Extended Runge-Kutta method of order four for hybrid fuzzy differential equations

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ABSTRACT

In this paper we give a numerical algorithm for solving hybrid fuzzy differential equations based on Seikkala's derivative of fuzzy process. We discuss in detail a numerical method based on extended Runge-Kutta like formulae of order four. The algorithm is illustrated by an example.

Keywords: Numerical solution; Hybrid fuzzy differential equations; Extended Runge-Kutta like formulae.

1. Introduction

The topics of fuzzy differential equations has been rapidly growing in recent years. Fuzzy set theory is a tool that makes possible to describe vague and uncertain notions. The differential systems containing fuzzy valued functions and iteration with a discrete time controller are named hybrid fuzzy differential systems. These systems are devoted to modeling, design and validation of interactive systems of computer programs and continuous systems. That is, control systems that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics can be modeled by hybrid systems.

The concept of a fuzzy derivative was first introduced by Chang and Zadeh [2] and later Dubois and Prade [3] defined the fuzzy derivative by using Zadeh's extension principle. Fuzzy differential equations have been suggested as a way of modeling uncertain and incompletely specified systems and were studied by many researchers [5, 10, 16, 17]. It is difficult to obtain exact solution for fuzzy differential equations and hence several numerical methods where proposed [1, 7, 8, 11, 15]. Pederson and Sambandham [12, 14] have investigated the numerical solution of hybrid fuzzy differential equations by using Euler method, Runge-Kutta method and discussed the characterization theorem for solving hybrid fuzzy differential IVPs. Recently Ghazanfari and Shakerami [4] studied the numerical solution of fuzzy differential equations by extended Runge-Kutta-like formulae of order four. In this paper we apply extended Runge-Kutta-like formulae of order four to solve hybrid fuzzy differential equations and establish that this method gives better solution than the Euler method [12]. The structure of the paper is organized as follows:

In section 2, we give some results for fuzzy valued functions. In section 3, we define the hybrid fuzzy differential systems. In section 4, we discuss the extended Runge-Kutta-like formulae of order four to solve the hybrid fuzzy differential equations and a convergence theorem. Finally in section 5, we provide an example to illustrate our results.

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2. Preliminaries

Let $P_K(R^n)$ denote the family of all non-empty compact, convex subsets of R^n. If $\alpha, \beta \in R$ and $A, B \in P_K(R^n)$, then

$$\alpha(A + B) = \alpha A + \alpha B, \ \ \alpha(\beta A) = (\alpha\beta)A, \ \ 1A = A$$

and if $\alpha, \ \beta \geq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$.

Denote E^n the set of $u : R^n \to [0, 1]$ such that u satisfies (i)-(iv) mentioned below:

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semi continuous,

(iv) $[u]^0 = \text{cl}\{x \in R^n : u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, we denote $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$. Then from (i)-(iv) it follows that α-level set $[u]^\alpha \in P_K(R^n)$ for $0 < \alpha \leq 1$. An example of a $u \in E$ is given by

$$u(x) = \begin{cases} 4x - 3, & \text{if } x \in (0.75, 1], \\ -2x + 3, & \text{if } x \in (1, 1.5), \\ 0, & \text{if } x \notin (0.75, 1.5). \end{cases} \tag{2.1}$$

The $\alpha-$level sets are given by

$$[u]^\alpha = [0.75 + 0.25\alpha, 1.5 - 0.5\alpha]. \tag{2.2}$$

Let I be a real interval. A mapping $y : I \to E$ is called a fuzzy process and its $\alpha-$level set is denoted by $[y(t)]^\alpha = [\underline{y}^\alpha(t), \overline{y}^\alpha(t)]$, $t \in I$, $\alpha \in (0, 1]$.

Triangular fuzzy numbers are those fuzzy sets in E which are characterized by an ordered triple $(x^l, \ x^c, \ x^r) \in R^3$ with $x^l \leq x^c \leq x^r$ such that $[U]^0 = [x^l, x^r]$ and $[U]^1 = \{x^c\}$, then

$$[U]^\alpha = [x^c - (1 - \alpha)(x^c - x^l), \ x^c + (1 - \alpha)(x^r - x^c)] \tag{2.3}$$

for any $\alpha \in I$.

