

EXISTENCE OF POSITIVE SOLUTIONS FOR SUPERLINEAR SEMIPOSITONE m -POINT BOUNDARY-VALUE PROBLEMS

RUYUN MA

*Department of Mathematics, Northwest Normal University,
Lanzhou 730070, Gansu, People's Republic of China (mary@nwnu.edu.cn)*

(Received 18 April 2002)

Abstract In this paper we consider the existence of positive solutions to the boundary-value problems

$$\begin{aligned}(p(t)u')' - q(t)u + \lambda f(t, u) &= 0, \quad r < t < R, \\ au(r) - bp(r)u'(r) &= \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(R) + dp(R)u'(R) &= \sum_{i=1}^{m-2} \beta_i u(\xi_i),\end{aligned}$$

where λ is a positive parameter, $a, b, c, d \in [0, \infty)$, $\xi_i \in (r, R)$, $\alpha_i, \beta_i \in [0, \infty)$ (for $i \in \{1, \dots, m-2\}$) are given constants satisfying some suitable conditions. Our results extend some of the existing literature on superlinear semipositone problems. The proofs are based on the fixed-point theorem in cones.

Keywords: multipoint boundary-value problems; positive solutions; fixed-point theorem; cones

2000 *Mathematics subject classification:* Primary 34B10, 34B18, 34B15

1. Introduction

Multipoint boundary-value problems (BVPs) for ordinary differential equations arise in a variety of areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multipoint BVP [7]; also, many problems in the theory of elastic stability can be handled by multipoint problems [9].

In [5], Il'in and Moiseev studied the existence of solutions for a linear multipoint BVP. Motivated by that study, Gupta [3] studied certain three-point BVPs for nonlinear ordinary differential equations. Since then, more general nonlinear multipoint BVPs have been studied by several authors. We refer the reader to [3, 4, 6, 10] for some references.

In this paper, we are interested in the existence of positive solutions for the second-order m -point BVP

$$\left. \begin{aligned} (p(t)u')' - q(t)u + \lambda f(t, u) &= 0, \quad r < t < R, \\ au(r) - bp(r)u'(r) &= \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(R) + dp(R)u'(R) &= \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{aligned} \right\} \quad (1.1)$$

where $p, q \in C([r, R], (0, \infty))$, $a, b, c, d \in [0, \infty)$, $\xi_i \in (Y, R)$, $\alpha_i, \beta_i \in (0, \infty)$ (for $i \in \{1, \dots, m-2\}$) are given constants. If $q \equiv 0$ and $\alpha_i = \beta_i = 0$ for $i = 1, \dots, m-2$, then the m -point BVP (1.1) reduces to the two-point BVP

$$\left. \begin{aligned} (p(t)u')' + \lambda f(t, u) &= 0, \quad r < t < R, \\ au(r) - bp(r)u'(r) &= 0, \\ cu(R) + dp(R)u'(R) &= 0. \end{aligned} \right\} \quad (1.2)$$

In 1996, Anuradha, Hai and Shivaji [1] studied the existence of positive solutions for (1.2) under the assumptions:

- (A1) $p \in C([r, R], (0, \infty))$;
- (A2) $a, b, c, d \in [0, \infty)$ with $ac + ad + bc > 0$;
- (A3) $f : [r, R] \times [0, \infty) \rightarrow R$ is continuous and there exists an $M > 0$ such that $f(t, u) \geq -M$ for every $t \in [r, R], u \geq 0$; and
- (A4) $\lim_{u \rightarrow \infty} (f(t, u)/u) = \infty$ uniformly on a compact subinterval $[\alpha, \beta]$ of (r, R) .

They established the following result for (1.2).

Theorem 1.1 (see Theorem 1 in [1]). Suppose that (A1)–(A4) hold. Then (1.2) has a positive solution for $\lambda > 0$ sufficiently small.

If $r = 0$, $R = 1$, $\lambda = 1$, $p(t) \equiv 1$, $q(t) \equiv 0$, $f(t, u) = h(t)\bar{f}(u)$, $a = c = 1$, $b = d = 0$, $\alpha_i = 0$ for $i = 1, \dots, m-2$, and $\beta_j = 0$ for $j = 2, \dots, m-2$, then (1.1) reduces to the three-point BVP

$$\left. \begin{aligned} u'' + h(t)\bar{f}(u) &= 0, \quad 0 < t < 1, \\ u(0) &= 0, \quad u(1) = \beta_1 u(\xi_1). \end{aligned} \right\} \quad (1.3)$$

In 1999, Ma [6] obtained the following result for (1.3).

Theorem 1.2 (see Theorem 1 in [6]).

