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Green element numerical solution of generalized MHD Couette flow with heat transfer

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ABSTRACT

This paper provides a Green element method (GEM) numerical analysis of the effects of a uniform transverse magnetic field on fluid flow. The Green element method is a robust numerical scheme that evolved essentially from the singular integral theory of the boundary element method (BEM) with the unique variety of numerically implementing the theory by the finite element procedure. One of the advantages inherent in this approach is that the coefficient matrix from the discrete equations of the assembled element equations is banded and amenable to numerical solution. For the purposes of this study, the fluid is incompressible, and electrically conducting, and flows between two parallel plates, one of which is moving with a uniform speed while the other is stationary. The depth of the channel is taken to be much smaller than the width and the channel is considered to be very long in the horizontal direction. As a result, the flow is assumed to be fully developed and driven by a pressure gradient in a uniform magnetic field. Numerical solutions obtained with GEM closely match analytical results. In order to validate the physics and numerics of the problem formulation, comprehensive parametric studies are carried out to show the effects on flow and electromagnetic fields of Hartmann number, pressure gradient, current distributions, and temperature .

Keywords: Green element method, singular integral theory, boundary element method, electrically conducting fluid, Hartmann number, pressure gradient, pressure distribution

Introduction

When a moving conducting fluid interacts with electric and magnetic fields, it yields a rich variety of phenomena associated with magnetohydrodynamics (MHD) energy conversion. This type of interaction can be observed in fluids or in plasmas. Practical applications of these can be found in such areas as the heat transfer characteristics of an MHD generator channel, or in a number of industrial processes such as MHD power pump, power generation from high temperature gases, heating and flow control in metal processing (Tillack and Morley[1]).
The study of heat transfer in an electrically conducting fluid in the presence of a magnetic field has generated a renewed interest. This can be attributed to the fact that alternative energy sources have become a major source of concern in recent years. Major studies and development in this area have been fully referenced in Soundalgekar [2]; as can be witnessed by the production of such devices like MHD generators which possess advantages of being safer to operate, more efficiency, and less hazardous to the environment.

Sutton and Sherman [3] obtained a closed form solution to the problem of transient or steady state magneto hydrodynamic flow of a viscous incompressible electrical conducting fluid under the influence of a constant pressure gradient. Soundalgekar[4] analytically solved the problem of an MHD Couette flow between two parallel plates subjected to a fixed temperature and a pressure gradient. He carried out some parametric studies to decide the influence of certain dimensionless quantities on the flow and temperature distribution. Early work in this area includes those of Illingworth[5], Leadon [6], Blevis[7] and Schlichting[8]. A more comprehensive work involving the effects of both magnetic and electric fields on the flow field was carried out by Sutton and Shermann [10].

In the present work, we are adopting a Green element numerical technique on a generalized MHD Couette flow with heat transfer. Particularly the effects of a transversely applied external magnetic field on a fully developed laminar flow of an incompressible, viscous, electrically conducting fluid in a differentially heated horizontal channel. The numerical solutions for the fluid velocity, magnetic and electrical fields as well as the temperature distributions and heat transfer are obtained an interpreted in line with the influence brought about by the Hartmann number, current density and the effects of the external circuit.

Mathematical Formulation

For a constant property, steady, incompressible, viscous, electrically conducting Newtonian fluid, the complete set of magnetohyrodynamics equations comprises the Navier-Stokes equations , the Maxwell's equations, the Ohm's Law and the equation of mass continuity. In differential form, these equations are represented by:

$$\rho(U \bullet \nabla)U = -\nabla \bullet p + \mu\nabla^2 \bullet U + J \times B \tag{0.1}$$

where $U(u, v, w)$ is the velocity vector, p is the pressure, ρ, μ are density and coefficient of viscosity respectively. $J(j_x, J_y, J_z)$, $B(b_x, b_y, b_(z))$ are current density and magnetic induction vectors and $J \times B$ is the electromotive force. The equation of Continuity is given by:

