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SPECIALIZATION OF GRADED MODULES

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Abstract The paper shows that specializations of finitely generated graded modules are also graded and that many important invariants of graded modules and ideals are preserved by specializations.

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1. Introduction

The first step towards an algebraic theory of specialization was the introduction of the specialization of an ideal by Krull [**8, 9**]. Seidenberg [**17**], Kuan [**10–12**] and Trung [**20**] used specializations of ideals to prove that hyperplane sections of normal varieties are normal again under certain conditions. Using specializations of finitely generated free modules and of homomorphisms between them, we defined in [**13**] the specialization of a finitely generated module, and we showed that basic properties and operations on modules are preserved by specializations. In [**14**] we followed the same approach to introduce and to study specializations of finitely generated modules over a local ring.

The aim of this paper is to show that specializations of finitely generated graded modules and of graded homomorphisms are also graded and that many important invariants of graded modules and ideals are preserved by specializations. Moreover, we will show that specializations can be used to prove Bertini Theorems for projective varieties.

This paper is divided into four sections. In § 1 we recall the definition of the specialization of a module. There we shall see that specializations of finitely generated graded modules and of graded homomorphisms are also graded over the ring R_α. In § 2 we will first prove the preservation of a graded minimal free resolution by specializations. We shall see that various degrees and cohomological invariants of graded modules are preserved by specializations which include the a-invariants and the Castelnuovo regularity. In § 3 we will give two non-trivial applications of specializations of graded ideals. Firstly, we use a recent result of Trung [**22**] to study the preservation of the reduction number of an homogeneous ideal. Secondly, we shall prove that the specialization of a filter-regular sequence is again a filter-regular sequence. This settles a question of Herzog (personal communication to N. V. Trung, 1998). In § 4 we will study hypersurface sections of projective varieties. There we will give a simple proof for the global Bertini Theorem of Flenner [**5**].

Throughout this paper we assume that all modules are finitely generated.

2. Definition and basic properties

Let k be an infinite field of arbitrary characteristic. Denote by K an extension field of k. Let $u = (u_1, \dots, u_m)$ be a family of indeterminates and $\alpha = (\alpha_1, \dots, \alpha_m)$ a family of elements of K. We denote the polynomial rings in $n+1$ variables $x_0, \dots, x_n$ over $k(u)$ and $k(\alpha)$ by $R = k(u)[x]$ and by $R_\alpha = k(\alpha)[x]$, respectively. Let $\mathfrak{m}$ and $\mathfrak{m}_\alpha$ be the maximal graded ideals of R and R_α, respectively. We shall say that a property holds for almost all α if it holds for all points of a Zariski-open non-empty subset of K^m. For convenience we shall often omit the phrase 'for almost all α' in the proofs of the results of this paper.

Following [**20**] we define the specialization of I with respect to the substitution $u \to \alpha$ as the ideal I_α of R_α generated by elements of the set $\{f(\alpha, x) \mid f(u, x) \in I \cap k[u, x]\}$.

This definition is slightly different than that considered by Krull and Seidenberg, who choose $\alpha \in k^m$. However, if some property holds for almost all $\alpha \in K^m$ in the sense of Krull and Seidenberg, then it holds for the extensions of I_α in the polynomial ring $K[x]$ for almost all $\alpha \in K^m$. Since $k(\alpha)[x] \to K[x]$ is a flat extension, we can often deduce that this property also holds for almost all I_α in our sense.

Example 2.1. Let $I = (f_1, \dots, f_s)$ be a homogeneous ideal in R, where $f_1, \dots, f_s$ are homogeneous polynomials. By [**17**, Appendix, Theorem 1] we have

$$I_\alpha K[x] = ((f_1)_\alpha, \dots, (f_s)_\alpha)K[x].$$

Since $k(\alpha)[x] \to K[x]$ is flat, we can deduce that $I_\alpha = ((f_1)_\alpha, \dots, (f_s)_\alpha)$. As $(f_1)_\alpha, \dots, (f_s)_\alpha$ are homogeneous, I_α is again a homogeneous ideal for almost all α.

The specialization of ideals can be generalized to modules. First, each element $a(u, x)$ of R can be written in the form

$$a(u, x) = \frac{p(u, x)}{q(u)}$$

with $p(u, x) \in k[u, x]$ and $q(u) \in k[u] \setminus \{0\}$. For any α such that $q(\alpha) \neq 0$ we define

$$a(\alpha, x) = \frac{p(\alpha, x)}{q(\alpha)}.$$

Let F be a free R-module of finite rank. The specialization F_α of F is a free R_α-module of the same rank. Let $\phi : F \to G$ be a homomorphism of free R-modules. We can represent ϕ by a matrix $A = (a_{ij}(u, x))$ with respect to fixed bases of F and G. Set $A_\alpha = (a_{ij}(\alpha, x))$. Then A_α is well defined for almost all α. The specialization $\phi_\alpha : F_\alpha \to G_\alpha$ of ϕ is given by the matrix A_α provided that A_α is well defined. We note that the definition of ϕ_α depends on the chosen bases of F_α and G_α.

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Definition 2.2 (see [13]). Let L be an R-module. Let $F_1 \xrightarrow{\phi} F_0 \to L \to 0$ be a finite free presentation of L. Let $\phi_\alpha : (F_1)_\alpha \to (F_0)_\alpha$ be a specialization of ϕ. We call $L_\alpha := \operatorname{Coker} \phi_\alpha$ a *specialization* of L (with respect to ϕ).

If we choose a different finite free presentation $F'_1 \to F'_0 \to L \to 0$, we may get a different specialization L'_α of L, but L_α and L'_α are canonically isomorphic. Hence L_α is uniquely determined up to isomorphisms [**13**, Proposition 2.2].

Let R be naturally graded. For a graded R-module L, we denote by L_t the homogeneous component of L of degree t. For an integer h we let $L(h)$ be the same module as L with grading shifted by h, that is, we set $L(h)_t = L_{h+t}$.

Let $F = \bigoplus_{j=1}^{s} R(-h_j)$ be a free graded R-module. We make the specialization F_α of F a free graded R_α-module by setting $F_\alpha = \bigoplus_{j=1}^{s} R_\alpha(-h_j)$. Let

$$\phi : \bigoplus_{j=1}^{s_1} R(-h_{1j}) \to \bigoplus_{j=1}^{s_0} R(-h_{0j})$$

be a graded homomorphism of degree 0 given by a homogeneous matrix $A = (a_{ij}(u, x))$. Since

$$\deg(a_{i1}(u, x)) + h_{01} = \cdots = \deg(a_{is_0}(u, x)) + h_{0s_0} = h_{1i},$$

$A_\alpha = (a_{ij}(\alpha, x))$ is a homogeneous matrix with

$$\deg(a_{i1}(\alpha, x)) + h_{01} = \cdots = \deg(a_{is_0}(\alpha, x)) + h_{0s_0} = h_{1i}.$$

Therefore, the homomorphism

$$\phi_\alpha : \bigoplus_{j=1}^{s_1} R_\alpha(-h_{1j}) \to \bigoplus_{j=1}^{s_0} R_\alpha(-h_{0j})$$

given by the matrix A_α is a graded homomorphism of degree 0.