- (H1) $0 < \beta_1 \xi_1 < 1$.
- (H2) $\bar{f} \in C([0, \infty), [0, \infty))$.
- (H3) $h \in C([0, 1], [0, \infty))$ and there exists $t_0 \in [\xi_1, 1]$ such that $h(t_0) > 0$.

Then (1.3) has at least one positive solution in one of the two following cases:

- (i) $\bar{f}_0 = 0$ and $\bar{f}_\infty = \infty$,
- (ii) $\bar{f}_0 = \infty$ and $\bar{f}_\infty = 0$,

where

$$\bar{f}_0 := \lim_{u \rightarrow 0^+} \frac{\bar{f}(u)}{u}, \quad \bar{f}_\infty := \lim_{u \rightarrow \infty} \frac{\bar{f}(u)}{u}.$$

Theorem 1.2 has been improved by Webb [10]. We remark that in the proof of Theorem 1.2 we rewrite (1.3) as the following equivalent integral equation:

$$\begin{aligned} u(t) = & - \int_0^t (t-s)h(s)\bar{f}(u(s)) \, ds - \frac{\beta_1 t}{1-\beta_1 \xi_1} \int_0^{\xi_1} (\xi_1-s)h(s)\bar{f}(u(s)) \, ds \\ & + \frac{t}{1-\beta_1 \xi_1} \int_0^1 (1-s)h(s)\bar{f}(u(s)) \, ds \\ := & (Au)(t). \end{aligned} \quad (1.4)$$

Clearly, $(Au)(t)$ contains one positive term and two negative terms. This form is not convenient for studying the existence of positive solutions. In fact, in order to apply the fixed-point theorem in cones, we need to show that

$$(Ay)(t) \geq 0, \quad \text{for all } y \in C([0, 1], [0, \infty)) \text{ and } t \in [0, 1]. \quad (1.5)$$

Since Ay contains two negative terms, it is not easy to show that (H1)–(H3) imply that (1.5) holds.

In this paper, we consider the more general m -point BVP (1.1). To deal with (1.1), we give a new integral equation which is equivalent to

$$\begin{aligned} (p(t)u')' - q(t)u + y(t) &= 0, \quad r < t < R, \\ au(r) - bp(r)u'(r) &= \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(R) + dp(R)u'(R) &= \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{aligned}$$

and contains two positive terms if $y \geq 0$. Our most important result (see Theorem 3.1 below) extends the main results of [1] in two directions:

- (i) the m -point BVP (1.1) is considered; and
- (ii) the case $q(t) > 0$ is studied.

By a positive solution of (1.1) we understand a function $u(t)$ which is positive on (r, R) and satisfies the differential equation and the boundary conditions in (1.1).

The main tool of this paper is the following well-known Guo–Krasnoselskii fixed-point theorem.

Theorem 1.3 (see [2]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$$

be a completely continuous operator such that

- (i) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$; or
- (ii) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. Preliminary lemmas

In the rest of the paper, we make the following assumptions:

(C1) $p \in C^1([r, R], (0, \infty))$, $q \in C([r, R], (0, \infty))$; and

(C2) $a, b, c, d \in [0, \infty)$ with $ac + ad + bc > 0$, $\alpha_i, \beta_i \in [0, \infty)$ for $i \in \{1, \dots, m-2\}$.

To state and prove the main results of this paper, we need the following lemmas.

Lemma 2.1. *Let (C1) and (C2) hold. Let ψ and ϕ be the solutions of the linear problems*

$$\left. \begin{aligned} (p(t)\psi'(t))' - q(t)\psi(t) &= 0, \\ \psi(r) &= b, \quad p(r)\psi'(r) = a \end{aligned} \right\} \quad (2.1)$$

and

$$\left. \begin{aligned} (p(t)\phi'(t))' - q(t)\phi(t) &= 0, \\ \phi(R) &= d, \quad p(R)\phi'(R) = -c, \end{aligned} \right\} \quad (2.2)$$

respectively. Then

- (i) ψ is strictly increasing on $[r, R]$, and $\psi(t) > 0$ on $(r, R]$; and
- (ii) ϕ is strictly decreasing on $[r, R]$, and $\phi(t) > 0$ on $[r, R)$.

Proof. We shall give a proof for (i) only. The proof of (ii) follows in a similar manner. It is easy to see that (2.1) is equivalent to the problem

$$\left. \begin{aligned} \psi''(t) + \frac{p'(t)}{p(t)}\psi'(t) - \frac{q(t)}{p(t)}\psi(t) &= 0, \\ \psi(r) &= b, \quad \psi'(r) = \frac{a}{p(r)}. \end{aligned} \right\} \quad (2.3)$$

Now we divide the proof into three steps.

Step 1. We show that there exists $\sigma \in (0, R - r)$ such that ψ is strictly increasing on $(r, r + \sigma)$.

