

PRIME RINGS WITH FINITENESS PROPERTIES ON ONE-SIDED IDEALS

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Abstract Let R be a prime ring with extended centroid C , ρ a non-zero right ideal of R and let $f(X_1, \dots, X_t)$ be a polynomial, having no constant term, over C . Suppose that $f(X_1, \dots, X_t)$ is not central-valued on RC . We denote by $f(\rho)$ the additive subgroup of RC generated by all elements $f(x_1, \dots, x_t)$ for $x_i \in \rho$. The main goals of this note are to prove two results concerning the extension properties of finiteness conditions as follows.

- (I) If $f(\rho)$ spans a non-zero finite-dimensional C -subspace of RC , then $\dim_C RC$ is finite.
- (II) If $f(\rho) \neq 0$ and is a finite set, then R itself is a finite ring.

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1. Introduction and results

Throughout this note, R always denotes a prime ring with extended centroid C and ρ a non-zero right ideal of R . The goal of this note is to study the extension properties of certain finiteness conditions from one-sided ideals of a prime ring to the whole prime ring. We were motivated by two elementary observations: (I) if $\dim_C \rho C$ is finite, then so is $\dim_C RC$; and (II) if ρ is a finite subset of R , then so is R . For (I) see [3, Lemma 1]. For (II), since, by the primeness of R , R can be embedded in $\text{End}(\rho, +, 0)$ via right multiplications and $\text{End}(\rho, +, 0)$ is a finite set, R itself is a finite ring. In a recent paper Bell proved the following theorem.

Theorem. *Suppose that ρ is of finite index in R and $[\rho, \rho]$ is finite. Then R is either finite or commutative (see [1, Theorem 2.2]).*

Since, in Bell's Theorem, $[\rho, \rho]$ contains all elements $xy - yx$ for all $x, y \in \rho$, we want to extend these results above to more generalized forms. In particular, we shall see that, in Bell's Theorem, the assumption that ρ is of finite index in R is superfluous. Our point of view in this note is different from that of [1]. To state our results we require some notation. For a polynomial $f(X_1, \dots, X_t)$ over C , where the X_i are non-commuting indeterminates,

we denote by $f(\rho)$ the additive subgroup of RC generated by all elements $f(x_1, \dots, x_t)$ for $x_i \in \rho$. For $x, y \in R$, set $[x, y]_0 = x$, $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$. Also, $[\rho, \rho]_k$ denotes the additive subgroup of R generated by all elements $[x, y]_k$ for $x, y \in \rho$. We are now ready to state our results.

Theorem 1.1. *Let R be a prime ring with extended centroid C , ρ a non-zero right ideal of R and $f(X_1, \dots, X_t)$ a polynomial, having no constant term, over C . Suppose that $f(X_1, \dots, X_t)$ is not central-valued on RC .*

- (I) *Suppose that $f(\rho)$ spans a non-zero finite-dimensional C -subspace of RC . Then $\dim_C RC$ is finite.*
- (II) *Suppose that $f(\rho) \neq 0$ and is a finite set. Then R itself is a finite ring.*

Corollary 1.2. *Let R be a prime ring with extended centroid C , ρ a non-zero right ideal of R , and k a non-negative integer.*

- (I) *Suppose that $[\rho, \rho]_k$ spans a finite-dimensional C -subspace of RC . Then $\dim_C RC$ is finite.*
- (II) *Suppose that $[\rho, \rho]_k$ is a finite set. Then R is either commutative or finite.*

Proof. In view of Theorem 1.1, it suffices to prove that $[\rho, \rho]_k \not\subseteq C$ unless R is commutative. Indeed, suppose that $[\rho, \rho]_k \subseteq C$. Then $[x, y]_{k+1} = 0$ for all $x, y \in \rho$. In view of [5, Lemma 1], R is commutative. This proves the corollary. \square

We remark that Corollary 1.2 (II) gives a generalization of Bell's Theorem [1, Theorem 2.2].

2. Proof of Theorem 1.1

To prove Theorem 1.1 we need the following two theorems [2, Theorem 1 (II) (i) and Theorem 2]. Recall that a prime ring R is called *centrally closed* if $R = RC$ and that a right ideal of a ring is called a polynomial identity (PI) right ideal if the right ideal is itself a PI-ring. We write $M_2(\text{GF}(2))$ to stand for the 2 by 2 matrix ring over $\text{GF}(2)$, the Galois field of two elements.