3. Hybrid fuzzy differential systems

Consider the hybrid fuzzy differential systems

$$\begin{cases} y'(t) = f(t, y(t), \lambda_k(y_k)), \\ y(t_k) = y_k, \end{cases} \tag{3.1}$$

where $0 \leq t_0 < t_1 < \cdots < t_k < \cdots, \ t_k \to \infty, \ f \in C[R_+ \times E \times E, E], \ \lambda_k \in C[E, E]$. Here we assume the existence and uniqueness of solutions of hybrid fuzzy system hold on each $[t_k, t_{k+1}]$.

To be specific the system would look like:

$$y'(t) = \begin{cases} y_0'(t) = f(t, y_0(t), \lambda_0(y_0)), \ y_0(t_0) = y_0, \ t_0 \leq t \leq t_1, \\ y_1'(t) = f(t, y_1(t), \lambda_1(y_1)), \ y_1(t_1) = y_1, \ t_1 \leq t \leq t_2, \\ \quad \vdots \\ y_k'(t) = f(t, y_k(t), \lambda_k(y_k)), \ y_k(t_k) = y_k, \ t_k \leq t \leq t_{k+1}, \\ \quad \vdots \end{cases}$$

By the solution of Equation (3.1) we mean the following function

$$y(t) = y(t, t_0, y_0) = \begin{cases} y_0(t_0) = y_0, & t_0 \leq t \leq t_1, \\ y_1(t_1) = y_1, & t_1 \leq t \leq t_2, \\ \vdots \\ y_k(t_k) = y_k, & t_k \leq t \leq t_{k+1}, \\ \vdots \end{cases} \tag{3.2}$$

We note that the solution of Equation (3.1) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $y_k \in E$ and $k = 0, 1, 2, \ldots$.

Using a representation of fuzzy numbers studied by Goetschel and Voxman and Wu and Ma , we may represent $y \in E$ by a pair of functions $(\underline{y}(r), \overline{y}(r)), 0 \leq r \leq 1$, such that (i) $\underline{y}(r)$ is bounded, left continuous, and nondecreasing, (ii) $\overline{y}(r)$ is bounded, left continuous, and nonincreasing, and (iii) $\underline{y}(r) \leq \overline{y}(r)$, $0 \leq r \leq 1$. For example, $u \in E$ given in (2.1) is represented by $(\underline{u}(r), \overline{u}(r)) = (0.75 + 0.25r, 1.5 - 0.5r)$, $0 \leq r \leq 1$, which is similar to $[u]^\alpha$ given by (2.2).

Therefore we may replace (3.1) by an equivalent system

$$\begin{cases} \underline{y}'(t) = \underline{f}(t, y, \lambda_k(y_k)) \equiv F_k(t, \underline{y}, \overline{y}), & \underline{y}(t_k) = \underline{y}_k, \\ \overline{y}'(t) = \overline{f}(t, y, \lambda_k(y_k)) \equiv G_k(t, \underline{y}, \overline{y}), & \overline{y}(t_k) = \overline{y}_k, \end{cases} \tag{3.3}$$

which possesses a unique solution $(\underline{y}, \overline{y})$ which is a fuzzy function. That is for each t, the pair $[\underline{y}(t; r), \overline{y}(t; r)]$ is a fuzzy number, where $\underline{y}(t; r), \overline{y}(t; r)$ are respectively the solution of the parametric form given by

$$\begin{cases} \underline{y}'(t; r) = F_k[t, \underline{y}(t; r), \overline{y}(t; r)], & \underline{y}(t_k; r) = \underline{y}_k(r), \\ \overline{y}'(t; r) = G_k[t, \underline{y}(t; r), \overline{y}(t; r)], & \overline{y}(t_k; r) = \overline{y}_k(r), \end{cases} \tag{3.4}$$

for $r \in [0, 1]$.

4. Fourth-order extended Runge-Kutta like formula

In this section for a hybrid fuzzy differential equation (3.1), we develop the Fourth order Runge-Kutta like formula via an application of the Fourth order Runge-Kutta like formula for fuzzy differential equation in [4] when f and λ_k in equation (3.1) can be obtained via the Zadeh extension principle from $f \in C[R_+ \times R \times R, R]$, $\lambda_k \in C[R, R]$. We assume that the existence and uniqueness of solutions of Equation (3.1) hold for each $[t_k, t_{k+1}]$.