If $a > 0$, then we are done. If $a = 0$, then we know from (C2) that $b > 0$. Therefore, we have from (2.3) that

$$\psi''(r) = \frac{q(r)}{p(r)}\psi(r) > 0,$$

which implies that there exists $\sigma > 0$ such that $\psi'(t) > 0$ on $(r, r + \sigma)$. Thus $\psi(t)$ is strictly increasing on $(r, r + \sigma)$.

Step 2. We show that ψ has no local maxima on all of (r, R) .

In fact, by Step 1, ψ is positive and strictly increasing on $(r, r + \sigma)$. So we can apply the maximum principle (see [8, Theorem 1 of Chapter 1]) to show that there are no local maxima on (r, R) . Moreover, ψ is non-decreasing on (r, R) .

Step 3. We show that ψ is strictly increasing on $[r, R]$.

If there exists $t_2, t_3 \subset [r, R]$ with $t_2 < t_3$ such that $\psi(t_2) = \psi(t_3)$, then

$$\psi(t) \equiv \psi(t_3), \quad t \in [t_2, t_3].$$

This implies

$$\psi'(t) = \psi''(t) = 0, \quad t \in [t_2, t_3].$$

We note that by Steps 1 and 2, $\psi(t_3) > 0$. Thus from (2.3) we get

$$\psi''(t_3) = \frac{q(t_3)}{p(t_3)}\psi(t_3) > 0.$$

This contradicts the fact that $\psi''(t_3) = 0$. □

Notation. Set

$$\rho := p(r) \begin{vmatrix} \phi(r) & \psi(r) \\ \phi'(r) & \psi'(r) \end{vmatrix}, \quad \Delta := \begin{vmatrix} -\sum_{i=1}^{m-2} \alpha_i \psi(\xi_i) & \rho - \sum_{i=1}^{m-2} \alpha_i \phi(\xi_i) \\ \rho - \sum_{i=1}^{m-2} \beta_i \psi(\xi_i) & -\sum_{i=1}^{m-2} \beta_i \phi(\xi_i) \end{vmatrix}.$$

Lemma 2.2. Let (C1) and (C2) hold. Assume that

(C3) $\Delta \neq 0$.

Then for $y \in C[r, R]$, the problem

$$\left. \begin{aligned} (p(t)u'(t))' - q(t)u(t) + y(t) &= 0, \quad r < t < R, \\ au(r) - bu'(r) &= \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(R) + du'(R) &= \sum_{i=1}^{m-2} \beta_i u(\xi_i) \end{aligned} \right\} \quad (2.4)$$

has a unique solution

$$u(t) = \int_r^R G(t, s)y(s) \, ds + A(y)\psi(t) + B(y)\phi(t), \quad (2.5)$$

where

$$G(t, s) = \frac{1}{\rho} \begin{cases} \phi(t)\psi(s), & r \leq s \leq t \leq R, \\ \phi(s)\psi(t), & r \leq t \leq s \leq R, \end{cases} \quad (2.6)$$

$$A(y) := \frac{1}{\Delta} \begin{vmatrix} \sum_{i=1}^{m-2} \alpha_i \int_r^R G(\xi_i, s)y(s) \, ds & \rho - \sum_{i=1}^{m-2} \alpha_i \phi(\xi_i) \\ \sum_{i=1}^{m-2} \beta_i \int_r^R G(\xi_i, s)y(s) \, ds & - \sum_{i=1}^{m-2} \beta_i \phi(\xi_i) \end{vmatrix} \quad (2.7)$$

and

$$B(y) := \frac{1}{\Delta} \begin{vmatrix} - \sum_{i=1}^{m-2} \alpha_i \psi(\xi_i) & \sum_{i=1}^{m-2} \alpha_i \int_r^R G(\xi_i, s)y(s) \, ds \\ \rho - \sum_{i=1}^{m-2} \beta_i \psi(\xi_i) & \sum_{i=1}^{m-2} \beta_i \int_r^R G(\xi_i, s)y(s) \, ds \end{vmatrix}. \quad (2.8)$$

Proof. The proof follows by routine calculations. \square

Lemma 2.3. Let (C1) and (C2) hold. Assume

$$(C4) \quad \Delta < 0, \quad \rho - \sum_{i=1}^{m-2} \alpha_i \phi(\xi_i) > 0, \quad \rho - \sum_{i=1}^{m-2} \beta_i \psi(\xi_i) > 0.$$

Then for $y \in C[r, R]$ with $y \geq 0$, the unique solution u of the problem (2.4) satisfies

$$u(t) \geq 0, \quad \text{for } t \in [r, R]. \quad (2.9)$$

Proof. This is an immediate consequence of the facts that $G \geq 0$ on $[r, R] \times [r, R]$ and

$$A(y) \geq 0, \quad B(y) \geq 0. \quad (2.10)$$

\square

Lemma 2.4. Let (C1), (C2) and (C4) hold. Let

$$\tilde{q}(t) := \min \left\{ \frac{\phi(t)}{\phi(r)}, \frac{\psi(t)}{\psi(R)} \right\}. \quad (2.11)$$