Lemma 2.3. *Let L be a finitely generated graded R-module. Then L_α is a graded R_α-module for almost all α.*

Proof. This follows from the definition of L_α and the above observation. $\square$

We now recall some facts from [**13**] which we shall need later. Let

$$\boldsymbol{F}_\bullet : 0 \to F_\ell \xrightarrow{\phi_\ell} F_{\ell-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0$$

be a complex of free R-modules finite ranks. Then we obtain a complex of free R_α-modules

$$(\boldsymbol{F}_\bullet)_\alpha : 0 \to (F_\ell)_\alpha \xrightarrow{(\phi_\ell)_\alpha} (F_{\ell-1})_\alpha \to \cdots \to (F_1)_\alpha \xrightarrow{(\phi_1)_\alpha} (F_0)_\alpha$$

for almost all α.

Proposition 2.4 (see Theorem 1.5 of [13]). *Let $\boldsymbol{F}_\bullet$ be a finite exact complex of free R-modules of finite ranks. Then $(\boldsymbol{F}_\bullet)_\alpha$ is an exact complex of free R_α-modules of finite ranks for almost all α.*

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Proposition 2.5 (see Theorem 4.3 of [13]). *Let L, M be R-modules. Then, for almost all α,*

$$\operatorname{Ext}^i_R(L,M)_\alpha \cong \operatorname{Ext}^i_{R_\alpha}(L_\alpha, M_\alpha), \quad i \geqslant 0.$$

As observed in [**13**], the specialization of a submodule of L can be canonically identified with a submodule of L_α for almost all α.

Proposition 2.6 (see Proposition 3.2 of [13]). *Let L be an R-module and M, N submodules of L. Then, for almost all α,*

(i) $(L/M)_\alpha \cong L_\alpha/M_\alpha$,

(ii) $(M \cap N)_\alpha \cong M_\alpha \cap N_\alpha$,

(iii) $(M + N)_\alpha \cong M_\alpha + N_\alpha$.

Proposition 2.7 (see Proposition 3.6 of [13]). *Let L be an R-module and I an ideal of R. Then, for almost all α,*

(i) $(IL)_\alpha \cong I_\alpha L_\alpha$,

(ii) $(0_L : I)_\alpha \cong 0_{L_\alpha} : I_\alpha$.

Proposition 2.8 (see Theorem 3.4 of [13]). *Let L be an R-module. Then, for almost all α,*

(i) $\operatorname{Ann} L_\alpha = (\operatorname{Ann} L)_\alpha$,

(ii) $\dim L_\alpha = \dim L$.

Lemma 2.9. *Let L be an R-module. Then*

$$\sqrt{\operatorname{Ann} L_\alpha} = \sqrt{(\sqrt{\operatorname{Ann} L})_\alpha}$$

for almost all α.

Proof. There exists t such that

$$(\sqrt{\operatorname{Ann} L})^t \subseteq \operatorname{Ann} L \subseteq \sqrt{\operatorname{Ann} L}.$$

By Proposition 2.7 (i), $((\sqrt{\operatorname{Ann} L})^t)_\alpha = (\sqrt{\operatorname{Ann} L})^t_\alpha$. Therefore,

$$(\sqrt{\operatorname{Ann} L})^t_\alpha \subseteq (\operatorname{Ann} L)_\alpha \subseteq (\sqrt{\operatorname{Ann} L})_\alpha.$$

From this it follows that

$$\sqrt{(\operatorname{Ann} L)_\alpha} = \sqrt{(\sqrt{\operatorname{Ann} L})_\alpha}.$$

Since $\operatorname{Ann} L_\alpha = (\operatorname{Ann} L)_\alpha$ by Proposition 2.8,

$$\sqrt{\operatorname{Ann} L_\alpha} = \sqrt{(\sqrt{\operatorname{Ann} L})_\alpha}$$

for almost all α. □

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Corollary 2.10. *Let L be an R-module of dimension d. If*

$$I = \bigcap_{\substack{\mathfrak{p}\in \operatorname{Ass}(L),\\ \dim R/\mathfrak{p}=d}} \mathfrak{p},$$

then, for almost all α,

$$\sqrt{I_\alpha} = \bigcap_{\substack{\mathfrak{q}\in \operatorname{Ass}(L_\alpha),\\ \dim R_\alpha/\mathfrak{q}=d}} \mathfrak{q}.$$

Proof. By Proposition 2.8, $\dim L_\alpha = d$. Denote by J the intersection of all minimal associated primes of L of dimension $< d$. Then $\sqrt{\operatorname{Ann} L} = I \cap J$. By Proposition 2.6 (ii), $(I\cap J)_\alpha = I_\alpha \cap J_\alpha$. Therefore,

$$\sqrt{(\sqrt{\operatorname{Ann} L})_\alpha} = \sqrt{I_\alpha \cap J_\alpha} = \sqrt{I_\alpha} \cap \sqrt{J_\alpha}.$$

By Lemma 2.9,

$$\sqrt{\operatorname{Ann} L_\alpha} = \sqrt{(\sqrt{\operatorname{Ann} L})_\alpha} = \sqrt{I_\alpha} \cap \sqrt{J_\alpha}.$$

Since I_α is an unmixed ideal with $\dim R_\alpha/I_\alpha = d$ [**20**, Lemma 1.1] and since $\dim R_\alpha/J_\alpha = \dim R/J < d$, $\sqrt{I_\alpha}$ is the intersection of the minimal primes of dimension d, and $\sqrt{J_\alpha}$ is the intersection of minimal primes of dimension $< d$. Hence $\sqrt{I_\alpha}$ is the intersection of all minimal associated primes of L_α of dimension d. $\square$

3. Preservations of graded invariants

In this section we want to prove that specializations of graded modules preserve Betti numbers, various notions of degrees and the Castelnuovo–Mumford regularity.

Let L be a finitely generated graded R-module. Let

$$\boldsymbol{F}_\bullet : 0 \to F_\ell \xrightarrow{\phi_\ell} F_{\ell-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to L \to 0$$

be a minimal graded free resolution of L, where each free module F_i may be written in the form $\bigoplus_j R(-j)^{\beta_{ij}}$, and all graded homomorphisms have degree 0. The integers $\beta_{ij} \neq 0$ are called the *graded Betti numbers* of L. The following theorem shows that the graded Betti numbers are preserved by specializations.

Theorem 3.1. *Let $\boldsymbol{F}_\bullet$ be a minimal graded free resolution of L. Then the complex*

$$(\boldsymbol{F}_\bullet)_\alpha : 0 \to (F_\ell)_\alpha \xrightarrow{(\phi_\ell)_\alpha} (F_{\ell-1})_\alpha \to \cdots \to (F_1)_\alpha \xrightarrow{(\phi_1)_\alpha} (F_0)_\alpha \to L_\alpha \to 0$$

is a minimal graded free resolution of L_α with the same graded Betti numbers for almost all α.