$$\nabla U = 0 \tag{0.2}$$

The electro-magnetic of Maxwell equations are:
At steady state;

$$\nabla \times E = 0 \tag{0.3}$$

where $E(E_x, E_y, E_z)$ is the electric field vector

$$J = \frac{1}{\mu_m}(\nabla \times B) \tag{0.4}$$

where μ_m is the magnetic permeability. Implicit in equations (1) to (4) are the following additional relationships:

$$\nabla \bullet B = 0 \tag{0.5}$$

$$\nabla \bullet J = 0 \tag{0.6}$$

$$\nabla \times E = 0 \tag{0.7}$$

Neglecting Hall effects, the Ohms law is represented as:

$$J = \rho(E + V \times B) \tag{0.8}$$

The electromotive force, which constitutes a body force term arising from interaction of electromagnetic effects and which contributes to the Navier-Stokes equation is given by:

$$F_m = J \times B \tag{0.9}$$

The energy equation is

$$\kappa \nabla^2 T + \frac{J^2}{\sigma} + \Phi = 0 \tag{0.10}$$

where $\frac{J^2}{\sigma}$ represent the heat due to Joules and viscous dissipation respectively.
Implicit in all the assumptions arising from the complete set of the magnetohydrodnamic equations, and relating them to the problem geometry (Fig.1), the equations of motion and energy become:

$$\mu \left[\frac{d^2u}{dy^2}\right] - \sigma(E_z + uB_0)B_0 - \frac{dp}{dx} = 0 \tag{0.11}$$

$$\kappa \frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2 + \sigma(E_z + uB_0)^2 = 0 \tag{0.12}$$

The viscosity, the electrical conductivity and thermal conductivity μ, σ, κ are taken to be constant. We normalize the governing differential equations so that the relative strengths of the different terms can be inferred by the size of the multiplying factors. The equations of motion and energy can be put in dimensionless forms by making the following substitutions.

$$Y^* = \frac{y}{L}, \quad U^* = \frac{u}{u_w}, \quad \mu^* = \frac{\mu}{\mu_w}, \quad T^* = \frac{T}{T_w}$$

$$\kappa^* = Pr, \quad E = \frac{u_w^2 \mu_w}{\kappa T_w}, \quad L_e = \frac{E_z}{u_w B_0}, \quad \sigma^* = \frac{\sigma}{\sigma_w}$$

$$Ha = \sqrt{\left(\frac{\sigma_w}{\mu_w}\right) LB_0} \quad , PRe = \frac{dp/dx}{\mu_w u_w / L^2}$$

where L_e, Ha, pre are the electric field loading parameter, Hartmann number, and the non dimensional pressure gradient term. The non dimensional version of the governing equation now become (the stars are dropped for convenience):

$$\frac{d^2U}{dY^2} - Ha^2(U + L_e) - Pre = 0 \tag{0.13}$$

$$\frac{d^2T}{dY^2} + K\left(\frac{dU}{dY}\right) + KHa^2\left[(L_e + U)^2\right] = 0 \tag{0.14}$$

The following dimensionless boundary conditions are assumed for this problem:

$$U(0) = 0, \quad U(1) = 1.0, \quad T(0) = 0.5, \quad T(1) = 1.0$$

Other useful equations are
The total current per unit length of channel is given as:

$$I_t = \int_0^1 J_z dY = \int_0^1 (L_e + U) dY \tag{0.15}$$

The current density is:$J = (L_e + U)$, and the flow per unit width of channel is given by:

