $|u'|^{p-2}u' \in AC([a,b])$ if and only if $\varphi \circ u' \in AC([a,b])$, generalized and weak solutions to (P) coincide when f is an L^1-Carathéodory function.

Now, let X be the Sobolev space $W_0^{1,p}([a,b])$ endowed with the norm

$$\|u\| := \left(\int_a^b |u'(t)|^p \, \mathrm{d}t\right)^{1/p},$$

and define the functional $\Phi : X \to \mathbb{R}$ by putting, for every $u \in X$,

$$\Phi(u) := \int_a^b \left(\int_0^{u'(t)} \varphi(x) \, \mathrm{d}x\right) \mathrm{d}t.$$

Simple calculations show that, for every $u \in X$,

$$\frac{1}{M}\frac{\|u\|^p}{p} \leqslant \Phi(u) \leqslant \frac{1}{m}\frac{\|u\|^p}{p}. \tag{2.1}$$

Clearly, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_a^b \varphi(u'(t))v'(t) \, \mathrm{d}t \tag{2.2}$$

for every $v \in X$, and $\Phi' : X \to X^*$ is continuous. Moreover, taking into account that Φ is convex, from Proposition 25.20 (i) of [**7**] we obtain that Φ is a sequentially weakly lower semicontinuous functional.

The remainder of this section is devoted to proving that Φ' admits a continuous inverse on X^*.

Proposition 2.4. *Φ' is coercive for every $p > 1$.*
If $p \geqslant 2$, then Φ' is uniformly monotone.
If $1 < p < 2$, then there exists $c > 0$ such that

$$\langle \Phi'(u) - \Phi'(v), u - v \rangle \geqslant c\frac{\|u - v\|^2}{(\|u\| + \|v\|)^{2-p}} \tag{2.3}$$

for every $u, v \in X$; hence, in particular, Φ' is strictly monotone.

https://doi.org/10.1017/S0013091502000767

Proof. For every $u \in X \setminus \{0\}$ we have

$$\begin{aligned}
\frac{\langle \Phi'(u), u\rangle}{\|u\|} &= \frac{\int_a^b \varphi(u'(t))u'(t)\,\mathrm{d}t}{\|u\|} \\
&= \frac{\int_a^b (\int_0^{u'(t)} [(p-1)|s|^{p-2}/h(s)]\,\mathrm{d}s) u'(t)\,\mathrm{d}t}{\|u\|} \\
&\geqslant \frac{1}{M} \frac{\int_a^b (\int_0^{u'(t)} (p-1)|s|^{p-2}\,\mathrm{d}s) u'(t)\,\mathrm{d}t}{\|u\|} \\
&= \frac{1}{M} \frac{\int_a^b |u'(t)|^{p-2}(u'(t))^2\,\mathrm{d}t}{\|u\|} \\
&= \frac{1}{M}\|u\|^{p-1},
\end{aligned}$$

hence Φ' is coercive.

Moreover, given $u, v \in X$ we have

$$\begin{aligned}
\langle \Phi'(u) - \Phi'(v), u - v\rangle &= \int_a^b (\varphi(u'(t)) - \varphi(v'(t)))(u'(t) - v'(t))\,\mathrm{d}t \\
&= \int_a^b \biggl(\int_{v'(t)}^{u'(t)} \frac{(p-1)|s|^{p-2}}{h(s)}\,\mathrm{d}s\biggr)(u'(t) - v'(t))\,\mathrm{d}t \\
&\geqslant \frac{1}{M}\int_a^b \biggl(\int_{v'(t)}^{u'(t)} (p-1)|s|^{p-2}\,\mathrm{d}s\biggr)(u'(t) - v'(t))\,\mathrm{d}t \\
&= \frac{1}{M}\int_a^b (|u'(t)|^{p-2}u'(t) - |v'(t)|^{p-2}v'(t))(u'(t) - v'(t))\,\mathrm{d}t,
\end{aligned}$$

thus, by (2.2) of [**6**], there exists $c_p > 0$ such that

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle \geqslant \begin{cases} \dfrac{c_p}{M}\displaystyle\int_a^b |u'(t) - v'(t)|^p\,\mathrm{d}t, & \text{if } p \geqslant 2, \\ \dfrac{c_p}{M}\displaystyle\int_a^b \frac{|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}\,\mathrm{d}t, & \text{if } 1 < p < 2. \end{cases}$$

If $p \geqslant 2$, then it follows at once that

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle \geqslant (c_p/M)\|u - v\|^p,$$

thus Φ' is uniformly monotone.