Proof. By Proposition 2.4 and by the definition of L_α, $(\boldsymbol{F}_\bullet)_\alpha$ is also exact. Since all $(F_i)_\alpha$ are graded free R_α-modules and all homomorphisms are graded, $(\boldsymbol{F}_\bullet)_\alpha$ is a graded free resolution of L_α. Since every homogeneous element of the represented matrix of

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ϕ_i belongs to $\mathfrak{m}$, every $(\phi_i)_\alpha$ has a represented matrix with the elements in $\mathfrak{m}_\alpha$. Hence $(\boldsymbol{F}_\bullet)_\alpha$ is a minimal graded free resolution of L_α. If $F_i = \bigoplus_j R(-j)^{\beta_{ij}}$, then $(F_i)_\alpha = \bigoplus_j R_\alpha(-j)^{\beta_{ij}}$. Therefore, the graded Betti numbers are preserved by specializations. □

Corollary 3.2. *For almost all α, $\dim_{k(\alpha)}(L_\alpha)_t = \dim_{k(u)} L_t$ for all $t \in \mathbb{Z}$.*

Proof. Let $\boldsymbol{F}_\bullet$ be a minimal graded free resolution of L, with $F_i = \bigoplus_j R_\alpha(-j)^{\beta_{ij}}$. Then $(\boldsymbol{F}_\bullet)_\alpha$ is also a minimal graded free resolution of L_α by Theorem 3.1. Since $(F_i)_\alpha = \bigoplus_j R_\alpha(-j)^{\beta_{ij}}$, $\dim_{k(\alpha)}((F_i)_\alpha)_t = \dim_{k(u)}(F_i)_t$. Therefore,

$$\dim_{k(\alpha)}(L_\alpha)_t = \sum_{i=0}^{\ell}(-1)^i \dim_{k(\alpha)}((F_i)_\alpha)_t = \sum_{i=0}^{\ell}(-1)^i \dim_{k(u)}(F_i)_t = \dim_{k(u)} L_t.$$

□

Let L be a graded R-module of dimension d. Let $h_L(t)$ and $P_L(z)$ denote the Hilbert polynomial and the Hilbert series of L.

Corollary 3.3. *Let L be a graded R-module. Then, for almost all α, we have*

(i) $h_{L_\alpha}(t) = h_L(t)$,

(ii) $P_{L_\alpha}(z) = P_L(z)$.

Proof. By definitions we have

$$h_L(t) = \dim_{k(u)} L_t \quad \text{for } t \gg 0,$$
$$P_L(z) = \sum_{t\in\mathbb{Z}} \dim_{k(u)} L_t z^t.$$

Hence the conclusions follow from Corollary 3.2. □

Let L be a finitely generated graded R-module and I a homogeneous ideal of R. We set

$$\Gamma_I(L) := \bigcup_{m\geqslant 0}(0_L : I^m).$$

For each prime ideal $\wp$ of R, we denote the length $\ell(\Gamma_\wp(L_\wp))$ by $\operatorname{mult}_L(\wp)$. We will denote by $\operatorname{Ass}(L)$ the set of the associated prime ideals of L and by $\operatorname{Min}(L)$ the set of the minimal associated prime ideals of L. The *degree* $\deg(L)$ is the multiplicity of the graded module L. By the associativity formula we have

$$\deg(L) = \sum_{\substack{\wp\in\operatorname{Ass}(L),\\ \dim R/\wp = d}} \operatorname{mult}_L(\wp)\deg R/\wp.$$

The *arithmetic degree* and the *geometric degree* of L are defined as

$$\operatorname{adeg}(L) := \sum_{\wp\in\operatorname{Ass}(L)} \operatorname{mult}_L(\wp)\deg R/\wp,$$
$$\operatorname{gdeg}(L) := \sum_{\wp\in\operatorname{Min}(L)} \operatorname{mult}_L(\wp)\deg R/\wp.$$

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See, for example, [**18**] or [**23**] for more information on these generalizations of the degree of a module. To prove the preservation of the arithmetic degree, we need the following lemma.

Lemma 3.4. *Let L be a graded R-module and I a homogeneous ideal of R. Then $\Gamma_I(L)_\alpha \cong \Gamma_{I_\alpha}(L_\alpha)$ for almost all α.*

Proof. There is an integer t such that $\Gamma_I(L) = 0_L : I^t$ and $0_L : I^t = 0_L : I^m$ for all $m \geqslant t$. By Proposition 2.7 (ii), $(0_L : I^m)_\alpha \cong 0_{L_\alpha} : I^m_\alpha$. Therefore, $\Gamma_I(L)_\alpha = (0_L : I^t)_\alpha = 0_{L_\alpha} : I^t_\alpha$ and $0_{L_\alpha} : I^t_\alpha = 0_{L_\alpha} : I^m_\alpha$ for $m \geqslant t$. Hence $\Gamma_I(L)_\alpha \cong \Gamma_{I_\alpha}(L_\alpha)$ for almost all α. □

Theorem 3.5. *Let L be a graded R-module of dimension d. Then, for almost all α, we have*

(i) $\deg(L_\alpha) = \deg(L)$,

(ii) $\operatorname{adeg}(L_\alpha) = \operatorname{adeg}(L)$,

(iii) $\operatorname{gdeg}(L_\alpha) = \operatorname{gdeg}(L)$.

Proof. (i) Because the degree of L (respectively, L_α) is obtained from the Hilbert polynomial of L (respectively, L_α), (i) follows from Corollary 3.3 (i).

(ii) Set $L_i = \operatorname{Ext}^i_R(\operatorname{Ext}^i_R(L, R), R)$ and $M_i = \operatorname{Ext}^i_{R_\alpha}(\operatorname{Ext}^i_{R_\alpha}(L_\alpha, R_\alpha), R_\alpha)$ for all $i \geqslant 0$. From Proposition 2.5 it follows that $(L_i)_\alpha \cong M_i$ for all $i \geqslant 0$. By [**23**, Proposition 9.1.2], this implies

$$\operatorname{adeg}(L_\alpha) = \sum_{i=0}^{n+1} \deg(M_i) = \sum_{i=0}^{n+1} \deg(L_i) = \operatorname{adeg}(L).$$