$$I_t = \int_0^1 U dY$$

For the open-circuit case:$L_e = 0$

Green element discretization

Our primary aim is to provide reliable means of discretizing the governing transport equations by adopting a hybrid boundary element methodology based on the Greens function of the Laplace diffusion operator in 1-D . Application of the boundary element technique to transport equations has continued to receive a lot of attention because of its relevance to a broad spectrum of scientific and engineering fields and the need to effectively deal with the problem domain. Earlier attempts relied mainly on the use of the Laplace transform technique (LT) which has the capability of transforming a parabolic equation into an elliptic type and also avoids the time stepping associated with transient problems(A.H.-D Cheng, O.K. Morohunfola [10]). Although some of the earlier formulations experienced difficulties in inverting from the LT plane to the real plane, a lot of improvements seem to have been made in their use. The use of the Greens function (A.E. Taigbenu, J.A. Liggett[11]), and Poissons techniques (A.E. Taigbenu and Liggett[12]) and lately to the application of reciprocity formulations specially designed to deal with challenges arising from the problem domain(C.A. Brebbia, D. S. Nardini[13],L.C. Wrobel, C.A. Brebbia and D. Nardini [14]) have also featured prominently in boundary element method (BEM) discretization.

The Green element method (GEM) came into the picture when the boundary-only implementation of the boundary element theory could not be easily amenable to the solution of nonlinear, heterogeneous, and transient problems. We refer specifically to those problems whose numerical solutions demand an encounter with the problem domain. A lot of attention was therefore concentrated on a hybrid method which not only takes advantage of the second-order accuracy of BEM formulation but also incorporates the efficiency of the finite element method (FEM)in handling domain discretization(A.E. Taigbenu[15], A.E. Taigbenu and O.O. Onyejekwe[16] and Onyejekwe[17]). Domain integral in line with GEM has since then become a permanent feature of BEM (Sladek et al.[18])

Our current GEM discretization is based on the Fredholm singular integral theory which employs the free-space Green function of the term with the highest derivative, namely $d^2/dx6'2$. It is worthwhile to note that for the purposes of this paper, and in order to avoid any confusion that may arise as a result of nomenclature or departure from previous work, we have chosen to represent a generic independent variable. The main feature of this formulation is the conversion of the governing differential equation into its integral analog, and its eventual solution on each element of the problem domain. We initiate this procedure by proposing a differential equation complementary to the governing differential equations, namely: $d^2G/dx^2 = \delta(x - x_i), \quad (-\infty, \infty))$, whose solution often referred to as the free-space Green function or the unit response function is of the form: $G(x^{'}, x_i^{'}) = (\left|x^{'} - x_i^{'}\right| + \kappa)/2$ where κ is an arbitrary constant, and is the distance between the point of application of a unit input at and any other point. Both the Greens second identity as well as the free space Greens function are then applied to the governing differential equation to arrive at its integral analog . With this in mind, equation (13) is represented as:

$$\begin{aligned}&-2\lambda U_i + [H(x_2^{'} - x_i) - H(x_i^{'} - x_2^{'})]U_2 - [H(x_1^{'} - x_i) - H(x_i^{'} - x_1^{'})]U_1 - \\ &(|x_2^{'} - x_i^{'}| + l)\psi_2 + (|x_i^{'} - x_1^{'}| + l)\psi_1 + \textstyle\int_{x_1}^{x_2}(|x_2^{'} - x_i^{'}| + l)(Ha^2\{U + l_e\} \\ &+ Pre)dx^{'} = 0 \quad i = 1, 2\end{aligned} \tag{16a}$$

where $\psi \equiv dU/dx$, λ and takes on the value of 0.5 when $x_i^{'}$ is either at $x_i^{'}$ or $x_2^{'}$, H is the Heaviside function, and l is set to the length of the longest element of the problem domain (for non uniform element s) in order to guarantee the positive-definiteness of the coefficient matrix. By the same token, the integral analog of equation(14) becomes:

$$\begin{aligned}&-2\lambda T_i + [H(x_2^{'} - x_i) - H(x_i^{'} - x_2^{'})]T_2 - [H(x_1^{'} - x_i) - H(x_i^{'} - x_1^{'})]T_1 - \\ &(|x_2^{'} - x_i^{'}| + l)\varphi_2 + (|x_i^{'} - x_1^{'}| + l)\varphi_1 + \textstyle\int_{x_1}^{x_2}(|x_2^{'} - x_i^{'}| + l) \\ &\left(-K\{dU/dy\}^2 - KHa^2[(Le + U)]^2 +\right) dx^{'} = 0 \\ &i = 1, 2\end{aligned} \tag{16b}$$

