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Simultaneous determination of unknown two parameters in parabolic equation

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ABSTRACT

This paper is concerned with the inverse problem of determining two parameters simultaneously in parabolic equation subject to the overdeteminations of the solution at special points along with the usual initial boundary conditions, and the unknown parameters are only time-dependent. The method of using some transformations of the problem and finite difference method are used to solve the problem. Numerical examples with the method indicate that the accurate estimates of the unknown parameters can be obtained even in the presence of noise in the data.

Keywords: Inverse problem; Parabolic equations; Additional specification; Parameter identification; Finite difference method.

1. Introduction

In this paper,we consider the inverse identification of simultaneously determining two unknown parameters in a parabolic equation subject to the overdeterminations, and the unknown parameters are only time-dependent.

This problem is described as follows: find $p(t), g_1(t)$ and $u = u(x,t)$ which satisfy

$$\begin{cases} u_t(x,t) = k(u)u_{xx}(x,t) + q(t)u_x(x,t) + p(t)u(x,t) + \phi(x,t), & (x,t) \in Q_T, \\ u(x,0) = \varphi(x), & x \in (0,l), \\ u(0,t) = g_1(t), u_x(l,t) = g_2(t), & t \in (0,T), \end{cases} \tag{1.1}$$

where

$$Q_T = \{(x,t) \in R^2 : 0 < x < l, 0 < t < T\},$$

and the additional specifications:

$$u_x(0,t) = E_1(t),\ u(l,t) = E_2(t), \tag{1.2}$$

where $k(u), q(t), \phi(x,t), \varphi(x), g_2(t)$ and $E_1(t), E_2(t) \neq 0$ are known functions.

If the functions $p(t)$ and $g_1(t)$ are known, then the direct initial boundary value problem (1.1) has a unique smooth solution $u(x,t)$ [1].

Parabolic systems appear naturally in a number of physical and engineering settings, in particular in hydrology, material sciences, heat transfer, combustion systems, medical imaging and transport problems. Usually the function that characterizes a certain property of the system is unknown, and the interest is to identify the unknown

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function based on some time dependent measurements, which leads to an inverse problem for a parabolic system. A number of investigators have considered such problems for various applications using different methods, J.R.Cannon [2, 3], P. Du Chateau [4], M. Dehghan [5], A. Hasanov [6, 7], A. G. Fatullayev [8], Z.H. Liu [9], etc.

In this paper, the parabolic equation denotes heat transfer equation, and $u(x,t)$ represents temperature, the functions $p(t)$ and $g_1(t)$ are interpreted as the control parameter and heat boundary respectively. We will use the method of some transformation of the equation and the finite difference method to solve the problem.

The rest of this paper is organized as follows. In Section 2, we describe a numerical procedure for the solution of the formulated inverse problem using some transformations of the equation and the finite-difference method. In Section 3, the numerical experiments will be presented and discussed.

2. Mathematical formulation of the inverse problem

Consider the equation:

$$u_t(x,t) = k(u)u_{xx}(x,t) + q(t)u_x(x,t) + p(t)u(x,t) + \phi(x,t)$$

along with the initial boundary and additional conditions, and if we make the following transformation

$$v(x,t) = r(t)u(x,t), \quad r(t) = e^{-\int_0^t p(s)ds},$$

so we have

$$u(x,t) = \frac{v(x,t)}{r(t)}, \quad p(t) = -\frac{r'(t)}{r(t)}, \tag{2.1}$$

then for these new functions,we get the following equation from (1.1)

$$\begin{cases} v_t = k(\frac{v}{r})v_{xx} + q(t)v_x(x,t) + r(t)\phi(x,t), & (x,t) \in Q_T, \\ v(x,0) = \varphi(x), & x \in (0,l), \\ v(0,t) = r(t)g_1(t), v_x(l,t) = r(t)g_2(t), & t \in (0,T), \end{cases} \tag{2.2}$$

and

$$r(t) = \frac{v(x,t)}{u(x,t)} = \frac{v(l,t)}{u(l,t)} = \frac{v(l,t)}{E_2(t)}.$$

So the system (2.2) can be written as follows:

$$\begin{cases} v_t = k(\frac{v}{r})v_{xx} + q(t)v_x(x,t) + v(l,t)\frac{\phi(x,t)}{E_2(t)}, & (x,t) \in Q_T, \\ v(x,0) = \varphi(x), & x \in (0,l), \\ v_x(0,t) = v(l,t)\frac{E_1(t)}{E_2(t)}, v_x(l,t) = v(l,t)\frac{g_2(t)}{E_2(t)}, & t \in (0,T), \end{cases} \tag{2.3}$$

from the solution of the system, we can obtain the approximate solution $p(t)$ and $g_1(t)$ from (2.1).