(iii) Set $d = \dim L$. Then $\dim L_\alpha = d$ by Proposition 2.8. We first consider the case where all the minimal associated primes of L have dimension d. Since $\sqrt{\operatorname{Ann} L}$ is unmixed of dimension d, $(\sqrt{\operatorname{Ann} L})_\alpha$ is again unmixed of dimension d by [**8**, Satz 5]. By Lemma 2.9, $\sqrt{\operatorname{Ann} L_\alpha}$ is unmixed of dimension d. Since the geometric degree and the degree coincide for this case, we have

$$\operatorname{gdeg}(L_\alpha) = \deg(L_\alpha) = \deg(L) = \operatorname{gdeg}(L)$$

for almost all α. Suppose now that not all the minimal associated primes of L have dimension d. Denote by I the intersection of all minimal associated primes of L with dimension d. Since $\{\wp \in \operatorname{Min}(L) \mid \dim R/\wp < d\} = \operatorname{Min}(L/\Gamma_I(L))$ and $\operatorname{mult}_L(\wp) = \operatorname{mult}_{L/\Gamma_I(L)}(\wp)$ for all $\wp \in \operatorname{Ass}(L/\Gamma_I(L))$, we have

$$\operatorname{gdeg}(L/\Gamma_I(L)) = \sum_{\substack{\wp \in \operatorname{Min}(L), \\ \dim R/\wp < d}} \operatorname{mult}_L(\wp) \deg R/\wp.$$

Since

$$\begin{aligned}
\deg(L) &= \sum_{\substack{\wp\in \mathrm{Min}(L),\\ \dim R/\wp=d}} \mathrm{mult}_L(\wp)\deg R/\wp,\\
\mathrm{gdeg}(L) &= \sum_{\substack{\wp\in \mathrm{Min}(L),\\ \dim R/\wp=d}} \mathrm{mult}_L(\wp)\deg R/\wp + \sum_{\substack{\wp\in \mathrm{Min}(L),\\ \dim R/\wp<d}} \mathrm{mult}_L(\wp)\deg R/\wp\\
&= \deg(L) + \mathrm{gdeg}(L/\Gamma_I(L)).
\end{aligned}$$

By Corollary 2.10,

$$\{\mathfrak{q}\in \mathrm{Min}(L_\alpha) \mid \dim R/\mathfrak{q} < d\} = \mathrm{Min}(L_\alpha/\Gamma_{I_\alpha}(L_\alpha)).$$

Since $\mathrm{mult}_{L_\alpha}(\mathfrak{q}) = \mathrm{mult}_{L_\alpha/\Gamma_{I_\alpha}(L_\alpha)}(\mathfrak{q})$ for all $\mathfrak{q}\in \mathrm{Ass}(L_\alpha/\Gamma_{I_\alpha}(L_\alpha))$, we have

$$\mathrm{gdeg}(L_\alpha/\Gamma_{I_\alpha}(L_\alpha)) = \sum_{\substack{\mathfrak{q}\in \mathrm{Min}(L_\alpha),\\ \dim R_\alpha/\mathfrak{q}<d}} \mathrm{mult}_{L_\alpha}(\mathfrak{q})\deg R_\alpha/\mathfrak{q}.$$

Therefore,

$$\mathrm{gdeg}(L_\alpha) = \deg(L_\alpha) + \mathrm{gdeg}(L_\alpha/\Gamma_{I_\alpha}(L_\alpha)).$$

From (i) we obtain $\deg(L_\alpha) = \deg(L)$. By Proposition 2.6 (i) and by Lemma 3.4,

$$(L/\Gamma_I(L))_\alpha \cong L_\alpha/\Gamma_I(L)_\alpha \cong L_\alpha/\Gamma_{I_\alpha}(L_\alpha).$$

Since $\dim L/\Gamma_I(L) < d$, $\mathrm{gdeg}(L_\alpha/\Gamma_{I_\alpha}(L_\alpha)) = \mathrm{gdeg}(L/\Gamma_I(L))$ by induction on the dimension. Therefore,

$$\begin{aligned}
\mathrm{gdeg}(L_\alpha) &= \deg(L_\alpha) + \mathrm{gdeg}(L_\alpha/\Gamma_{I_\alpha}(L_\alpha))\\
&= \deg(L) + \mathrm{gdeg}(L/\Gamma_I(L)) = \mathrm{gdeg}(L).
\end{aligned}$$

□

Let M be a finitely generated graded module over the graded algebra A and let B be a Gorenstein graded algebra mapping onto A. Assume that $\dim B = n, \dim M = d$. In [**23**] the *homological degree* of M is defined as the integer

$$\mathrm{hdeg}(M) := \deg(M) + \sum_{i=n-d+1}^{n} \binom{d-1}{i-n+d-1} \mathrm{hdeg}(\mathrm{Ext}_B^i(M,B)).$$

We note that the homological degree is defined recursively on the dimension of the support of M. If $\dim M = 0$, then $\mathrm{hdeg}(M) = \deg(M)$. If $\dim M = 1$, $\mathrm{hdeg}(M) = \deg(M) + \ell(\mathrm{Ext}_B^1(M,B))$.

Proposition 3.6. *Let L be a graded R-module. Then, for almost all α,*

$$\mathrm{hdeg}(L_\alpha) = \mathrm{hdeg}(L).$$

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Proof. We want to prove the assertion by induction on the dimension of L. The rings R_α and R are Gorenstein rings. Set $d =: \dim L$. By Proposition 2.7, $\dim L_\alpha = d$. If $d = 0$, we have $\mathrm{hdeg}(L_\alpha) = \deg(L_\alpha)$ and $\mathrm{hdeg}(L) = \deg(L)$. Then $\mathrm{hdeg}(L_\alpha) = \mathrm{hdeg}(L)$ by Theorem 3.5. Now we consider the case $d \geqslant 1$. Assume that the assertion is true for all modules of dimension strictly less than d. We see that if $i \geqslant n - d + 2$, then $n + 1 - i \leqslant d - 1$. By [**2**, Corollary 3.5.11],

$$\dim \mathrm{Ext}^i_{R_\alpha}(L_\alpha, R_\alpha) = \dim \mathrm{Ext}^i_R(L, R) \leqslant n + 1 - i \leqslant d - 1.$$

By the induction hypothesis,

$$\mathrm{hdeg}(\mathrm{Ext}^i_{R_\alpha}(L_\alpha, R_\alpha)) = \mathrm{hdeg}(\mathrm{Ext}^i_R(L, R)).$$

Hence, for almost all α,

$$\begin{aligned} \mathrm{hdeg}\, L_\alpha &= \deg L_\alpha + \sum_{i=n-d+2}^{n+1} \binom{d-1}{i-n+d-2} \mathrm{hdeg}(\mathrm{Ext}^i_{R_\alpha}(L_\alpha, R_\alpha)) \\ &= \deg L + \sum_{i=n-d+2}^{n+1} \binom{d-1}{i-n+d-2} \mathrm{hdeg}(\mathrm{Ext}^i_R(L, R)) = \mathrm{hdeg}(L). \end{aligned}$$

□

For a graded R-module $L = \bigoplus_{t\in\mathbb{Z}} L_t$, the number $a(L)$ is defined as

$$a(L) := \begin{cases} \max\{t \mid L_t \neq 0\} & \text{if } L \neq 0, \\ -\infty & \text{if } L = 0. \end{cases}$$