If $1 < p < 2$, by the Hölder inequality, we obtain

$$\int_a^b |u'(t) - v'(t)|^p\,\mathrm{d}t \leqslant \biggl(\int_a^b \frac{|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}\,\mathrm{d}t\biggr)^{p/2} \biggl(\int_a^b (|u'(t)| + |v'(t)|)^p\,\mathrm{d}t\biggr)^{(2-p)/2},$$

and, since

$$\int_a^b (|u'(t)| + |v'(t)|)^p\,\mathrm{d}t \leqslant 2^{p-1}\int_a^b (|u'(t)|^p + |v'(t)|^p)\,\mathrm{d}t \leqslant 2^p(\|u\| + \|v\|)^p,$$

https://doi.org/10.1017/S0013091502000767

we get

$$\int_a^b |u'(t) - v'(t)|^p \, \mathrm{d}t \leqslant 2^{p(2-p)/2} \biggl(\int_a^b \frac{|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}} \, \mathrm{d}t \biggr)^{p/2} (\|u\| + \|v\|)^{p(2-p)/2},$$

thus

$$\int_a^b \frac{|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}} \, \mathrm{d}t \geqslant \frac{\|u - v\|^2}{2^{2-p}(\|u\| + \|v\|)^{2-p}},$$

and, in conclusion,

$$\langle \Phi'(u) - \Phi'(v), u - v \rangle \geqslant \frac{c_p}{M 2^{2-p}} \frac{\|u - v\|^2}{(\|u\| + \|v\|)^{2-p}}.$$

□

Corollary 2.5. *$\Phi' : X \to X^*$ admits a continuous inverse.*

Proof. Clearly, Φ' is hemicontinuous, since it is continuous. Moreover, it is coercive by the first part of Proposition 2.4.

If $p \geqslant 2$, then Φ' is uniformly monotone by Proposition 2.4, thus the conclusion follows directly by Theorem 26.A (d) of [**7**].

If $1 < p < 2$, then Φ' is strictly monotone by Proposition 2.4, thus by Theorem 26.A (d) of [**7**] we obtain that the inverse $[\Phi']^{-1}$ exists and is bounded. Furthermore, given $g_1, g_2 \in X^*$, by inequality 2.3 of Proposition 2.4, where $u := [\Phi']^{-1}(g_1)$, $v := [\Phi']^{-1}(g_2)$, we get

$$\|[\Phi']^{-1}(g_1) - [\Phi']^{-1}(g_2)\| \leqslant (1/c)(\|[\Phi']^{-1}(g_1)\| + \|[\Phi']^{-1}(g_2)\|)^{2-p} \|g_1 - g_2\|_{X^*},$$

from which the Lipschitz continuity of $[\Phi']^{-1}$ follows. □

3. Results

In this section we apply Theorem 1.1 to prove that under suitable assumptions the problem (P) admits at least three classical solutions. We present the main result, Theorem 3.1, and some of its consequences.

Here, f, h, λ, p, φ, X and Φ are as in previous section. Moreover, for each $(t, \xi) \in [a, b] \times \mathbb{R}$, put

$$g(t, \xi) := \int_0^\xi f(t, x) \, \mathrm{d}x.$$

Theorem 3.1. *Assume that there exist four positive constants c, d, μ and s, with $c < d$ and $s < p$, such that*

(j) *$g(t, \xi) \geqslant 0$ for each $(t, \xi) \in ([a, a + \frac{1}{4}(b - a)] \cup [b - \frac{1}{4}(b - a), b]) \times [0, d]$;*

(jj) $$\frac{\int_a^b \sup_{|\xi| \leqslant c} g(t, \xi) \, \mathrm{d}t}{c^p} < \frac{m}{M(1 + 2^{p-1})} \frac{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t, d) \, \mathrm{d}t}{d^p};$$