Let $\tau = \Delta t > 0$, and $h = \Delta x > 0$ be step-length on time and space coordinate $\{0 = t_0 < \cdots < t_m = T\}$ and $\{0 = x_0 < \cdots < x_n = l\}$, where $t_j = j\tau, x_i = ih$ denote a partitions of the $(0,T)$ and $(0,l)$ respectively.

So we can rewrite the system (2.3) as follows:

$$\begin{cases} \dfrac{v_i^{j+1}-v_i^j}{\tau} = k(\dfrac{v_i^j E_2(t_j)}{v_m^j})\dfrac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{h^2} & \\ \qquad +q(t_j)\dfrac{v_{i+1}^j - v_i^j}{h} + \dfrac{\phi(x_i,t_j)}{E_2(t_j)}v_m^j, & 2 \le i \le m-1, 1 \le j \le n-1, \\ v_i^1 = \varphi(x_i), & 1 \le i \le m, \\ \dfrac{v_2^j - v_1^j}{h} = \dfrac{E_1(t_j)}{E_2(t_j)}v_m^j, \ \dfrac{v_m^j - v_{m-1}^j}{h} = \dfrac{g(t_j)}{E_2(t_j)}v_m^j, & 1 \le j \le n, \end{cases} \tag{2.4}$$

where v_i^j denotes the approximation to $v(x_i,t_j)$.

3. Numerical experiments and discussions

In this section,we report and discuss some results with two examples of our numerical calculations using the numerical procedure described in the previous section.

Example 3.1. If we take the solution $u(x,t)$, the parameters $k(u), q(t), p(t)$ and l, T,as

$$u(x,t) = (1+x)te^t + 1, k(u) = u, q(t) = 1+t, p(t) = 0.5 - t, l = 1, T = 1,$$

then substituting in (1.1), it can be seen that the input data and additional conditions can be as follows:

$$u(x,0) = 1,$$

$$u(0,t) = 1 + te^t, u(1,t) = 1 + 2te^t,$$
$$u_x(0,t) = te^t, u_x(1,t) = te^t.$$

Example 3.2. If we take the solution $u(x,t)$, the parameters $k(u), q(t), p(t)$ and l, T,as

$$u(x,t) = t + xe^t, k(u) = u, q(t) = 1 + t^2, p(t) = 0.5 + t, l = 1, T = 1,$$

then substituting in (1.1), it can be seen that the input data and additional conditions can be as follows:

$$u(x,0) = x,$$

$$u(0,t) = t, u(1,t) = t + e^t,$$
$$u_x(0,t) = e^t, u_x(1,t) = e^t.$$

The numerical results are given in Figures 1-2.

In this paper,these measurements ensure that the inverse problem has a unique solution,but in most cases,the measurements have the noise. So in the next experiment,we have to illustrate the sensitivity of the solution of the inverse problem with respect to the noisy data.The artificial error were introduced into the additional specification data by defining function:

$$E_1^\gamma = E_1(t)(1+\gamma), E_2^\gamma = E_2(t)(1+\gamma).$$

where γ respects the noisy parameter,and in this paper,we let $\gamma = 0.005, 0.001$ respectively,and the numerical results are given in Figures 3-4.