Equations (16) are implemented on each element of the problem domain in a finite element sense. Unlike the BEM which relies on global support by seeking information from all the nodes in the problem domain to obtain solution at any of the boundary nodes, GEM obtains solution at any node by the implementation of integrations within each element to which belongs the node or any of the shared nodes without recourse to other nodes in the entire problem domain (local support). Adopting such an approach not only guarantees that any spatial variability or nonlinearity in the material properties of the problem domain be sufficiently addressed but also that the coefficient matrix is sparsely populated, banded and equipped to handle field problems.

The element-by-element solution of the coupled equations requires that the computational domain be discretized into suitable elements over which the primary variable assumes some functional distribution. For this study, the functional variables are assumed to have linear variation over the elements. For example the primary variable as well as its derivative can be approximated in space using linear interpolating functions, namely:

$$U \approx \Omega_j(\zeta)U_j \tag{17}$$

where $\Omega_j(\zeta)$ is the interpolating function, ζ is a local coordinate represented as $\zeta = (x^{'} - x_i^{'})/l$, and $l = x_2^{'} - x_1^{'}$ is the length of an element. Substituting the approximation for the distribution of the dependent variables and their functions into the governing equations and evaluating the line integrals yields a system of element equations which can be represented generically as:

$$R_{ij}\Psi_j + L_{ij}\Lambda_j + T_{ij}\Theta_j \tag{18}$$

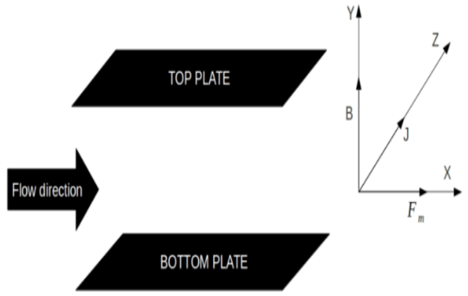

Figure 1: Problem geometry

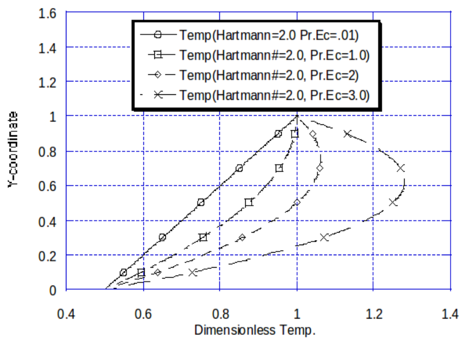

Figure 2: Effect of PrEc on temperature profile

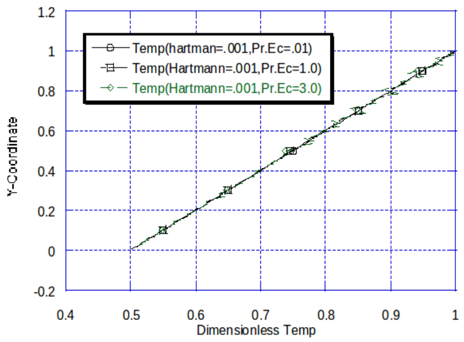

Figure 3: Relative effects of Hartmann number and PrEc on temperature profiles

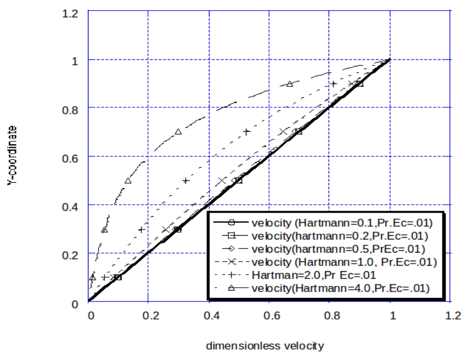

Figure 4: Effect of Hartmann number on velocity distribution

Values of the element matrices can be found in GEM literature(Onyejekwe (19)). Following a FEM approach, the solutions for each element in the computational domain are assembled to yield the global matrix equation