(jjj) *$g(t, \xi) \leqslant \mu(1 + |\xi|^s)$ for all $(t, \xi) \in [a, b] \times \mathbb{R}$.*

https://doi.org/10.1017/S0013091502000767

Then, for each

$$\lambda \in \left] \frac{2^{2p-1}d^p/((b-a)^{p-1}mp)}{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t,d)\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant c} g(t,\xi)\,\mathrm{d}t}, \frac{2^p c^p/((b-a)^{p-1}Mp)}{\int_a^b \sup_{|\xi|\leqslant c} g(t,\xi)\,\mathrm{d}t} \right[,$$

the problem (P) *admits at least three classical solutions.*

Proof. For each $u \in X$, put $\Psi(u) := -\int_a^b g(t,u(t))\,\mathrm{d}t$.

It is well known that Ψ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = -\int_a^b f(t,u(t))v(t)\,\mathrm{d}t$$

for every $v \in X$, and that $\Psi' : X \to X^*$ is a continuous and compact operator.

We have seen in the previous section that Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteax derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by (2.2) for every $v \in X$, and that $\Phi' : X \to X^*$ admits a continuous inverse on X^*.

Since, by Proposition 2.2, classical solutions to problem (P) coincide with weak ones, and these are exactly the critical points of the functional $\Phi + \lambda\Psi$, our aim is to apply Theorem 1.1 to Φ and Ψ.

Hypothesis (i) of Theorem 1.1 follows in a simple way, by (jjj), (2.1) and

$$|u(t)| \leqslant \tfrac{1}{2}(b-a)^{(p-1)/p}\|u\|$$

for all $u \in X$ and for all $t \in [a,b]$.

In order to prove (ii) of Theorem 1.1, we claim that

$$\varphi_1(r) \leqslant \frac{\int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t}{r} \tag{C1}$$

for each $r > 0$, and

$$\varphi_2(r) \geqslant mp\frac{\int_a^b g(t,y(t))\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t}{\|y\|^p} \tag{C2}$$

for each $r > 0$ and every $y \in X$ such that

$$\frac{1}{M}\frac{\|y\|^p}{p} \geqslant r \tag{3.1}$$

and

$$\int_a^b g(t,y(t))\,\mathrm{d}t \geqslant \int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t. \tag{3.2}$$

https://doi.org/10.1017/S0013091502000767

In fact, for $r>0$, and taking into account that the function $u\equiv 0$ on $[a,b]$ obviously belongs to $\Phi^{-1}(]-\infty,r[)$, and that $\Psi(0)=0$, we get

$$\varphi_1(r)\leqslant \frac{\sup_{\overline{\Phi^{-1}(]-\infty,r[)}^w}\int_a^b g(t,x(t))\,\mathrm{d}t}{r},$$

and, since $\overline{\Phi^{-1}(]-\infty,r[)}^w=\Phi^{-1}(]-\infty,r])$, we have

$$\frac{\sup_{\overline{\Phi^{-1}(]-\infty,r[)}^w}\int_a^b g(t,x(t))\,\mathrm{d}t}{r}=\frac{\sup_{\Phi^{-1}(]-\infty,r])}\int_a^b g(t,x(t))\,\mathrm{d}t}{r};$$

thus, from

$$\Phi^{-1}(]-\infty,r])\subset\left\{x\in X:\frac{1}{M}\frac{\|x\|^p}{p}\leqslant r\right\}$$

and

$$|x(t)|\leqslant \tfrac{1}{2}(b-a)^{(p-1)/p}\|x\|\leqslant([prM(b-a)^{p-1}]^{1/p})/2,$$

for every $x\in X$ such that

$$\frac{1}{M}\frac{\|x\|^p}{p}\leqslant r$$

and for each $t\in[a,b]$, we obtain

$$\frac{\sup_{\Phi^{-1}(]-\infty,r])}\int_a^b g(t,x(t))\,\mathrm{d}t}{r}\leqslant\frac{\int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t}{r}.$$

So, (C1) is proved.