Then for $y \in C[r, R]$ with $y \geq 0$, the unique solution u of the problem (2.4) satisfies

$$u(t) \geq \frac{1}{2} \gamma(t) \|u\|,$$

where $\|u\| = \max\{u(t) | t \in [r, R]\}$ and

$$\gamma(t) := \frac{1}{k_0} [\tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t)] \quad (2.12)$$

with $k_0 \in N$ a fixed integer such that

$$\frac{1}{k_0} [\tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t)] \leq 1, \quad \text{for all } t \in [r, R],$$

$$\tilde{A} := \frac{1}{\Delta} \begin{vmatrix} \sum_{i=1}^{m-2} \alpha_i \tilde{q}(\xi_i) & \rho - \sum_{i=1}^{m-2} \alpha_i \phi(\xi_i) \\ \sum_{i=1}^{m-2} \beta_i \tilde{q}(\xi_i) & - \sum_{i=1}^{m-2} \beta_i \phi(\xi_i) \end{vmatrix} \quad (2.13)$$

and

$$\tilde{B} := \frac{1}{\Delta} \begin{vmatrix} - \sum_{i=1}^{m-2} \alpha_i \psi(\xi_i) & \sum_{i=1}^{m-2} \alpha_i \tilde{q}(\xi_i) \\ \rho - \sum_{i=1}^{m-2} \beta_i \psi(\xi_i) & \sum_{i=1}^{m-2} \beta_i \tilde{q}(\xi_i) \end{vmatrix}. \quad (2.14)$$

Proof. We have from (2.6) that

$$0 \leq G(t, s) \leq G(s, s), \quad t \in [r, R],$$

which implies

$$u(t) \leq \int_r^R G(s, s)y(s) \, ds + A(y)\psi(t) + B(y)\phi(t), \quad \text{for all } t \in [r, R]. \quad (2.15)$$

Applying (2.6), we have that for $t \in [r, R]$

$$\begin{aligned} \frac{G(t, s)}{G(s, s)} &= \begin{cases} \frac{\phi(t)}{\phi(s)}, & r \leq s \leq t \leq R, \\ \frac{\psi(t)}{\psi(s)}, & r \leq t \leq s \leq R, \end{cases} \\ &\geq \begin{cases} \frac{\phi(t)}{\phi(r)}, & r \leq s \leq t \leq R, \\ \frac{\psi(t)}{\psi(R)}, & r \leq t \leq s \leq R, \end{cases} \\ &\geq \tilde{q}(t), \end{aligned} \quad (2.16)$$

where $\tilde{q}(t)$ is as in (2.11). Combining (2.16) with (2.7) and (2.8), we can conclude that

$$A(y) \geq \tilde{A} \int_r^R G(s, s)y(s) \, ds, \quad B(y) \geq \tilde{B} \int_r^R G(s, s)y(s) \, ds, \quad (2.17)$$

where \tilde{A} and \tilde{B} are as in (2.13) and (2.14), respectively. Thus for $t \in [r, R]$,

$$\begin{aligned}
 u(t) &= \int_r^R G(t, s)y(s) \, ds + A(y)\psi(t) + B(y)\phi(t) \\
 &\geq \frac{1}{2} \left[\int_r^R G(t, s)y(s) \, ds + A(y)\psi(t) + B(y)\phi(t) \right] + \frac{1}{2} [A(y)\psi(t) + B(y)\phi(t)] \\
 &= \frac{1}{2} \left[\int_r^R \frac{G(t, s)}{G(s, s)} G(s, s)y(s) \, ds + A(y)\psi(t) + B(y)\phi(t) \right] + \frac{1}{2} [A(y)\psi(t) + B(y)\phi(t)] \\
 &\geq \frac{1}{2} \left[\tilde{q}(t) \int_r^R G(s, s)y(s) \, ds + \tilde{A} \int_r^R G(s, s)y(s) \, ds \psi(t) \right. \\
 &\quad \left. + \tilde{B} \int_r^R G(s, s)y(s) \, ds \phi(t) \right] + \frac{1}{2} [A(y)\psi(t) + B(y)\phi(t)] \\
 &= \frac{1}{2} k_0 \gamma(t) \int_r^R G(s, s)y(s) \, ds + \frac{1}{2} [A(y)\psi(t) + B(y)\phi(t)] \\
 &\geq \frac{1}{2} \gamma(t) \left[\int_r^R G(s, s)y(s) \, ds + A(y)\psi(t) + B(y)\phi(t) \right] \\
 &\geq \frac{1}{2} \gamma(t) \|u\| \quad (\text{by (2.15)}),
 \end{aligned}$$

where

$$\gamma(t) := \frac{1}{k_0} [\tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t)].$$