For fixed r, we replace each interval by a set of $N_k + 1$ discrete equally spaced grid points(including the end-points) at which the exact solution $y(t; r) = (\underline{y}(t; r), \overline{y}(t; r))$ is approximated by some $(\underline{y}_k(t; r), \overline{y}_k(t; r))$. For the chosen grid points on $[t_k, t_{k+1}]$ at $t_{k,n} = t_k + nh_k$, $h_k = \frac{t_{k+1} - t_k}{N_k}, 0 \leq n \leq N_k$, let $(\underline{Y}_k(t; r), \overline{Y}_k(t; r)) \equiv (\underline{y}(t; r), \overline{y}(t; r))$. $(\underline{Y}_k(t; r), \overline{Y}_k(t; r))$ and $(\underline{y}_k(t; r), \overline{y}_k(t; r))$ may be denoted respectively by $(\underline{Y}_{k,n}(r), \overline{Y}_{k,n}(r))$ and $(\underline{y}_{k,n}(r), \overline{y}_{k,n}(r))$. We allow the $N_k's$ to vary over the $[t_k, t_{k+1}]'s$ so that the $h_k's$ may be comparable.

The extended Runge-Kutta like formula is a fourth-order approximation of $\underline{Y}_k'(t; r)$ and $\overline{Y}_k'(t; r)$. To develop the extended Runge-Kutta like formula for (3.1), we follow [4] and define

$$\underline{y}_{k,n+1}(r) - \underline{y}_{k,n}(r) = h\underline{k}_1^{(1)}(t_{k,n}, y_{k,n}(r)) + \frac{1}{6}h^2\underline{k}_1^{(2)}(t_{k,n}, y_{k,n}(r))$$
$$+ \frac{1}{3}h^2\underline{k}_3^{(2)}(t_{k,n}, y_{k,n}(r)),$$
$$\overline{y}_{k,n+1}(r) - \overline{y}_{k,n}(r) = h\overline{k}_1^{(1)}(t_{k,n}, y_{k,n}(r)) + \frac{1}{6}h^2\overline{k}_1^{(2)}(t_{k,n}, y_{k,n}(r))$$

$$+\frac{1}{3}h^2\overline{k}_3^{(2)}(t_{k,n},y_{k,n}(r)).$$

where

$$\underline{k}_1^{(1)}(t_{k,n},y_{k,n}(r)) = min\{f(t_{k,n},u,\lambda_k(u_k))|u \in [\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)], u_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\},$$

$$\overline{k}_1^{(1)}(t_{k,n},y_{k,n}(r)) = max\{f(t_{k,n},u,\lambda_k(u_k))|u \in [\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)], u_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\},$$

$$\underline{k}_1^{(2)}(t_{k,n},y_{k,n}(r)) = min\{f'(t_{k,n},u,\lambda_k(u_k))|u \in [\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)], u_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\},$$

$$\overline{k}_1^{(2)}(t_{k,n},y_{k,n}(r)) = max\{f'(t_{k,n},u,\lambda_k(u_k))|u \in [\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)], u_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\},$$

$$\underline{k}_3^{(2)}(t_{k,n},y_{k,n}(r)) = min\{f'(t_{k,n},w,\lambda_k(w_k))|w \in [\underline{z}_2(t_{k,n},y_{k,n}(r)),\overline{z}_2(t_{k,n},y_{k,n}(r))],$$
$$w_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\},$$

$$\overline{k}_3^{(2)}(t_{k,n},y_{k,n}(r)) = max\{f'(t_{k,n},w,\lambda_k(w_k))|w \in [\underline{z}_2(t_{k,n},y_{k,n}(r)),\overline{z}_2(t_{k,n},y_{k,n}(r))],$$
$$w_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\},$$

where

$$\underline{z}_2(t_{k,n},y_{k,n}(r)) = \underline{y}_{k,n}(r) + \frac{1}{2}h\underline{k}_2^{(1)}(t_{k,n},y_{k,n}(r)),$$

$$\overline{z}_2(t_{k,n},y_{k,n}(r)) = \overline{y}_{k,n}(r) + \frac{1}{2}h\overline{k}_2^{(1)}(t_{k,n},y_{k,n}(r)).$$