$$[A]\begin{Bmatrix}\Psi\\ \Lambda\end{Bmatrix} = \{S\} \tag{0.16}$$

where [A] is a matrix of coefficients, it comes with a half bandwidth of 2 and a row dimension that is twice the number of elements, while the unknown vector contains the primary variable and its flux Λ, the right side vector $\{S\}$contains boundary values of the primary variables Ψ or their derivatives, as well as any internal or externally imposed sources or sinks.

Results and Discussions

The numerical method discussed herein, is deployed to handle the governing differential equations for various values of Hartmann number, pressure gradient and the product of Prandtl and Eckert numbers and comments are made on their effects on the problem dependent variables. The influence of the external circuit configuration on

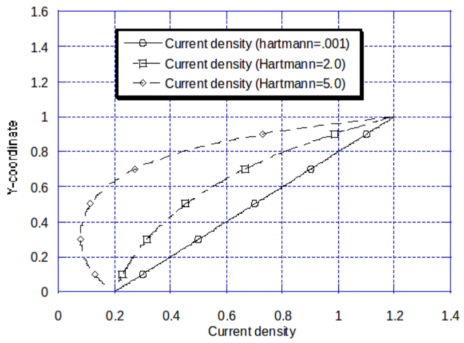

Figure 5: Effect of Hartmann number on current density

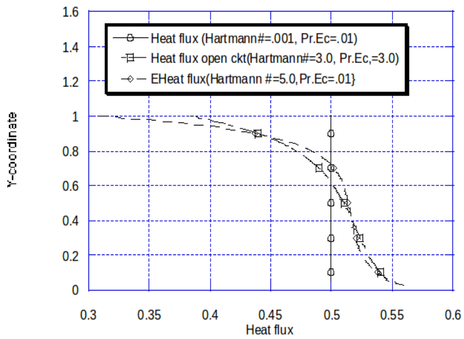

Figure 6: Heat flux for the closed circuit case (Hartmann number: 5.0, 3.0, .001 PrEc: .01)

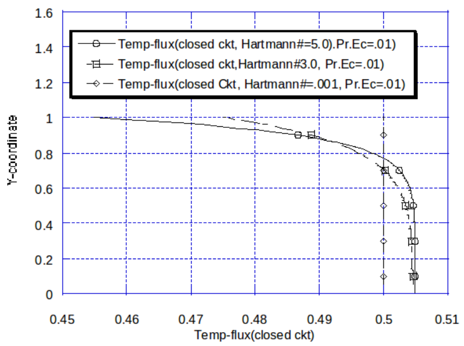

Figure 7: Relative effect of PrEc and Hartmann numbers on heat flux (open circuited case)

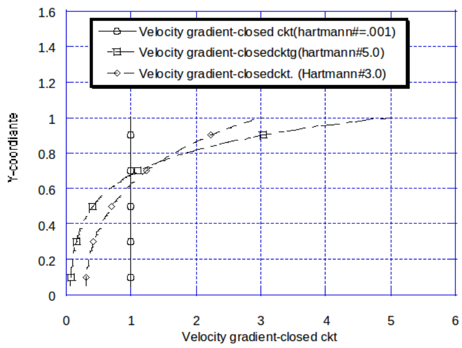

Figure 8: Profiles of velocity gradient for an closed circuited case (Hartmann number: 5.0, 3.0 .001)

the velocity, temperature and current density are also discussed considering the two extreme cases of short and open circuits.

The temperature distribution in the problem domain depends strongly on the product of Prandlt and Eckert numbers. The shape of the temperature profiles in Fig. 2 is similar to those found in standard fluid mechanics texts. It can be seen that when heat flows from the upper wall to the fluid but reverses when. This is in agreement with Setayeshpour[20]. However Fig.3 illustrates the relative importance of the Hartmans number on the temperature profile. Lower values of Hartman number have no significant effect on the temperature profile.