Moreover, for each $r>0$ and each $y\in X$ such that $\Phi(y)\geqslant r$, hence in particular for each $y\in X$ such that

$$\frac{1}{M}\frac{\|y\|^p}{p}\geqslant r,$$

we have

$$\varphi_2(r)\geqslant\inf_{x\in\Phi^{-1}(]-\infty,r[)}\frac{\int_a^b g(t,y(t))\,\mathrm{d}t-\int_a^b g(t,x(t))\,\mathrm{d}t}{\Phi(y)-\Phi(x)},$$

thus, from

$$\Phi^{-1}(]-\infty,r[)\subset\left\{x\in X:\frac{1}{M}\frac{\|x\|^p}{p}<r\right\}$$

and

$$|x(t)|\leqslant \tfrac{1}{2}(b-a)^{(p-1)/p}\|x\|<([prM(b-a)^{p-1}]^{1/p})/2,$$

for every $x\in X$ such that

$$\frac{1}{M}\frac{\|x\|^p}{p}<r$$

and for each $t \in [a,b]$, we obtain

$$\inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\int_a^b g(t,y(t))\,\mathrm{d}t - \int_a^b g(t,x(t))\,\mathrm{d}t}{\Phi(y)-\Phi(x)} \geqslant \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\int_a^b g(t,y(t))\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t}{\Phi(y)-\Phi(x)},$$

from which, since

$$0 < \Phi(y) - \Phi(x) \leqslant \Phi(y) \leqslant \frac{1}{m}\frac{\|y\|^p}{p}$$

for every $x \in \Phi^{-1}(]-\infty,r[)$, and under the further condition (3.2), we can write

$$\inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\int_a^b g(t,y(t))\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t}{\Phi(y)-\Phi(x)} \geqslant mp\frac{\int_a^b g(t,y(t))\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t}{\|y\|^p}.$$

So, (C2) is also proved.

Now, in order to prove (ii) of Theorem 1.1, taking into account (C1) and (C2), it suffices to find $r > 0$ and $y \in X$, which verifies (3.1), and

$$\frac{\int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t}{r} < mp\frac{\int_a^b g(t,y(t))\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t}{\|y\|^p}. \tag{3.3}$$

Notice that (3.2) is a consequence of (3.3).

To this end, we define

$$y(t) := \begin{cases} \dfrac{4}{b-a}d(t-a), & \text{if } t \in [a, a+\frac{1}{4}(b-a)[, \\ d, & \text{if } t \in [a+\frac{1}{4}(b-a), b-\frac{1}{4}(b-a)], \\ \dfrac{4}{b-a}d(b-t), & \text{if } t \in]b-\frac{1}{4}(b-a), b] \end{cases}$$

and

$$r := \frac{(2c)^p}{pM}\frac{1}{(b-a)^{p-1}}.$$

Clearly, $y \in X$ and

$$\|y\|^p = 2^{2p-1}\frac{1}{(b-a)^{p-1}}d^p.$$

Hence, since $c < d$, we have

$$\frac{1}{M}\frac{\|y\|^p}{p} > r.$$

From (jj), since $m/M \leqslant 1$, we get

$$\frac{\int_a^b \sup_{|\xi|\leqslant c} g(t,\xi)\,\mathrm{d}t}{2^{p-1}c^p} < \frac{1}{2^{p-1}(1+2^{p-1})}\frac{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t,d)\,\mathrm{d}t}{d^p}$$
$$= \left(\frac{1}{2^{p-1}} - \frac{1}{1+2^{p-1}}\right)\frac{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t,d)\,\mathrm{d}t}{d^p},$$

then

$$\frac{1}{1+2^{p-1}}\frac{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t,d)\,\mathrm{d}t}{d^p} + \frac{\int_a^b \sup_{|\xi|\leqslant c} g(t,\xi)\,\mathrm{d}t}{2^{p-1}c^p} < \frac{1}{2^{p-1}}\frac{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t,d)\,\mathrm{d}t}{d^p},$$

thus, since $c < d$,

$$\frac{1}{1+2^{p-1}}\frac{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t,d)\,\mathrm{d}t}{d^p} < \frac{1}{2^{p-1}}\frac{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t,d)\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant c} g(t,\xi)\,\mathrm{d}t}{d^p},$$

hence, using (jj) again,

$$\frac{\int_a^b \sup_{|\xi|\leqslant c} g(t,\xi)\,\mathrm{d}t}{c^p} < \frac{m}{M}\frac{1}{2^{p-1}}\frac{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t,d)\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant c} g(t,\xi)\,\mathrm{d}t}{d^p}.$$