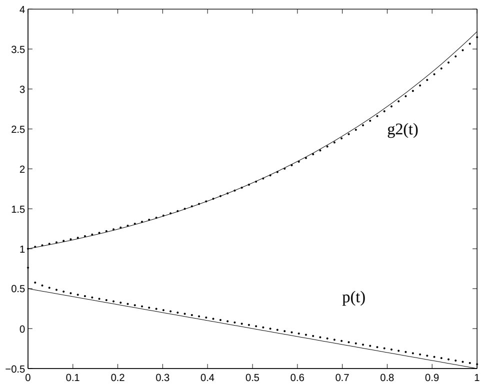

Figure 1: Example 1: (-)exact solution,(.)numerical solution

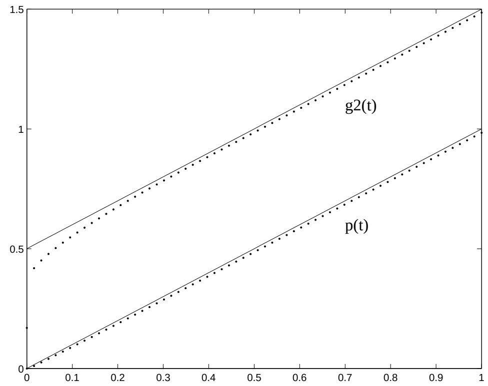

Figure 2: Example 2: (-)exact solution,(.)numerical solution

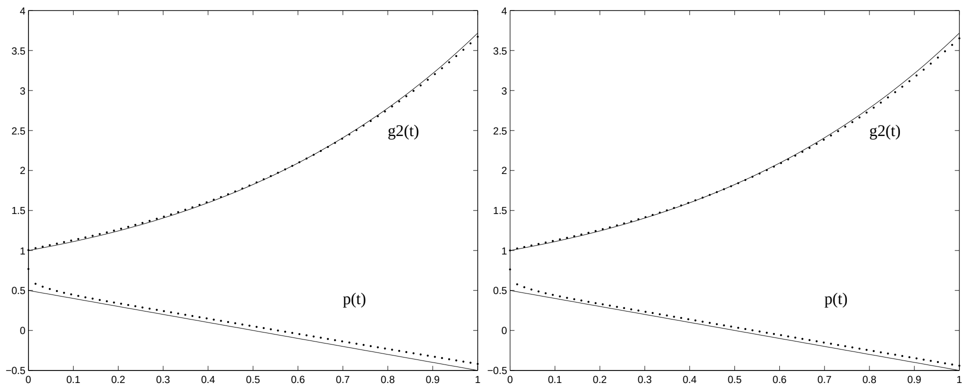

Figure 3: Example 1: $\gamma = 0.005, 0.001$ (-)exact solution,(.)numerical solution

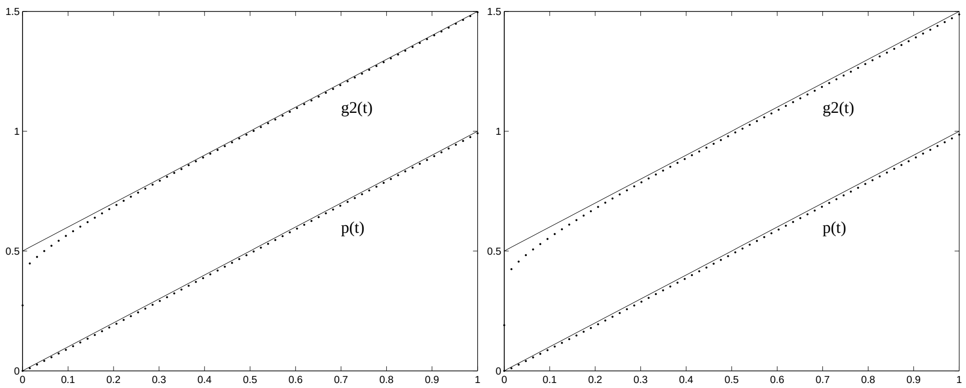

Figure 4: Example 2: $\gamma = 0.005, 0.001$ (-)exact solution,(.)numerical solution

In order to illustrate the accurate estimates of the unknown parameters,we define the value of norm of difference between numerical result and exact solution by

$$\|\Delta p\| = \frac{1}{n}\sqrt{\sum_{j=1}^{n}(p(t_j) - p_j)^2}, \quad \|\Delta g_1\| = \frac{1}{n}\sqrt{\sum_{j=1}^{n}(g_1(t_j) - g_{1j})^2},$$

here p_j is the numerical result to exact $p(t_j)$,and g_{1j} is the numerical result to exact $g_1(t_j)$, the data is in the table 1.

Table 1: The norm of the difference between numerical result and exact solution

γ	example 1		example 2	
	$p(t)$	$g_1(t)$	$p(t)$	$g_1(t)$
0	3.5×10^{-4}	2.6×10^{-2}	3.1×10^{-2}	1.6×10^{-2}
0.005	8.0×10^{-4}	2.6×10^{-2}	3.1×10^{-2}	1.6×10^{-2}
0.001	4.4×10^{-4}	2.6×10^{-2}	3.1×10^{-2}	1.6×10^{-2}

4. Conclusion

In this paper, we use some transformations of the problem and finite difference method to solve the inverse problem of the parabolic equation. As seen from the figures and the table that the solutions of the problem with respect to the noisy data is not much sensitive. so the applicability of the method is demonstrated on numerical examples successfully.

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