□

Lemma 2.5. *Let (C1)–(C4) hold and Let \bar{w} be the solution of*

$$\left. \begin{aligned}
 (p(t)u'(t))' - q(t)u(t) + 1 &= 0, \quad r < t < R, \\
 au(r) - bu'(r) &= \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\
 cu(R) + du'(R) &= \sum_{i=1}^{m-2} \beta_i u(\xi_i).
 \end{aligned} \right\} \quad (2.18)$$

Then there exists a positive number C such that $\bar{w}(t) \leq C\gamma(t)$ for every $t \in [r, R]$.

Proof. By Lemma 2.2, we know that

$$\begin{aligned}
 \bar{w}(t) &= \int_r^R G(t, s) \, ds + A(1)\psi(t) + B(1)\phi(t) \\
 &= \frac{1}{\rho} \left[\int_r^t \phi(t)\psi(s) \, ds + \int_r^t \psi(t)\phi(s) \, ds \right] + A(1)\psi(t) + B(1)\phi(t)
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\rho} \left[\int_r^t \phi(t)\psi(t) \, ds + \int_t^R \psi(t)\phi(t) \, ds \right] + A(1)\psi(t) + B(1)\phi(t) \\
&\leq \frac{1}{\rho} (R-r)\phi(t)\psi(t) + A(1)\psi(t) + B(1)\phi(t) \\
&\leq \frac{1}{\rho} (R-r)\phi(r)\psi(R)\tilde{q}(t) + A(1)\psi(t) + B(1)\phi(t) \\
&= \frac{1}{\rho} (R-r)\phi(r)\psi(R)\tilde{q}(t) + \frac{A(1)}{\tilde{A}}\tilde{A}\psi(t) + \frac{B(1)}{\tilde{B}}\tilde{B}\phi(t) \\
&\leq \mu[\tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t)] \\
&= C\gamma(t),
\end{aligned}$$

where $C := k_0\mu$ and

$$\mu := \begin{cases} \max\left\{\frac{1}{\rho}(R-r)\phi(r)\psi(R), \frac{A(1)}{\tilde{A}}, \frac{B(1)}{\tilde{B}}\right\}, & \text{if } \sum_{i=1}^{m-2} \alpha_i \neq 0, \sum_{i=1}^{m-2} \beta_i \neq 0, \\ \max\left\{\frac{1}{\rho}(R-r)\phi(r)\psi(R), \frac{B(1)}{\tilde{B}}\right\}, & \text{if } \sum_{i=1}^{m-2} \alpha_i \neq 0, \sum_{i=1}^{m-2} \beta_i = 0, \\ \max\left\{\frac{1}{\rho}(R-r)\phi(r)\psi(R), \frac{A(1)}{\tilde{A}}\right\}, & \text{if } \sum_{i=1}^{m-2} \alpha_i = 0, \sum_{i=1}^{m-2} \beta_i \neq 0. \end{cases} \quad (2.19)$$

We note that

$$\tilde{A} > 0 \quad \text{if } \sum_{i=1}^{m-2} \beta_i \neq 0$$

and

$$\tilde{B} > 0 \quad \text{if } \sum_{i=1}^{m-2} \alpha_i \neq 0.$$

So the constant C in (2.19) is well defined. \square

3. The main result

The main result of the paper is the following theorem.

Theorem 3.1. *Let (C1), (C2), (C4) and (A3) and (A4) hold. Then (1.1) has a positive solution for $\lambda > 0$ sufficiently small.*

Remark 3.2. Theorem 3.1 extends [1, Theorem 1] in two main directions:

- (i) the m -point BVPs (1.1) are considered; and
- (ii) the case $q(t) > 0$ is studied.

Proof of Theorem 3.1. Let λ satisfy

$$0 < \lambda < \min \left\{ \frac{1}{C_1 \|\bar{w}\|}, \frac{1}{2CM} \right\}, \quad (3.1)$$

where $C_1 = \sup\{g(t, u) \mid r \leq t \leq R, 0 \leq u \leq 1\}$, $g(t, u) := f(t, u) + M$ and C is the constant defined in Lemma 2.5. Let $w = \lambda M \bar{w}$. Then u is a positive solution of (1.1) if and only if $\tilde{u} = u + w$ is a solution of

$$\left. \begin{aligned} (p(t)u')' - q(t)u + \lambda \tilde{g}(t, u - w) &= 0, \quad r < t < R, \\ au(r) - bp(r)u'(r) &= \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(R) + dp(R)u'(R) &= \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{aligned} \right\} \quad (3.2)$$

with $\tilde{u}(t) > w(t)$ on (r, R) . Here

$$\tilde{g}(t, u) = \begin{cases} g(t, u), & \text{for } u \geq 0, \\ g(t, 0), & \text{for } u < 0. \end{cases} \quad (3.3)$$