Finally, taking into account the values of r and $\|y\|^p$, and using (j), we get

$$\frac{\int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t}{r}$$
$$< mp\frac{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t,y(t))\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t}{\|y\|^p}$$
$$\leqslant mp\frac{\int_a^b g(t,y(t))\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant([prM(b-a)^{p-1}]^{1/p})/2} g(t,\xi)\,\mathrm{d}t}{\|y\|^p}.$$

Thus, the conclusion follows by Theorem 1.1, by observing that

$$\frac{1}{\varphi_2(r)} \leqslant \frac{2^{2p-1}d^p/((b-a)^{p-1}mp)}{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t,d)\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant c} g(t,\xi)\,\mathrm{d}t}$$

and

$$\frac{1}{\varphi_1(r)} \geqslant \frac{2^p c^p/((b-a)^{p-1}Mp)}{\int_a^b \sup_{|\xi|\leqslant c} g(t,\xi)\,\mathrm{d}t}.$$

□

Remark 3.2. Taking into account Remark 2.3, if we assume that $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is L^1-Carathéodory, that there exist three positive constants c, d, s, with $c < d$ and $s < p$, such that (j) and (jj) in Theorem 3.1 hold and, furthermore, that there exists a function $\mu \in L^1([a,b])$ such that

(jjj)′ $g(t,\xi) \leqslant \mu(t)(1+|\xi|^s)$ for almost every $t \in [a,b]$ and for all $\xi \in \mathbb{R}$,

https://doi.org/10.1017/S0013091502000767

then, for each

$$\lambda \in \left] \frac{2^{2p-1}d^p/((b-a)^{p-1}mp)}{\int_{a+(b-a)/4}^{b-(b-a)/4} g(t,d)\,\mathrm{d}t - \int_a^b \sup_{|\xi|\leqslant c} g(t,\xi)\,\mathrm{d}t}, \frac{2^p c^p/((b-a)^{p-1}Mp)}{\int_a^b \sup_{|\xi|\leqslant c} g(t,\xi)\,\mathrm{d}t} \right[,$$

the problem (P) admits at least three generalized solutions.

The following is an example of an application of Theorem 3.1.

Example 3.3. The problem

$$\begin{cases} -(|u'|^2u')' = \lambda[\mathrm{e}^{-tu}u^{12}(13-tu)(1+\frac{1}{2}\sin u') + (1+\frac{1}{2}\sin u')], \\ u(0) = u(1) = 0, \end{cases}$$

admits at least three non-trivial classical solutions for each $\lambda \in]\frac{2}{3}, \frac{4}{3}[$. In fact, the function $g(t,u) = \mathrm{e}^{-tu}u^{13} + u$ satisfies all assumptions of Theorem 3.1 by choosing, for instance, $c = 1$ and $d = 2$.

We now point out a particular case of Theorem 3.1, in which the function f has separated variables and h is equal to 1. Given two non-negative continuous functions $\alpha : [a,b] \to \mathbb{R}$, $\beta : \mathbb{R} \to \mathbb{R}$, put

$$A := \int_{a+(b-a)/4}^{b-(b-a)/4} \alpha(t)\,\mathrm{d}t, \qquad \|\alpha\|_1 := \int_a^b \alpha(t)\,\mathrm{d}t, \qquad B(\xi) := \int_0^\xi \beta(x)\,\mathrm{d}x \quad (\xi \in \mathbb{R}).$$

Theorem 3.4. *Assume that there exist four positive constants c, d, μ, s, with $c < d$ and $s < p$, such that*

(k) $\|\alpha\|_1 \dfrac{B(c)}{c^p} < \dfrac{A}{1+2^{p-1}} \dfrac{B(d)}{d^p}$;

(kk) $B(\xi) \leqslant \mu(1+|\xi|^s)$ *for all* $\xi \in \mathbb{R}$.

Then, for each

$$\lambda \in \left] \frac{2^{2p-1}d^p/((b-a)^{p-1}p)}{AB(d) - \|\alpha\|_1 B(c)}, \frac{2^p c^p/((b-a)^{p-1}p)}{\|\alpha\|_1 B(c)} \right[,$$

the problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda\alpha(t)\beta(u), \\ u(a) = u(b) = 0 \end{cases} \tag{P1}$$

admits at least three classical solutions.