Let $H^i_{\mathfrak{m}}(L)$ denote the ith local cohomology module of L with respect to $\mathfrak{m}$. We set

$$\begin{aligned} a_i(L) &= a(H^i_{\mathfrak{m}}(L)), \\ \mathrm{reg}(L) &= \max\{a_i(L) + i \mid i \geqslant 0\}, \\ a^*(L) &= \max\{a_i(L) \mid i \geqslant 0\}. \end{aligned}$$

The number $\mathrm{reg}(L)$ is called the *Castelnuovo–Mumford regularity* [**4**, **16**], and $a^*(L)$ the a^*-invariant of L [**21**] (cf. [**6**]). Note that the Castelnuovo–Mumford regularity and the a^*-invariant can be viewed as special cases of the more general invariants

$$\begin{aligned} \mathrm{reg}_p(L) &:= \max\{a_i(L) + i \mid 0 \leqslant i \leqslant p\}, \\ a^*_p(L) &:= \max\{a_i(L) \mid i \leqslant p\}, \quad p = 0, \dots, d. \end{aligned}$$

If $F_i = \bigoplus_j R(-j)^{\beta_{ij}}$ is the ith term of a minimal graded free resolution of L, then

$$\begin{aligned} \mathrm{reg}_p(L) &= \max\{j - i \mid i \geqslant n + 1 - p,\ \beta_{ij} \neq 0\}, \\ a^*_p(L) &= \max\{j \mid i \geqslant n + 1 - p,\ \beta_{ij} \neq 0\} - n - 1 \end{aligned}$$

(see, for example, [**22**]). It will be shown that these invariants are preserved by specializations.

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Proposition 3.7. *Let L be a graded R-module. Then, for almost all α, we have*

(i) $a_p(L_\alpha) = a_p(L)$,

(ii) $\operatorname{reg}_p(L_\alpha) = \operatorname{reg}_p(L)$,

(iii) $a_p^*(L_\alpha) = a_p^*(L)$.

Proof. Since R is a Gorenstein ring, the local duality theorem of Grothendieck says that $H^p_{\mathfrak{m}}(L) \cong \operatorname{Ext}_R^{n+1-p}(L,R)^v$, where ^{v} is the Matlis dual functor (see [**2**, Theorem 3.5.8]). Since $\operatorname{Ext}_{R_\alpha}^{n+1-p}(L_\alpha, R_\alpha) \cong \operatorname{Ext}_R^{n+1-p}(L,R)_\alpha$ by Proposition 2.5,

$$\begin{aligned} a_p(L_\alpha) &= \max\{t \mid \operatorname{Ext}_{R_\alpha}^{n+1-p}(L_\alpha, R_\alpha)_{-t-n-1} \neq 0\} \\ &= \max\{t \mid \operatorname{Ext}_R^{n+1-p}(L,R)_{-t-n-1} \neq 0\} = a_p(L). \end{aligned}$$

This proves (i). Clearly, (ii) and (iii) follow from (i). □

4. Specialization of reductions and filter-regular sequences

In this section we will show that the reduction number and filter-regular sequences are preserved by specialization.

Let I be an arbitrary homogeneous ideal of R. Set $d = \dim R/I$. Then $\dim R_\alpha/I_\alpha = d$ by Proposition 2.6. We denote by $\mathfrak{n}$ and $\mathfrak{n}_\alpha$ the maximal graded ideals of R/I and R_α/I_α, respectively. Let $\mathfrak{a}$ be a homogeneous ideal of R/I. Recall that $\mathfrak{a}$ is said to be a *reduction* of $\mathfrak{n}$ if $\mathfrak{a}\mathfrak{n}^r = \mathfrak{n}^{r+1}$ for some non-negative integer r and the least integer r with this property is called the reduction number of R/I with respect to $\mathfrak{a}$ [**15**]. This number is denoted by $r_{\mathfrak{a}}(R/I)$, and it is the largest non-vanishing degree of R/I. A reduction of $\mathfrak{n}$ is said to be minimal if it does not contain any other reduction of $\mathfrak{n}$. Since k is an infinite field, a reduction of $\mathfrak{n}$ is minimal if and only if it is generated by d elements. The reduction number $r(R/I)$ of R/I is defined as the minimum $r_{\mathfrak{a}}(R/I)$ of all minimal reductions $\mathfrak{a}$ of $\mathfrak{n}$.

Now we will prove that the reduction number $r(R/I)$ does not change when we specialize u to α. The main difficulty is how to locate a reduction which gives the reduction number of R/I and of R_α/I_α by specializations. We overcome this difficulty by taking the generic reduction. The following result is due to Trung (see the proof of Lemma 4.2 in [**22**]).

Lemma 4.1. *Let J be a homogeneous ideal of $S = k[x]$ and $d = \dim S/J$. Define $z_i = v_{i0}x_0 + \cdots + v_{in}x_n$, $i = 1, \ldots, d$, where $v = (v_{ij})$ is a family of $d(n+1)$ indeterminates. Put $S_v = k(v)[x]$, $J_v = JS_v$, $\mathfrak{a} = (J_v, z_1, \ldots, z_d)/J_v$. Then*

$$r(S/J) = r_{\mathfrak{a}}(S_v/J_v).$$

Lemma 4.2. *If a homogeneous ideal $\mathfrak{a}$ is a minimal reduction of $\mathfrak{n}$, then $\mathfrak{a}_\alpha$ is also a minimal reduction of $\mathfrak{n}_\alpha$ with $r_{\mathfrak{a}_\alpha}(R_\alpha/I_\alpha) = r_{\mathfrak{a}}(R/I)$ for almost all α.*

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Proof. We first note that a reduction $\mathfrak{a}$ is a minimal reduction of $\mathfrak{n}$ if and only if it is generated by d homogeneous elements of R/I of degree 1 and there exists a non-negative integer r such that $\mathfrak{a}_{r+1} = (R/I)_{r+1}$ and $\mathfrak{a}_s \neq (R/I)_s$ for all $s \leqslant r$. Suppose that $y_1, \dots, y_d \in R_1$ such that $\mathfrak{a} = (I, y_1, \dots, y_d)/I$. Then

$$\dim_{k(u)}(I, y_1, \dots, y_d)_{r+1} = \dim_{k(u)} R_{r+1},$$
$$\dim_{k(u)}(I, y_1, \dots, y_d)_s < \dim_{k(u)} R_s$$

for all $s \leqslant r$. By Corollary 3.2,

$$\dim_{k(\alpha)}(I_\alpha, (y_1)_\alpha, \dots, (y_d)_\alpha)_{r+1} = \dim_{k(\alpha)}(R_\alpha)_{r+1},$$
$$\dim_{k(\alpha)}((I)_\alpha, (y_1)_\alpha, \dots, (y_d)_\alpha)_s < \dim_{k(\alpha)}(R_\alpha)_s$$

for all $s \leqslant r$. By Proposition 2.8, $\mathfrak{a}_\alpha = (I_\alpha, (y_1)_\alpha, \dots, (y_d)_\alpha)/I_\alpha$ is again a minimal reduction of $\mathfrak{n}_\alpha$ and we obtain $r_{\mathfrak{a}_\alpha}(R_\alpha/I_\alpha) = r_{\mathfrak{a}}(R/I)$ for almost all α. □