Fig.4 illustrates the influence of the Hartmann number on the velocity profile. Since the Hartmann number gives the measure of the relative importance between the drag forces resulting from magnetic body force and the viscous forces, a lager value of Hartmann number indicates a relatively small value of velocity.

The influence of the external circuit configuration on the flow and temperature profiles will be discussed by considering the two extreme cases of short and open circuits. For the short circuit consideration, there is no electric field, and consequently the electric field loading parameter, the electromotive or Lorentz force is perpendicular to

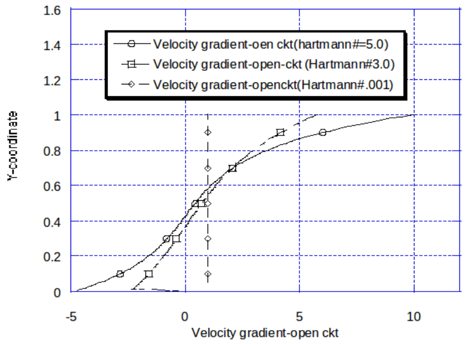

Figure 9: Profiles of velocity gradient for an closed circuited case (Hartmann number: 5.0, 3.0 .001)

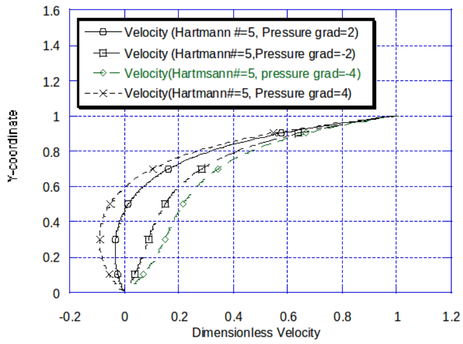

Figure 10: Effect of pressure gradient on velocity distribution

the flow direction, and is a retarding force. The retarding effect of this force is clearly demonstrated in Fig. 5 for the case of current density where for each particular channel position a smaller value of current density is recorded for a greater value of Hartmann number.

Figures 6 and 7 show a comparison between the temperature gradient profiles for closed and open configuration of the external circuit. Note that for small values of Hartmanns number, the temperature gradient throughout the flow domain is infinite . In the short-circuit configuration, both the electric field and the electric loading parameter are zero; hence the induced current density is in the positive z direction, and the resulting electromotive force is a retarding force. This causes a build up of the fluid particles and a consequential rise in the temperature gradient. This effect is easily noticeable from the bottom of the plate to slightly more than fifty percent of the total channel depth where temperature gradient profile is almost infinite. Whereas in the in the open-circuited case, the total current throughout the channel is zero, it can be seen from Fig. 8 that the temperature gradients are flatter comparatively because the current density is less for most parts of the channel.

A similar trend follows Figs.8 and 9 for the velocity gradient involving closed and open circuit configurations. Since less viscous forces attend to lower values of Hartman number, relatively higher velocities are recorded for small Hartmann numbers and a high velocity gradient builds up from the bottom of the plate because of the no-slip boundary condition. For Fig. 9 or the short circuited case, a higher value of Hartmann number guarantees a higher viscosity, a lower velocity and a lower value of the velocity gradient. On the other hand, for the open-circuit configuration , different signs are recorded in the upper and lower parts of the channel, when the velocity gradient is negative, it indicates a flow from the upper to the bottom part of the plate, an vice versa.

Fig. 10 illustrates the effect of the influence of the pressure gradient on the flow profile. For the pressure gradient values of 2 and 4 and for a fixed value of Hartmann number, the velocity profile changes from negative to positive within the channel depth of Y=0.4 to Y= 0.6. This underscores the influence of the retarding force to almost fifty percent of the channel depht. There is no change in flow direction for negative values of pressure gradient. All the velocity profiles irrespective of the values of pressure gradient reflect the effect of the upper moving plate

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