Remark 3.5. In similar assumptions to those of Theorem 3.4 (with α constant), in [**3**] (see Theorem 2.3) it was proved that there exists an open interval $\Lambda \subseteq]0, +\infty[$ such that, for each $\lambda \in \Lambda$, the problem (P1) admits at least three solutions, which are uniformly bounded in norm with respect to λ. In the present case, we establish a precise interval of parameters λ for which the problem (P1) has three solutions.

As a consequence of Theorem 3.4, we obtain the proof of Theorem 1.2 in the introduction.

Proof of Theorem 1.2. Fix $b > 0$ and pick

$$\lambda > \frac{1}{b^p} \frac{2^{2p}}{p} \frac{d^p}{B(d)}.$$

Since

$$\lim_{x \to 0^+} \frac{\beta(x)}{x^{p-1}} = 0,$$

there exists $c > 0$ such that $c < d$,

$$\frac{B(c)}{c^p} < \min\left\{ \frac{1}{1+2^{p-1}} \frac{1}{2} \frac{B(d)}{d^p}, \frac{2^p}{b^p p \lambda} \right\},$$

and

$$\lambda > \frac{2^{2p} d^p}{b^p p (B(d) - 2B(c))}.$$

Moreover, assumption (kk) of Theorem 3.4 follows easily from

$$\lim_{x \to +\infty} \frac{\beta(x)}{x^q} \in \mathbb{R}.$$

Hence, for each

$$\lambda > \frac{1}{b^p} \frac{2^{2p}}{p} \frac{d^p}{B(d)},$$

the problem

$$\begin{cases} -(|u'|^{p-2} u')' = \lambda \beta(u), \\ u(0) = u(b) = 0 \end{cases} \tag{P2}$$

admits at least two non-trivial classical solutions and, as a consequence, by choosing

$$\bar{b} := \left(\frac{2^{2p}}{p} \frac{d^p}{B(d)} \right)^{1/p}, \tag{3.4}$$

we have the conclusion. □

Finally, we present a very easy example of an application of Theorem 1.2.

Example 3.6. Let $\beta : \mathbb{R} \to \mathbb{R}$ be the function defined as follows:

$$\beta(u) := \begin{cases} u^4, & \text{if } u \leqslant 1, \\ u^2, & \text{if } u > 1. \end{cases}$$

Taking into account that $\bar{b} = 2(20)^{1/4}$ (see (3.4) with $p = 4$ and $d = 1$), from Theorem 1.2 we obtain that, for every $b > \bar{b}$, the problem

$$\begin{cases} -(|u'|^2 u')' = \beta(u), \\ u(0) = u(b) = 0 \end{cases} \tag{Pb}$$

admits at least two non-trivial classical solutions.

https://doi.org/10.1017/S0013091502000767

References

1. I. Addou, Multiplicity results for classes of one-dimensional p-Laplacian boundary-value problems with cubic-like nonlinearities, *Electron. J. Diff. Eqns* **2000** (2000), 1–42.
2. D. Averna and G. Bonanno, A three critical point theorem and its applications to the ordinary Dirichlet problem, *Topolog. Meth. Nonlin. Analysis* **22** (2003), 93–103.
3. G. Bonanno and R. Livrea, Multiplicity theorems for the Dirichlet problem involving the p-Laplacian, *Nonlin. Analysis* **54** (2003), 1–7.
4. H. Dang, K. Schmitt and R. Shivaji, On the number of solutions of boundary value problems involving the p-Laplacian, *Electron. J. Diff. Eqns* **1996** (1996), 1–9.
5. L. Kong and J. Wang, Multiple positive solutions for the one-dimensional p-Laplacian, *Nonlin. Analysis* **42** (2000), 1327–1333.
6. J. Simon, *Regularitè de la solution d'une equation non lineaire dans* $\mathbb{R}^n$ (ed. P. Benilan and J. Robert), Lecture Notes in Mathematics, no. 665 (Springer, 1978).
7. E. Zeidler, *Nonlinear functional analysis and its applications*, vol. II/B (Springer, 1990).