Theorem 4.3. *Let I be a homogeneous ideal of R. Then, for almost all α, we have*

$$r(R_\alpha/I_\alpha) = r(R/I).$$

Proof. Define $z_i = \sum_{j=0}^{n} v_{ij} x_j$, $i = 1, \dots, d$, where all v_{ij} are indeterminates. Put $S_v = k(\alpha, v)[x]$, $J_v = I_\alpha S_v$ and $\mathfrak{a} = (J_v, z_1, \dots, z_d)/J_v$. By Lemma 4.1 we have

$$r(R_\alpha/I_\alpha) = r_{\mathfrak{a}}(S_v/J_v).$$

Similarly, if we put $R_v = k(u, v)[x]$, $I_v = IR_v$, and $\mathfrak{b} = (I_v, z_1, \dots, z_d)/I_v$, then

$$r(R/I) = r_{\mathfrak{b}}(R_v/I_v).$$

By Lemma 4.2,

$$r_{\mathfrak{a}}(S_v/J_v) = r_{\mathfrak{b}}(R_v/I_v)$$

for almost all α. Summing up we get $r(R_\alpha/I_\alpha) = r(R/I)$. □

Let $f_1, \dots, f_h$ be a sequence of homogeneous elements of a finitely generated graded algebra $A = \bigoplus_{i \geqslant 0} A_i$ over a field A_0. Let A_+ denote the ideal generated by the elements of positive degree of A. Let L be an A-module. The sequence $f_1, \dots, f_h$ is called *filter-regular* for L if $f_i \notin P$ for all associated prime ideals P of $(f_1, \dots, f_{i-1})L$, $P \neq M$, where M denotes the maximal graded ideal of A. This is equivalent to saying that the A-modules

$$(f_1, \dots, f_{i-1})L : f_i/(f_1, \dots, f_{i-1})L, \quad i = 1, \dots, h,$$

are of finite lengths. The notion of filter-regular sequences plays an important role in the theory of generalized Cohen–Macaulay rings [**3**].

Proposition 4.4. *Let $f_1, \dots, f_h$ be a filter-regular sequence of homogeneous elements of R/I with $h \geqslant 1$. Then the sequence $(f_1)_\alpha, \dots, (f_h)_\alpha$ is also a filter-regular sequence of R_α/I_α for almost all α.*

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Proof. Assume that $f_1, \dots, f_h$ is a filter-regular sequence of R/I. Then

$$(I, f_1, \dots, f_{i-1}) : f_i/(I, f_1, \dots, f_{i-1}), \quad i = 1, \dots, h,$$

are of finite length. The R-modules

$$(I, f_1, \dots, f_{i-1}) : f_i/(I, f_1, \dots, f_{i-1})$$

will be denoted by N_i for all $i = 1, \dots, h$. By Proposition 2.6,

$$(N_i)_\alpha \cong (I_\alpha, (f_1)_\alpha, \dots, (f_{i-1})_\alpha) : (f_i)_\alpha/(I_\alpha, (f_1)_\alpha, \dots, (f_{i-1})_\alpha)$$

for $i = 1, \dots, h$ and for almost all α. By Proposition 2.7,

$$\dim(N_i)_\alpha = \dim N_i = 0, \quad i = 1, \dots, h,$$

for almost all α. Hence $(I_\alpha, (f_1)_\alpha, \dots, (f_{i-1})_\alpha) : (f_i)_\alpha/(I_\alpha, (f_1)_\alpha, \dots, (f_{i-1})_\alpha)$, $i = 1, \dots, h$, are R_α-modules of finite length. Therefore $(f_1)_\alpha, \dots, (f_h)_\alpha$ is also a filter-regular sequence of R_α/I_α. □

The following consequence of Proposition 4.4 gives a positive answer to a question raised by Herzog (personal communication to N. V. Trung, 1998) which concerns the existence of filter-regular sequences of homogeneous elements of degree 1 in a graded algebra over an infinite field without taking generic elements [**1**, Proposition 2.1].

Corollary 4.5. *Let J be a homogeneous ideal of $k[x]$. We put*

$$y_i = \alpha_{i0}x_0 + \cdots + \alpha_{in}x_n, \quad i = 1, \dots, d,$$

where $\alpha = (\alpha_{ij}) \in k^{d(n+1)}$ and $d \geqslant 1$. Then the sequence $y_1, \dots, y_d$ is a filter-regular sequence of $k[x]/J$ for almost all α.

Proof. We define

$$z_i = u_{i0}x_0 + \cdots + u_{in}x_n, \quad i = 1, \dots, d,$$

where $u = (u_{ij})$ is a family of $d(n+1)$ indeterminates. By Proposition 4.4, we only need to show that $z_1, \dots, z_d$ is a filter-regular sequence of $A = k(u)[x]/Jk(u)[x]$. It suffices to prove the case $d = 1$. Put $S = k[x]/J$. We note that

$$\mathrm{Ass}_{k[v,x]}(S[v]) = \{P = \mathfrak{p}k[v, x] \mid \mathfrak{p} \in \mathrm{Ass}_{k[x]}(S)\}.$$

If $z_1 \in P = \mathfrak{p}k[v, x]$, then $(x_0, \dots, x_n) \subseteq \mathfrak{p}$. Therefore $\mathfrak{p} = \mathfrak{m}$ and $P = \mathfrak{m}k[v, x]$. Since A is a localization of $S[v]$, we can deduce that $z_1 \notin P$ for all associated prime ideals P of A which are different from the maximal graded ideal of A. Therefore, z_1 is a filter-regular element of A. □

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5. Bertini Theorems

Let V be a closed subscheme of the projective space $\mathbb{P}^n_k$. Let H_α be the hypersurface defined by a form $f_\alpha = \alpha_1 f_1 + \cdots + \alpha_m f_m$ in $\mathbb{P}^n_k$, where $\alpha = (\alpha_1, \dots, \alpha_m) \in k^m$ and $f_1, \dots, f_m$ is a family of forms of the same degree in $k[X] = k[X_0, \dots, X_n]$. Let I be the defining homogeneous ideal of V in $k[X]$. To study $V \cap H_\alpha$ means to study the local ring of the graded ring $k[X]/(I, f_\alpha)$ at its maximal graded ideal. This ring can be considered as a specialization of the local ring of the graded ring $R/(I, f)$ at its maximal graded ideal, where $R = k(u)[x]$, $u = (u_1, \dots, u_m)$ is a family of indeterminates and $f = u_1 f_1 + \cdots + u_m f_m$. Note that the latter ring corresponds to what one calls the generic hypersurface section of V. It is not hard to find conditions on I which allow a given property to be transferred from I to (I, f) (see [**7**, **19**]) and to check whether this property is preserved by specializations. We will demonstrate this method by reproving some Bertini Theorems.