So that

$$\underline{k}_2^{(1)}(t_{k,n},y_{k,n}(r)) = min\{f(t_{k,n},v,\lambda_k(v_k))|v \in [\underline{z}_1(t_{k,n},y_{k,n}(r)),\overline{z}_2(t_{k,n},y_{k,n}(r))],$$
$$v_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\},$$

$$\overline{k}_2^{(1)}(t_{k,n},y_{k,n}(r)) = max\{f(t_{k,n},v,\lambda_k(v_k))|v \in [\underline{z}_1(t_{k,n},y_{k,n}(r)),\overline{z}_2(t_{k,n},y_{k,n}(r))],$$
$$v_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)]\},$$

where

$$\underline{z}_1(t_{k,n},y_{k,n}(r)) = \underline{y}_{k,n}(r) + \frac{1}{4}h\underline{k}_1^{(1)}(t_{k,n},y_{k,n}(r)),$$

$$\overline{z}_1(t_{k,n},y_{k,n}(r)) = \overline{y}_{k,n}(r) + \frac{1}{4}h\overline{k}_1^{(1)}(t_{k,n},y_{k,n}(r)).$$

Next we define

$$S_k[t_{k,n},\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)] = h\underline{k}_1^{(1)}(t_{k,n},y_{k,n}(r)) + \frac{1}{6}h^2\underline{k}_1^{(2)}(t_{k,n},y_{k,n}(r))$$
$$+\frac{1}{3}h^2\underline{k}_3^{(2)}(t_{k,n},y_{k,n}(r)),$$

$$T_k[t_{k,n},\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)] = h\overline{k}_1^{(1)}(t_{k,n},y_{k,n}(r)) + \frac{1}{6}h^2\overline{k}_1^{(2)}(t_{k,n},y_{k,n}(r))$$
$$+\frac{1}{3}h^2\overline{k}_3^{(2)}(t_{k,n},y_{k,n}(r)).$$

The exact solution at $t_{k,n+1}$ is given by

$$\begin{cases} \underline{Y}_{k,n+1}(r) \approx \underline{Y}_{k,n}(r) + S_k[t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)], \\ \overline{Y}_{k,n+1}(r) \approx \overline{Y}_{k,n}(r) + T_k[t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)]. \end{cases}$$

The approximate solution is given by

$$\begin{cases} \underline{y}_{k,n+1}(r) = \underline{y}_{k,n}(r) + S_k[t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)], \\ \overline{y}_{k,n+1}(r) = \overline{y}_{k,n}(r) + T_k[t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)]. \end{cases} \tag{4.1}$$

By an application of methods suggested in [4](Theorem 4.3) and [12](Lemma 3.1 and Theorem 3.2), the following result can be proved (convergence is pointwise in r for a fixed k).

Theorem 4.1 ([13])**.** *Consider the systems* (3.3) *and* (4.1)*. For a fixed* $k \in Z^+$ *and* $r \in [0, 1]$,

$$\lim_{h_0,\ldots,h_k \to 0} \underline{y}_{k,N_k}(r) = \underline{Y}(t_{k+1}; r), \tag{4.2}$$

$$\lim_{h_0,\ldots,h_k \to 0} \overline{y}_{k,N_k}(r) = \overline{Y}(t_{k+1}; r). \tag{4.3}$$

5. Numerical example

Before illustrating the numerical solution of a hybrid fuzzy IVP, first we recall the fuzzy IVP Example 3 of [11];

$$y'(t) = y(t), \ \ y(0,r) = [0.75 + 0.25r, 1.125 - 0.125r], \ \ 0 \le r \le 1. \tag{5.1}$$