Let

$$K = \{u \in C[r, R] : u(t) \geq \frac{1}{2}\gamma(t)\|u\|, t \in [r, R]\}, \quad (3.4)$$

where γ is as in (2.12). For each $v \in K$, let $u = Tv$ be the solution of

$$\left. \begin{aligned} (p(t)u')' - q(t)u + \lambda \tilde{g}(t, v - w) &= 0, \quad r < t < R, \\ au(r) - bp(r)u'(r) &= \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(R) + dp(R)u'(R) &= \sum_{i=1}^{m-2} \beta_i u(\xi_i). \end{aligned} \right\} \quad (3.5)$$

By Lemma 2.2,

$$Tv = \lambda \left[\int_r^R G(t, s) \tilde{g}(s, v(s) - w(s)) ds + A(\tilde{g}(\cdot, v - w))\psi(t) + B(\tilde{g}(\cdot, v - w))\phi(t) \right]. \quad (3.6)$$

From Lemma 2.4, we know that $T : K \rightarrow K$. It is easy to check that T is completely continuous. We shall prove that T has a fixed point in K by using Theorem 1.3.

Define $\Omega_1 = \{u \in C[r, R] : \|u\| < 1\}$. For $u \in \partial\Omega_1 \cap K$,

$$\begin{aligned} (Tv)(t) &= \lambda \left[\int_r^R G(t, s) \tilde{g}(s, v(s) - w(s)) ds + A(\tilde{g}(\cdot, v - w))\psi(t) + B(\tilde{g}(\cdot, v - w))\phi(t) \right] \\ &\leq \lambda C_1 \left[\int_r^R G(t, s) ds + A(1)\psi(t) + B(1)\phi(t) \right] \\ &= \lambda C_1 \bar{w}(t) \\ &\leq 1, \end{aligned}$$

since $0 \leq v - w \leq v \leq 1$. Thus

$$\|Tu\| \leq \|u\|, \quad \text{for } u \in \partial\Omega_1 \cap K.$$

Now choose a constant $\tilde{M} > 0$ such that

$$1 \leq \frac{1}{4}\lambda\tilde{M}\Gamma \inf_{r \leq t \leq R} \int_{\alpha}^{\beta} G(t, s) \, ds, \quad (3.7)$$

where

$$\Gamma := \min_{\alpha \leq t \leq \beta} \gamma(t).$$

By (A4), we know that there is a constant $D > 0$ such that

$$\frac{\tilde{g}(t, s)}{s} \geq \tilde{M}, \quad \text{for } (t, s) \in [\alpha, \beta] \times [D, \infty). \quad (3.8)$$

Set

$$\rho_2 = \max \left\{ 4, 4\lambda CM, \frac{4D}{\Gamma} \right\}$$

and define

$$\Omega_2 = \{u \in C[r, R] : \|u\| < \rho_2\}.$$

For $u \in \partial\Omega_2 \cap K$, we have from Lemmas 2.5 and 2.4 that

$$\begin{aligned} u(s) - w(s) &= u(s) - \lambda M \bar{w}(s) \\ &\geq u(s) - \lambda M C \gamma(s) \\ &\geq u(s) - \frac{\lambda CM}{\rho_2} 2u(s) \\ &\geq \frac{1}{2}u(s) \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} \min_{\alpha \leq s \leq \beta} (u(s) - w(s)) &\geq \min_{\alpha \leq s \leq \beta} \frac{1}{2}u(s) \\ &\geq \min_{\alpha \leq s \leq \beta} \frac{1}{4}\|u\|\gamma(s) \\ &= \frac{1}{2}\rho_2\Gamma \geq D. \end{aligned} \quad (3.10)$$

Therefore, for $u \in \partial\Omega_2 \cap K$, we have

$$\begin{aligned} \min_{t \in [\alpha, \beta]} (Tu)(t) &= \lambda \min_{t \in [\alpha, \beta]} \int_r^R G(t, s) \tilde{g}(s, u - w) \, ds \\ &\quad + A(\tilde{g}(\cdot, u - w))\psi(t) + B(\tilde{g}(\cdot, u - w))\phi(t) \\ &\geq \lambda \min_{t \in [\alpha, \beta]} \int_r^R G(t, s) \tilde{g}(s, u - w) \, ds \end{aligned}$$

$$\begin{aligned}
&\geq \lambda \min_{t \in [\alpha, \beta]} \int_r^R G(t, s) \tilde{M}(u(s) - w(s)) \, ds \\
&\geq \lambda \min_{t \in [\alpha, \beta]} \int_r^R G(t, s) \tilde{M} \frac{1}{2} u(s) \, ds \\
&\geq \lambda \min_{t \in [\alpha, \beta]} \int_r^R G(t, s) \tilde{M} \frac{1}{4} \gamma(s) \, ds \|u\| \\
&\geq \lambda \min_{t \in [\alpha, \beta]} \int_r^R G(t, s) \tilde{M} \frac{1}{4} \Gamma \, ds \|u\| \\
&\geq \|u\|.
\end{aligned} \tag{3.11}$$