Let $A = \bigoplus_{i \geqslant 0} A_i$ be a graded algebra over a field A_0. We denote the maximal homogeneous ideal of A by A_+. We put

$$\operatorname{Proj}(A) = \{P \in \operatorname{Spec}(A) \mid P \text{ is homogeneous and } P \neq A_+\}.$$

The non-Cohen–Macaulay locus and the singular locus of a factor ring B of A in $\operatorname{Proj}(A)$ are defined by

$$\begin{aligned} \mathrm{N_{CM}}(B) &= \{P \in \operatorname{Proj}(A) \mid B_P \text{ is not Cohen–Macaulay}\}, \\ \operatorname{Sing}(B) &= \{P \in \operatorname{Proj}(A) \mid B_P \text{ is not regular}\}. \end{aligned}$$

Given any ideal C of ring A we will denote by $V_+(C)$ the set of homogeneous prime ideals P containing C, $P \neq A_+$, and we define $D_+(C) = \operatorname{Proj}(A) - V_+(C)$. The following lemmas can be proved similarly as in the local case (see [**14**]); hence we omit the proofs.

Lemma 5.1 (cf. Lemma 4.3 of [14]). *Let I be an arbitrary homogeneous ideal of R. There is a homogeneous ideal $J \supseteq I$ such that for almost all α,*

$$\begin{aligned} \mathrm{N_{CM}}(R/I) &= V_+(J), \\ \mathrm{N_{CM}}(R_\alpha/I_\alpha) &= V_+(J_\alpha). \end{aligned}$$

Lemma 5.2 (cf. Lemma 4.4 of [14]). *Let k be a field of characteristic zero. Let I be a homogeneous ideal of R. There is a homogeneous ideal $J \supseteq I$ such that for almost all α,*

$$\begin{aligned} \operatorname{Sing}(R/I) &= V_+(J), \\ \operatorname{Sing}(R_\alpha/I_\alpha) &= V_+(J_\alpha). \end{aligned}$$

Let $t \geqslant 0$ be a fixed integer. We say that a ring A satisfies Serre's condition (S_t) if $\operatorname{depth} A_\mathfrak{p} \geqslant \min\{\dim A_\mathfrak{p}, t\}$ for any prime ideal $\mathfrak{p}$ or Serre's condition (R_t) if $A_\mathfrak{p}$ is regular for any prime ideal $\mathfrak{p}$ with height $\mathfrak{p} \leqslant t$.

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Lemma 5.3 (cf. Theorem 4.5 of [14]). *Let k be a field of characteristic zero. Let I be a homogeneous ideal of R. Assume that R/I satisfies one of the following properties:*

(i) (S_t),

(ii) (R_t),

(iii) *R/I is reduced,*

(iv) *R/I is normal.*

Then R_α/I_α has the same property for almost all α.

Using the above lemmas we will give simple proofs to the following Bertini Theorems.

Theorem 5.4 (see Hauptsatz of [19]). *Let k be a field of characteristic zero. Let A be a graded k-algebra generated by elements of degree 1. Let $f_1, \dots, f_m$ be a family of homogeneous elements of the same degree in A and $f_\alpha = \alpha_1 f_1 + \cdots + \alpha_m f_m$, where $\alpha = (\alpha_1, \dots, \alpha_m) \in k^m$. Assume that A is a normal ring and* $\operatorname{grade}(f_1, \dots, f_m) \geqslant 3$. *Then $A/f_\alpha A$ is a normal ring for almost all α.*

Proof. Let $A = k[X]/I$, where I is a homogeneous ideal of $k[X]$. Let $B = k(u)[X]/(I)$ and $f = u_1 f_1 + \cdots + u_m f_m$. Then $A/f_\alpha A = k[X]/(I, f_\alpha)$ is a specialization of $B/fB = k(u)[X]/(I, f)$. By Lemma 5.3 we only need to show that B/fB is a normal ring. But this follows from the assumptions by [**19**, Korollar 4.4]. □

The following result is the global Bertini Theorem of Flenner.

Theorem 5.5 (see Satz 5.4 of [5]). *Let k be a field of characteristic zero. Let A be a graded k-algebra finitely generated by elements of degree 1. Let $f_1, \dots, f_m$ be homogeneous elements in A of the same degree. Let $U \subseteq D_+(f_1, \dots, f_m)$ be an open set with one of the following properties:*

(i) *U satisfies (S_t),*

(ii) *U satisfies (R_t),*

(iii) *U is reduced,*

(iv) *U is normal,*

(v) *U is regular.*

Let $f_\alpha = \alpha_1 f_1 + \cdots + \alpha_m f_m$, where $\alpha = (\alpha_1, \dots, \alpha_m) \in k^m$. Then $U \cap V_+(f_\alpha) \subseteq$ $\operatorname{Proj}(A/f_\alpha A)$ *has the same property as U for almost all α.*

Proof. Let $A = k[X]/I$, where I is a homogeneous ideal of $k[X]$. Let $B = k(u)[X]/(I)$ and $f = u_1 f_1 + \cdots + u_m f_m$. Then $A/f_\alpha A = k[X]/(I, f_\alpha)$ is a specialization of $B/fB = k(u)[X]/(I, f)$. By Lemma 5.1, there is a homogeneous ideal $\mathfrak{b} \supseteq fB$ of B such that $V_+(\mathfrak{b})$ is the projective non-Cohen–Macaulay locus of B/fB in $\operatorname{Proj}(B)$ and $V_+(\mathfrak{b}_\alpha)$ is the projective non-Cohen–Macaulay locus of $A/f_\alpha A$ in $\operatorname{Proj}(A)$. Let $\mathfrak{a}$ be a homogeneous ideal of A such that $U = D_+(\mathfrak{a})$. Let $\mathfrak{P}$ be an arbitrary homogeneous prime ideal of $V_+(\mathfrak{b} : \mathfrak{a})$. Then $\mathfrak{P}$ does not contain $\mathfrak{a}$ and the local ring $(B/fB)_{\mathfrak{P}}$ is not Cohen–Macaulay. Let $\mathfrak{p}$ denote the contraction of $\mathfrak{P}$ in A. Then $\mathfrak{p}$ does not contain $\mathfrak{a}$. Hence $\mathfrak{p} \in U$. Since $U \subseteq D_+(f_1, \dots, f_m)$, $\mathfrak{p}$ does not contain $(f_1, \dots, f_m)$. Hence $\operatorname{grade}(f_1, \dots, f_m)A_{\mathfrak{p}} = \infty$. If U satisfies (S_t), then $A_{\mathfrak{p}}$ satisfies (S_t). By [**19**, Proposition 3.1], $A_{\mathfrak{p}}[u]/fA_{\mathfrak{p}}[u]$ also satisfies (S_t). Since $(B/fB)_{\mathfrak{P}}$ is the local ring of $A_{\mathfrak{p}}[u]/fA_{\mathfrak{p}}[u]$ at a prime ideal, $\operatorname{depth}(B/fB)_{\mathfrak{P}} > t$. By [**2**, Proposition 1.2.10], this implies $\operatorname{grade}(\mathfrak{b} : \mathfrak{a}/fB) > t$. By [**14**, Lemma 2.5] and [**14**, Corollary 3.4],

$$\operatorname{grade}(\mathfrak{b}_\alpha : \mathfrak{a}/f_\alpha A) = \operatorname{grade}(\mathfrak{b} : \mathfrak{a}/fB)_\alpha = \operatorname{grade}(\mathfrak{b} : \mathfrak{a}/fB) > t.$$