By the Euler method with N=10 in [5],

$$y(1.0; r) = \left[(0.75 + 0.25r)\left(1 + \frac{1}{10}\right)^{10}, (1.125 - 0.125r)\left(1 + \frac{1}{10}\right)^{10}\right], \ \ 0 \le r \le 1, \tag{5.2}$$

where $y(t; r)$ denote an approximate solution of (5.1). Since the exact solution of (5.1) is

$$Y(t; r) = \left[(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t\right], \ \ 0 \le r \le 1, \tag{5.3}$$

we see that at $t = 1.0$

$$Y(1; r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e], \ \ 0 \le r \le 1, \tag{5.4}$$

Now by ERK4 with $N = 10$ in [4], (5.1) gives

$$y(1.0; r) = \left[(0.75 + 0.25r)(c_{0,1})^{10}, (1.125 - 0.125r)(c_{0,1})^{10}\right], \ \ 0 \le r \le 1, \tag{5.5}$$

where

$$c_{0,1} = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}$$

now we proceed with (5.5) to ERK4 for hybrid fuzzy IVP.

Example 5.1. Next consider the following hybrid fuzzy IVP from [12];

$$
\begin{cases}
y'(t) = y(t) + m(t)\lambda_k(y(t_k)), \ t \in [t_k, t_{k+1}], \ t_k = k, \ k = 0, 1, 2, \ldots, \\
y(0, r) = [0.75 + 0.25r, 1.125 - 0.125r], \ 0 \le r \le 1,
\end{cases}
\tag{5.6}
$$

where

$$
m(t) = \begin{cases}
2(t(mod\,1)), & \text{if } t(mod\,1) \le 0.5 \\
2(1 - t(mod\,1)), & \text{if } t(mod\,1) > 0.5,
\end{cases}
$$

$$
\lambda_k(\mu) = \begin{cases}
\hat{0}, & \text{if } k = 0 \\
\mu, & k \in \{1, 2, 3, \ldots\}.
\end{cases}
$$

In (5.6), $y(t) + m(t)\lambda_k(y(t_k))$ is a continuous function of t, x and $\lambda_k(y(t_k))$. Therefore by [13], for each $k = 0, 1, 2, \ldots$ the fuzzy IVP

$$
\begin{cases}
y'(t) = y(t) + m(t)\lambda_k(y(t_k)), \ t \in [t_k, t_{k+1}], \ t_k = k, \\
y(t_k) = y_{t_k},
\end{cases}
\tag{5.7}
$$

has a unique solution on $[t_k, t_{k+1}]$. To solve the hybrid fuzzy IVP (5.6) numerically we apply the ERK4 method for hybrid fuzzy differential equations from Section 4 with $N = 10$ to obtain $y_{1,10}(r)$ approximating $Y(2.0; r)$. Let $f : [0, \infty) \times R \times R \to R$ be given by

$$
f(t, y, \lambda_k(y(t_k))) = y(t) + m(t)\lambda_k(y(t_k)), \ t_k = k, \ k = 0, 1, 2, \ldots,
\tag{5.8}
$$

where $\lambda_k : R \to R$ is given by

$$
\lambda_k(y) = \begin{cases}
0, & if \ k = 0 \\
x, & if \ k \in \{1, 2, \ldots\}.
\end{cases}
$$

Using the hybrid fuzzy ERK4, we have

$$
\begin{aligned}
\underline{y}_{k,n+1}(r) = {}& \underline{y}_{k,n}(r) + h\underline{k}_1^{(1)}(t_{k,n}, y_{k,n}(r)) + \frac{1}{6}h^2\underline{k}_1^{(2)}(t_{k,n}, y_{k,n}(r)) \\
& + \frac{1}{3}h^2\underline{k}_3^{(2)}(t_{k,n}, y_{k,n}(r)),
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
\overline{y}_{k,n+1}(r) = {}& \overline{y}_{k,n}(r) + h\overline{k}_1^{(1)}(t_{k,n}, y_{k,n}(r)) + \frac{1}{6}h^2\overline{k}_1^{(2)}(t_{k,n}, y_{k,n}(r)) \\
& + \frac{1}{3}h^2\overline{k}_3^{(2)}(t_{k,n}, y_{k,n}(r)).
\end{aligned}
\tag{5.10}
$$