Theorem 2.1. *Let R be a centrally closed prime C -algebra and $f(X_1, \dots, X_t)$ a non-zero polynomial over C . Suppose that ρ is a PI right ideal of R such that $f(\rho) \neq 0$. Then there exists an idempotent e in the socle of R such that $\rho = eR$ and $eR(1 - e) \subseteq f(\rho)$.*

Theorem 2.2. *Let R be a prime ring with extended centroid C and I a non-zero ideal of R . Suppose that $f(X_1, \dots, X_t)$ is a polynomial over C which is not central-valued on RC . Then $[M, R] \subseteq f(I)$ for some non-zero ideal M of R , except when $R \cong M_2(\text{GF}(2))$ and $f(R) = \{0, e_{12} + e_{21}, 1 + e_{12}, 1 + e_{21}\}$ or $\{0, 1, e_{11} + e_{12} + e_{21}, e_{22} + e_{12} + e_{21}\}$.*

From now on, R is always a prime ring with extended centroid C , ρ a non-zero right ideal and $f(X_1, \dots, X_t)$ a non-zero polynomial, having no constant term, over C . We denote by $d_n(X_1, \dots, X_n; Y_1, \dots, Y_{n-1})$ the Capelli polynomial of degree $2n - 1$, that is

$$d_n(X_1, \dots, X_n; Y_1, \dots, Y_{n-1}) = \sum_{\sigma \in S_n} (-1)^\sigma X_{\sigma(1)} Y_1 X_{\sigma(2)} \cdots Y_{n-1} X_{\sigma(n)},$$

where X_i and Y_j are non-commuting indeterminates. To apply Theorem 2.1 to our case, we require the following lemma.

Lemma 2.3. *Suppose that $f(\rho) \neq 0$. Then*

- (I) *if the additive subgroup $f(\rho)$ spans a finite-dimensional C -space, then $f(\rho)C = f(\rho C)C$; and*
- (II) *if $f(\rho)$ is finite, then so are both C and $f(\rho C)$.*

Proof. (I) Suppose that $\dim_C f(\rho)C = n < \infty$. Then

$$d_{n+1}(f(x_{11}, \dots, x_{1t}), \dots, f(x_{n+1,1}, \dots, x_{n+1,t}); y_1, \dots, y_n) = 0 \quad (2.1)$$

for all $x_{ij} \in \rho$ and all $y_i \in R$. Since ρ and ρC satisfy the same generalized polynomial identities (GPIs) [4, Lemma 2], equation (2.1) still holds for all $x_{ij} \in \rho C$ and all $y_i \in RC$. In view of [7, Theorem 7.6.16], for $x_{ij} \in \rho C$ the $n + 1$ elements $f(x_{11}, \dots, x_{1t}), \dots, f(x_{n+1,1}, \dots, x_{n+1,t})$ are C -dependent and so $\dim_C f(\rho C)C \leq n$. But since $f(\rho)C \subseteq f(\rho C)C$, we conclude that $f(\rho)C = f(\rho C)C$ as asserted.

(II) Suppose that $f(\rho)$ is a finite set. Write

$$f(X_1, \dots, X_t) = \sum_{i=1}^m f_i(X_1, \dots, X_t), \quad (2.2)$$

where $f_i(X_1, \dots, X_t)$ is the homogeneous part of $f(X_1, \dots, X_t)$ of degree i for $1 \leq i \leq m$. Suppose on the contrary that C is infinite. Choose m distinct elements β_1, \dots, β_m in C and then a non-zero ideal I of R such that $\beta_i I \subseteq \rho$. Then $\beta_i \rho I \subseteq \rho$ for each i . Applying a standard determinant argument to equation (2.2) we see that $f_i(\rho I)$ is finite for each i . Since $f(\rho) \neq 0$, we see that $f_j(\rho) \neq 0$ for some j . Thus we may assume from the start that $f(X_1, \dots, X_t)$ is homogeneous of degree $m \geq 1$. Suppose that $f(\rho)$ consists of ℓ elements. Since C is an infinite field, we can choose $\mu_1, \dots, \mu_{\ell+1}$ in C such that $\mu_i^m \neq \mu_j^m$ for $i \neq j$. Let J be a non-zero ideal of R such that $\mu_i J \subseteq \rho$ for each i . In view of [4, Lemma 2], there exist $x_i \in \rho J$, $1 \leq i \leq t$, such that $f(x_1, \dots, x_t) \neq 0$. Now, we see that $\mu_i^m f(x_1, \dots, x_t) = f(\mu_i x_1, \dots, \mu_i x_t) \in f(\rho)$ for each i . This derives a contradiction, as the set $\{\mu_i^m f(x_1, \dots, x_t) \mid 1 \leq i \leq \ell + 1\}$ consists of $\ell + 1$ elements. Thus C is a finite field. By (I) we see that $f(\rho)C = f(\rho C)C$. But since $f(\rho)$ and C are finite sets, this implies that $f(\rho C)$ is a finite set, proving (II). \square