This implies

$$\|Au\| \geq \|u\| \quad \text{for } u \in \partial\Omega_2 \cap K.$$

By Theorem 1.3, T has a fixed point \tilde{u} with $1 \leq \|\tilde{u}\| \leq \rho_2$. It follows that

$$\tilde{u}(t) \geq \frac{1}{2} \gamma(t) \geq \frac{1}{2} (2\lambda CM) \gamma(t) \geq \lambda M \bar{w}(t) = w(t),$$

and so $u = \tilde{u} - w$ is a positive solution of (1.1), completing the proof of Theorem 3.1. \square

4. An example

Let us consider the three-point BVP

$$\left. \begin{aligned} u'' - u + \lambda(u^5 - 2) &= 0, \quad 0 < t < 1, \\ u(0) &= \frac{1}{2} u(\tfrac{1}{2}), \quad u(1) = \frac{1}{2} u(\tfrac{1}{2}). \end{aligned} \right\} \tag{4.1}$$

Clearly, (C1) and (C2) hold. It is easy to check that

$$\psi(t) = \frac{1}{2}(e^t - e^{-t}), \quad \phi(t) = \frac{1}{2}(e^{1-t} - e^{t-1})$$

and

$$\rho = \begin{vmatrix} \phi(0) & \psi(0) \\ \phi'(0) & \psi'(0) \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(e - e^{-1}) & 0 \\ \frac{1}{2}(-e - e^{-1}) & 1 \end{vmatrix} = \frac{1}{2}(e - e^{-1}).$$

Since

$$\Delta = \begin{vmatrix} -\frac{1}{2}\psi(\frac{1}{2}) & \rho - \frac{1}{2}\phi(\frac{1}{2}) \\ \rho - \frac{1}{2}\psi(\frac{1}{2}) & -\frac{1}{2}\phi(\frac{1}{2}) \end{vmatrix} = -\rho(\frac{1}{2}(e - e^{-1}) - \frac{1}{2}(e^{1/2} - e^{-1/2})) < 0,$$

$$\rho - \frac{1}{2}\phi(\tfrac{1}{2}) = \frac{1}{2}(e - e^{-1}) - \frac{1}{4}(e^{1/2} - e^{-1/2}) > 0$$

and

$$\rho - \frac{1}{2}\psi(\tfrac{1}{2}) = \frac{1}{2}(e - e^{-1}) - \frac{1}{4}(e^{1/2} - e^{-1/2}) > 0,$$

we know that (C4) is satisfied. Let \bar{w} be the unique solution of

$$\left. \begin{aligned} u'' - u + 1 &= 0, \quad 0 < t < 1, \\ u(0) &= \frac{1}{2} u(\tfrac{1}{2}), \quad u(1) = \frac{1}{2} u(\tfrac{1}{2}), \end{aligned} \right\} \tag{4.2}$$

then

$$\bar{w} = \frac{(1 - e^{-1})e^t + (e - 1)e^{-t}}{2[(e^{1/2} - e^{-1/2}) + (e - e^{-1})]} + 1.$$

Moreover,

$$\|w\| = w(\tfrac{1}{2}) \quad (4.3)$$

and

$$\|w\| \doteq 0.203\,347\,172\,171\,906\,298\,02. \quad (4.4)$$

From (2.11),

$$\tilde{q}(t) = \min\left\{\frac{\phi(t)}{\phi(0)}, \frac{\psi(t)}{\psi(1)}\right\} = \min\left\{\frac{e^{1-t} - e^{t-1}}{e - e^{-1}}, \frac{e^t - e^{-t}}{e - e^{-1}}\right\} \quad (4.5)$$

and

$$\tilde{q}(t) \leq \tilde{q}(\tfrac{1}{2}) = \frac{1}{e^{1/2} + e^{-1/2}}. \quad (4.6)$$