Here

$$
\begin{aligned}
\underline{k}_1^{(1)}(t_{k,n}, y_{k,n}(r)) &= min\{f(t_{k,n}, u, \lambda_k(u_k)) | u \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)], u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)]\}, \\
&= \underline{y}_{k,n} + m(t_n)\underline{y}_k, \\
\overline{k}_1^{(1)}(t_{k,n}, y_{k,n}(r)) &= max\{f(t_{k,n}, u, \lambda_k(u_k)) | u \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)], u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)]\}, \\
&= \overline{y}_{k,n} + m(t_n)\overline{y}_k, \\
\underline{k}_1^{(2)}(t_{k,n}, y_{k,n}(r)) &= min\{f'(t_{k,n}, u, \lambda_k(u_k)) | u \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)], u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)]\}, \\
&= \underline{y}_{k,n} + (m(t_n) + m'(t_n))\underline{y}_k, \\
\overline{k}_1^{(2)}(t_{k,n}, y_{k,n}(r)) &= max\{f'(t_{k,n}, u, \lambda_k(u_k)) | u \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)], u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)]\},
\end{aligned}
$$

$$= \overline{y}_{k,n} + (m(t_n) + m'(t_n))\overline{y}_k,$$

$$\underline{k}_3^{(2)}(t_{k,n}, y_{k,n}(r)) = min\{f'(t_{k,n}, w, \lambda_k(w_k))| w \in [\underline{z}_2(t_{k,n}, y_{k,n}(r)), \overline{z}_2(t_{k,n}, y_{k,n}(r))],$$
$$w_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)]\},$$

$$\overline{k}_3^{(2)}(t_{k,n}, y_{k,n}(r)) = max\{f'(t_{k,n}, w, \lambda_k(w_k))| w \in [\underline{z}_2(t_{k,n}, y_{k,n}(r)), \overline{z}_2(t_{k,n}, y_{k,n}(r))],$$
$$w_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)]\},$$

where

$$\underline{z}_2(t_{k,n}, y_{k,n}(r)) = \underline{y}_{k,n}(r) + \frac{1}{2}h\underline{k}_2^{(1)}(t_{k,n}, y_{k,n}(r)),$$

$$\overline{z}_2(t_{k,n}, y_{k,n}(r)) = \overline{y}_{k,n}(r) + \frac{1}{2}h\overline{k}_2^{(1)}(t_{k,n}, y_{k,n}(r)).$$

So that

$$\underline{k}_2^{(1)}(t_{k,n}, y_{k,n}(r)) = min\{f(t_{k,n}, v, \lambda_k(v_k))| v \in [\underline{z}_1(t_{k,n}, y_{k,n}(r)), \overline{z}_2(t_{k,n}, y_{k,n}(r))],$$
$$v_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)]\},$$

$$\overline{k}_2^{(1)}(t_{k,n}, y_{k,n}(r)) = max\{f(t_{k,n}, v, \lambda_k(v_k))| v \in [\underline{z}_1(t_{k,n}, y_{k,n}(r)), \overline{z}_2(t_{k,n}, y_{k,n}(r))],$$
$$v_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)]\},$$

where

$$\underline{z}_1(t_{k,n}, y_{k,n}(r)) = \underline{y}_{k,n}(r) + \frac{1}{4}h\underline{k}_1^{(1)}(t_{k,n}, y_{k,n}(r)),$$

$$= \underline{y}_{k,n}(r) + \frac{1}{4}h\left(\underline{y}_{k,n}(r) + m(t_n)\underline{y}_k\right),$$

$$= \underline{y}_{k,n}(r)\left(1 + \frac{h}{4}\right) + \frac{h}{4}m(t_n)\underline{y}_k,$$

$$\overline{z}_1(t_{k,n}, y_{k,n}(r)) = \overline{y}_{k,n}(r) + \frac{1}{4}h\overline{k}_1^{(1)}(t_{k,n}, y_{k,n}(r)),$$

$$= \overline{y}_{k,n}(r) + \frac{1}{4}h\left(\overline{y}_{k,n}(r) + m(t_n)\overline{y}_k\right),$$