Lemma 2.4. *Let R be a centrally closed prime C -algebra with non-zero socle H . Suppose that e is a non-trivial idempotent in H .*

(I) *If $\dim_C eR(1-e) < \infty$, then R is finite dimensional over C .*

(II) *If $eR(1-e)$ is a finite set, then R is a finite ring.*

Proof. (I) Suppose that $\dim_C eR(1-e) = n < \infty$. Note that e and $1-e$ are C -independent. Thus we see that

$$S_{n+1}(eX_1(1-e)Y, \dots, eX_{n+1}(1-e)Y)$$

is a non-trivial GPI for R , where $S_{n+1}(X_1, \dots, X_{n+1})$ is the standard polynomial of degree $n+1$. Since R is centrally closed, it follows from Martindale's Theorem [6] that R is a strongly primitive ring. By assumption, $e \in H$ and so $\dim_C eRe < \infty$. Thus $\dim_C eR = \dim_C eRe + \dim_C eR(1-e) < \infty$. That is, R contains a non-zero right ideal eR , which is finite dimensional over C . In view of [3, Lemma 1], $\dim_C R < \infty$ follows. This proves (I).

(II) Since $eR(1-e) \neq 0$, we choose $x_0 \in R$ such that $ex_0(1-e) \neq 0$. But $Cex_0(1-e) \subseteq eR(1-e)$, so the finiteness of $eR(1-e)$ implies that C is a finite field. By (I), R is finite dimensional over C and so R is a finite ring, proving (II). \square

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. (I) Suppose that $f(\rho)$ spans a non-zero finite-dimensional C -subspace of RC . It follows from Lemma 2.3 (I) that $0 \neq f(\rho C)$ is finite dimensional over C . Clearly, ρC is a PI right ideal of RC . Note that RC is a centrally closed prime C -algebra. In view of Theorem 2.1, there exists an idempotent e in the socle of RC such that $\rho C = eRC$ and $eRC(1-e) \subseteq f(\rho C)$ as $f(\rho C) \neq 0$. If $e \neq 1$, applying Lemma 2.4 we conclude that $\dim_C RC$ is finite. Suppose next that $e = 1$. Then $\rho C = RC$. Since $f(X_1, \dots, X_t)$ is not central-valued on RC , by Theorem 2.2 there exists a non-zero ideal I of RC such that $[I, RC] \subseteq f(\rho C)$ unless $R = RC \cong M_2(\text{GF}(2))$. There is nothing to prove for the exceptional case. Therefore, we may assume that $[I, RC] \subseteq f(\rho C)$ and so $\dim_C [I, RC] < \infty$, implying that R itself is a PI-ring. Applying Posner's Theorem yields that $\dim_C RC$ is finite.

(II) Suppose that $f(\rho) \neq 0$ and is a finite set. By Lemma 2.3, $f(\rho C)$ is a finite set. Clearly, in this case ρC must be a PI right ideal of R . In view of Theorem 2.1, there exists an idempotent e in the socle of RC such that $\rho C = eRC$ and $eRC(1-e) \subseteq f(\rho C)$. If $e \neq 1$, Lemma 2.4 implies that R is a finite ring. Otherwise, $e = 1$ follows and so $\rho C = RC$. As in the argument given in (I), we may assume that $R \not\cong M_2(\text{GF}(2))$. By Theorem 2.2 there exists a non-zero ideal I of RC such that $[I, RC] \subseteq f(\rho C)$ and so $[I, RC]$ is a finite set. But since R is not commutative, $[I, RC] \neq 0$ and so C must be finite. Now, RC is a PI-ring with C a finite field. It is now clear that R itself is a finite ring, proving (II). \square

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