From (2.13) and (2.14), we know that

$$\tilde{A} := \frac{1}{\Delta} \begin{vmatrix} \frac{1}{2}\tilde{q}(\frac{1}{2}) & \rho - \frac{1}{2}\phi(\frac{1}{2}) \\ \frac{1}{2}\tilde{q}(\frac{1}{2}) & -\frac{1}{2}\phi(\frac{1}{2}) \end{vmatrix} = -\frac{1}{2\Delta}\rho\tilde{q}(\tfrac{1}{2}) \quad (4.7)$$

and

$$\tilde{B} := \frac{1}{\Delta} \begin{vmatrix} -\frac{1}{2}\psi(\frac{1}{2}) & \frac{1}{2}\tilde{q}(\frac{1}{2}) \\ \rho - \frac{1}{2}\psi(\frac{1}{2}) & \frac{1}{2}\tilde{q}(\frac{1}{2}) \end{vmatrix} = -\frac{1}{2\Delta}\rho\tilde{q}(\tfrac{1}{2}). \quad (4.8)$$

Clearly,

$$\tilde{A} = \tilde{B} \doteq 0.338\,943\,166\,556\,021\,992\,19.$$

Thus from (2.12)

$$\begin{aligned} \tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t) &\leq \tilde{q}(\tfrac{1}{2}) + \tilde{A}\psi(1) + \tilde{B}\phi(0) \\ &= \tilde{q}(\tfrac{1}{2}) \left\{ 1 + \frac{\rho}{-2\Delta} \left[\frac{e^1 - e^{-1}}{2} + \frac{e^1 - e^{-1}}{2} \right] \right\} \\ &= \frac{1}{e^{1/2} + e^{-1/2}} \left\{ 1 + \frac{e^1 - e^{-1}}{e - e^{-1} - e^{1/2} + e^{-1/2}} \right\} \\ &\doteq 0.886\,818\,883\,970\,073\,908\,68. \end{aligned} \quad (4.9)$$

So we can take $k_0 = 1$ and

$$\gamma(t) = \tilde{q}(t) + \tilde{A}\psi(t) + \tilde{B}\phi(t) \quad (4.10)$$

in Lemma 2.4. By (4.10)

$$\begin{aligned} \gamma(t) &\geq \tilde{A}\psi(t) + \tilde{B}\phi(t) \\ &= \tilde{A}(\psi(t) + \phi(t)) \\ &\geq \tilde{A}\psi(\tfrac{1}{2}) \end{aligned} \quad (4.11)$$

for all $t \in [0, 1]$. This together with (4.3) imply that

$$\bar{w}(t) \leq C^* \gamma(t), \quad t \in [0, 1], \quad (4.12)$$

where

$$C^* = \frac{\|w\|}{\bar{A}\phi(\frac{1}{2})} \doteq 1.151\,314\,817\,609\,928\,832\,4.$$

Now, by the proof of Theorem 3.1, we know that (4.1) has at least one positive solution for each $\lambda \in (0, A)$ with

$$A = \min \left\{ \frac{1}{C_1 \|\bar{w}\|}, \frac{1}{2C^*M} \right\} \doteq 0.217\,143\,040\,440\,482\,925\,60,$$

where $C_1 = 1$, $M = 2$.

Acknowledgements. The author was supported by the NSFC, GG-110-10736-1003, NWNK-KJCXGC-212 and the Foundation of Major Projects of Science and Technology of the Chinese Education Ministry. The author is very grateful to the referee for his/her valuable suggestions.

References

1. V. ANUHADHA, D. D. HAI AND R. SHIVAJI, Existence results for superlinear semipositone boundary value problems, *Proc. Am. Math. Soc.* **124** (1996), 757–763.
2. D. GUO AND V. LAKSHMIKANTHAM, *Nonlinear problems in abstract cones* (Academic, 1988).
3. C. P. GUPTA, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation, *J. Math. Analysis Applic.* **168** (1992), 540–551.
4. C. P. GUPTA, A generalized multi-point boundary value problem for second order ordinary differential equations, *Appl. Math. Computat.* **89** (1998), 133–146.
5. V. A. IL'IN AND E. I. MOISEEV, Nonlocal boundary value problem of the first kind for a Sturm–Liouville operator in its differential and finite difference aspects, *Diff. Eqns* **23** (1987), 803–810.
6. R. MA, Positive solutions of a nonlinear three-point boundary-value problems, *Electron. J. Diff. Eqns* **34** (1999), 1–8.
7. M. MOSHINSKY, Sobre los problemas de condiciones a la frontera en una dimension de características discontinuas, *Bol. Soc. Mat. Mexicana* **7** (1950), 1–25.
8. M. H. PROTTER AND H. F. WEINBERGER, *Maximum principles in differential equations* (Springer, 1984).
9. S. TIMOSHENKO, *Theory of elastic stability* (McGraw-Hill, New York, 1961).
10. J. R. L. WEBB, Positive solutions of some three-point boundary value problems via fixed point theory, *Nonlin. Analysis* **47** (2001), 4319–4332.