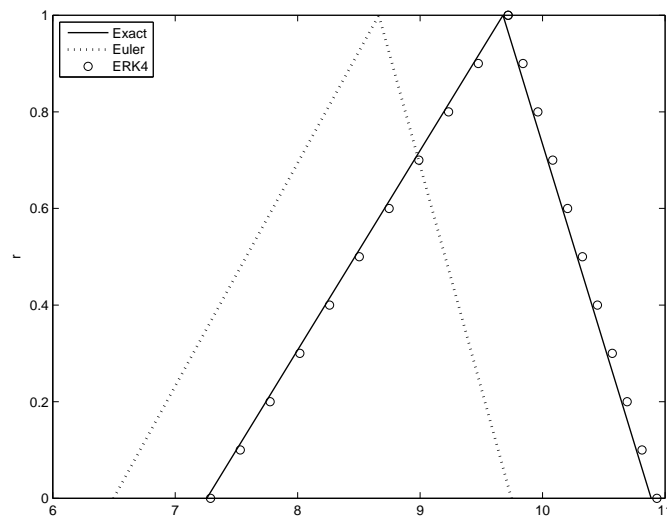$$= \overline{y}_{k,n}(r)\left(1 + \frac{h}{4}\right) + \frac{h}{4}m(t_n)\overline{y}_k.$$

Using these with $N = 10$ and $k = 1$ in (5.9) and (5.10), we get the approximate solution for hybrid fuzzy IVP (5.6) as

$$\underline{y}_{1,10}(r) = c_{0,1}{}^{10}c_{1,10}(0.75 + 0.25r), \tag{5.11}$$

$$\overline{y}_{1,10}(r) = c_{0,1}{}^{10}c_{1,10}(1.125 - 0.125r), \tag{5.12}$$

where

$$c_{1,10} = c_{0,1}c_{1,9} + \frac{1}{5}\left(h - \frac{9}{2}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right),$$

$$c_{1,9} = c_{0,1}c_{1,8} + \frac{2}{5}\left(h - 2h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right),$$

$$c_{1,8} = c_{0,1}c_{1,7} + \frac{3}{5}\left(h - \frac{7}{6}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right),$$

Figure 1: At $t = 2$ with $h = 0.1$.

$$c_{1,7} = c_{0,1}c_{1,6} + \frac{4}{5}\left(h - \frac{3}{4}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right),$$

$$c_{1,6} = c_{0,1}c_{1,5} + h + \frac{3}{2}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4,$$

$$c_{1,5} = c_{0,1}c_{1,4} + \frac{4}{5}\left(h + \frac{3}{4}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right),$$

$$c_{1,4} = c_{0,1}c_{1,3} + \frac{3}{5}\left(h + \frac{13}{6}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right),$$

$$c_{1,3} = c_{0,1}c_{1,2} + \frac{2}{5}\left(h + 3h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right),$$

$$c_{1,2} = c_{0,1}c_{1,1} + \frac{1}{5}\left(h + \frac{11}{2}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right),$$

$$c_{1,1} = 1 + h + \frac{3}{2}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4.$$

For $t \in [1, 1.5]$, the exact solution of Equation (5.6) satisfies

$$Y(t; r) = Y(1; r)\left(3e^{t-1} - 2t\right), \ 0 \le r \le 1.$$

Then $Y(1.5; 1)$ is approximately 5.290 and $y(1.5; 1)$ is approximately 5.294
For $t \in [1.5, 2]$, the exact solution of Equation (5.6) satisfies

$$Y(t; r) = Y(1; r)\left(2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)\right), \ 0 \le r \le 1.$$

Then $Y(2; 1)$ is approximately 9.677 and $y(2; 1)$ is approximately 9.678

6. Conclusion

In this work we applied extended Runge-Kutta-like formulae of order 4 for finding the numerical solution of hybrid fuzzy differential equations. Comparison of solutions of Example 5.1 shows that our proposed method

Table 1: Comparison of solutions to Example 1 when $r = 1$ the Euler method, the extended Runge-Kutta of order four(ERK4), the exact solution and difference.

t	Euler	ERK4	Exact	Exact-Euler	Exact-ERK4
1.0	2.5937	2.7183	2.7183	0.1246	0
1.5	4.7539	5.2627	5.2902	0.5363	0.0275
2.0	8.6631	9.6785	9.6770	1.0139	-0.0415

gives better solution than Euler method.

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