Thus, $\operatorname{depth}(A/f_\alpha A)_{\mathfrak{q}} > t$ for any homogeneous prime ideal $\mathfrak{q} \supseteq \mathfrak{b}_\alpha$ which does not contain $\mathfrak{a}$. Since $(A/f_\alpha A)_{\mathfrak{q}}$ is a Cohen–Macaulay ring for any prime ideal $\mathfrak{q} \not\supseteq \mathfrak{b}_\alpha$, we get $\operatorname{depth}(A/f_\alpha A)_{\mathfrak{q}} > \min\{\dim(A/f_\alpha A)_{\mathfrak{q}}, t\}$ for any homogeneous prime ideal $\mathfrak{q} \in U$. Hence $U \cap V_+(f_\alpha) \subseteq \operatorname{Proj}(A/f_\alpha A)$ satisfies (S_t).

Similarly, using Lemma 5.2 we can find a homogeneous ideal $\mathfrak{c} \supseteq fB$ of B such that $V_+(\mathfrak{c})$ is the projective singular locus of B/fB in $\operatorname{Proj}(B)$ and $V_+(\mathfrak{c}_\alpha)$ is the projective singular locus of $A/f_\alpha A$ in $\operatorname{Proj}(A)$. If U satisfies (R_t), using [**19**, Proposition 3.8] we can show that $\operatorname{height}(\mathfrak{c} : \mathfrak{a}/fB) > t$. By [**14**, Lemma 2.5] and [**14**, Proposition 2.6],

$$\operatorname{height}(\mathfrak{c}_\alpha : \mathfrak{a}/f_\alpha A) = \operatorname{height}(\mathfrak{c} : \mathfrak{a}/fB)_\alpha = \operatorname{height}(\mathfrak{c} : \mathfrak{a}/fB) > t.$$

From this it follows that $U \cap V_+(f_\alpha) \subseteq \operatorname{Proj}(A/f_\alpha A)$ satisfies (R_t).

As in the above proof we see that if U satisfies (S_t) and (R_h), then $U \cap V_+(f_\alpha) \subseteq \operatorname{Proj}(A/f_\alpha A)$ also satisfies (S_t) and (R_h). If U is reduced (normal), then U and therefore $U \cap V_+(f_\alpha) \subseteq \operatorname{Proj}(A/f_\alpha A)$ satisfies (S_1) and (R_0) ((S_2) and (R_1)). Hence $U \cap V_+(f_\alpha) \subseteq \operatorname{Proj}(A/f_\alpha A)$ is reduced (normal). Let $d = \dim A$. If U is regular, then U satisfies (R_d). By (ii), $U \cap V_+(f_\alpha)$ satisfies (R_d). Since $\dim A/f_\alpha A \leqslant d$, $U \cap V_+(f_\alpha) \subseteq \operatorname{Proj}(A/f_\alpha A)$ is regular. □

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References

1. A. Aramova and J. Herzog, Almost regular sequences and Betti numbers, *Am. J. Math.* **122** (2000), 689–719.
2. W. Bruns and J. Herzog, *Cohen–Macaulay rings* (Cambridge University Press, 1993).
3. N. T. Cuong, P. Schenzel and N. V. Trung, Verallgemeinerte Cohen–Macaulay moduln, *Math. Nachr.* **85** (1978), 57–73.
4. D. Eisenbud and S. Goto, Linear free resolutions and minimal multiplicity, *J. Algebra* **88** (1984), 89–133.

5. H. Flenner, Die Sätze von Bertini für lokale Ringe, *Math. Annln* **229** (1977), 97–111.
6. S. Goto and K. Watanabe, On graded rings, I, *J. Math. Soc. Jpn* **30** (1978), 179–213.
7. M. Hochster, Properties of Noetherian rings stable under general grade reduction, *Arch. Math.* **24** (1973), 393–396.
8. W. Krull, Parameterspezialisierung in Polynomringen, *Arch. Math.* **1** (1948), 56–64.
9. W. Krull, Parameterspezialisierung in Polynomringen, II, Grundpolynom, *Arch. Math.* **1** (1948), 129–137.
10. W. E. Kuan, A note on a generic hyperplane section of an algebraic variety, *Can. J. Math.* **22** (1970), 1047–1054.
11. W. E. Kuan, Specialization of a generic hyperplane section through a rational point of an algebraic variety, *Annli Mat. Pura Appl.* **94** (1972), 75–82.
12. W. E. Kuan, Some results on normality of a graded ring, *Pac. J. Math.* **64** (1976), 455–463.
13. D. V. Nhi and N. V. Trung, Specialization of modules, *Commun. Algebra* **27** (1999), 2959–2978.
14. D. V. Nhi and N. V. Trung, Specialization of modules over a local ring, *J. Pure Appl. Algebra* **152** (2000), 275–288.
15. D. G. Northcott and D. Rees, Reductions of ideals in local rings, *Math. Proc. Camb. Phil. Soc.* **50** (1954), 145–158.
16. A. Ooishi, Castelnuovo's regularity and graded rings and modules, *Hiroshima Math. J.* **12** (1982), 627–644.
17. A. Seidenberg, The hyperplane sections of normal varieties, *Trans. Am. Math. Soc.* **69** (1950), 375–386.
18. B. Sturmfels, N. V. Trung and W. Vogel, Bounds on degrees of projective schemes, *Math. Annln* **302** (1995), 417–432.
19. N. V. Trung, Über die Übertragung der Ringeigenschaften zwischen R und $R[u]/(F)$, *Math. Nachr.* **92** (1979), 215–229.
20. N. V. Trung, Spezialisierungen allgemeiner Hyperflächenschnitte und Anwendungen, in *Seminar D. Eisenbud, B. Singh and W. Vogel*, vol. 1, *Teubner-Texte Math.* **29** (1980), 4–43.
21. N. V. Trung, The largest non-vanishing degree of graded local cohomology modules, *J. Algebra* **215** (1999), 481–499.
22. N. V. Trung, Gröbner bases, local cohomology and reduction number, *Proc. Am. Math. Soc.* **129** (2001), 9–18.
23. W. V. Vasconcelos, *Computational methods in commutative algebra and algebraic geometry* (Springer, 1998).