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The neighbor coloring set in graphs

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ABSTRACT

Given a graph $G = (V, E)$, a set $S \subseteq V$ is a neighborhood set of G, if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by v and all vertices adjacent to v. A neighborhood set $S \subseteq V$ is said to be a neighbor coloring set of G if each color class $V_i, 1 \leq i \leq k$ contains at least one vertex, which belongs to S. The minimum cardinality taken over all neighbor coloring set of a graph G is called neighbor chromatic number and is denoted by $\chi_\eta(G)$. In this paper, we study the properties of $\chi_\eta(G)$ and also its relationship with other graph theoretic parameters are explored.

Keywords: Graph, neighborhood set, neighborhood number, neighbor-coloring, neighbor chromatic number

1. INTRODUCTION

All the graph considered here are finite, undirected and connected with no loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices X. $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively. Let $deg(v)$ be the degree of vertex v and as usual $\delta(G)$, the minimum degree and $\triangle(G)$, the maximum degree of a graph G. For a real number $x > 0$, let $\lceil x \rceil$ be the least integer not less than x and $\lfloor x \rfloor$ be the greatest integer not greater than x. For graph-theoretical terminology and notation refer to [2][5].

A set $S \subseteq V$ is a neighborhood set of G, if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by v and all vertices adjacent to v. The neighborhood number $\eta(G)$ of G is the minimum cardinality of a neighborhood set of G, [12]. A set $S \subseteq V$ is a minimal neighborhood set, if $S - v$ for all $v \in S$, is not a neighborhood set of G, [3]. The nomatic number of G, $N(G)$ is the largest number of sets in a partition of V into disjoint minimal neighborhood sets of a graph G, [7]. Further, a neighborhood set $S \subseteq V$ is called an independent neighborhood set, if $\langle S \rangle$ is an independent and neighborhood set of G, [10] and maximal neighborhood set, if $V - S$ does not contain a neighborhood set of G, [13][14]. The minimum cardinality taken over all independent and maximal neighborhood set in G is called independent and maximal neighborhood number of a graph G and is denoted by $\eta_i(G)$ and $\eta_m(G)$, respectively. A neighborhood set S with minimum cardinality is called η - set of a graph G. Similarly, the other sets can be expected. For more details on neighborhood sets and its related parameters, refer to [1][4].

Graph coloring theory is one of the major areas within graph theory which have been extensively studied. Graph coloring deals with the fundamental problem of partitioning a set objects into classes according to certain rules. Time tabling, sequencing and scheduling problems in their many terms are basically of this nature. A coloring of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The set of

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all vertices with any one color is independent and is called a color class. A k-coloring of a graph G uses k-colors: its there by partitions V into k-color classes. The chromatic number $\chi(G)$ is defined as the minimum k for which G has an k-coloring. For complete review on the topic on theory of coloring and its related parameters, [8].

A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G, [6][15]. A dominator coloring of a graph G is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes required in a dominator coloring of a graph G, [11].

Analogously, we now define the neighbor chromatic number as follows: A neighborhood set $S \subseteq V$ is said to be a neighbor coloring set of G if each color class $V_i, 1 \leq i \leq k$ contains at least one vertex, which belongs to S. The minimum cardinality taken over all neighbor coloring set of a graph G is called neighbor chromatic and is denoted by $\chi_\eta(G)$.

The following known results are used in the sequel.

Observation 1.1. For every bipartite graph $\beta(G) \geq \lceil p/2 \rceil$ and therefore for every tree $\beta(T) \geq \lceil p/2 \rceil > (p-1)/2$. Similarly $\alpha'(T) > (p-1)/2$.

Theorem 1.1 (Harary [5])**.** *For any graph G,*

(i) $p^2/(p^2 - 2q) \leq \chi(G)$

(ii) $p/(p - \delta(G)) \leq \chi(G)$.

2. NEIGHBOR COLORING FOR CLASSES OF GRAPH

Proposition 2.1. For any complete graph K_p with $p \geq 2$ vertices,

$$\chi_\eta(K_p) = p. \tag{2.1}$$

Proof. Obviously, we have $\chi(K_p) = p \leq \chi_\eta(K_p) \leq p$, since every neighbor coloring set is a vertex coloring set of a graph G. Hence (2.1) follows. □

Proposition 2.2. For any path P_p with $p \geq 1$ vertices,

$$\chi_\eta(P_2) = 2\,;\chi_\eta(P_p) = \lceil p/2 \rceil \; if p > 2. \tag{2.2}$$

Proof. For proving the above result, we consider the following cases. **Case 1.** If $p = 2$. By the definition of $\chi(G)$, we have $\chi(P_2) = 2$ and $\chi_\eta(P_2) \leq p$ with $p = 2$. Hence $\chi(P_2) = 2 \leq \chi_\eta(P_2) \leq 2$. There fore $\chi(P_2) = 2$.

Case 2. If $p > 2$ and even, then we have two color classes say V_1 and V_2 such that $V_1 = V_2 = p/2$. So, the two pendent vertices belong to different classes, one in V_1 and the other in V_2. By taking all vertices in V_1 of degree 2 and the one vertex of V_2 adjacent with the pendent vertex of V_1, we get a minimum neighbor coloring set of P_p with cardinality $p/2$.

Case 3. If $p > 2$ and odd, then the two pendent vertices belong in a same color class say V_1 and hence $V_1 = V_2 + 1 = (p+1)/2$. By taking all V_2's vertices and one vertex from V_1, we get a minimum neighbor coloring set of P_p with cardinality $p/2$.

Hence from all the above cases, (2.2) follows. □

Proposition 2.3. For any cycle C_p with $p \geq 3$ vertices,

$$\chi_\eta(C_p) = (p/2) + 1 if p = 2r; r \geq 2 \tag{2.3}$$

$$\chi_\eta(C_p) = \lceil p/2 \rceil if p = 2r + 1; r \geq 2. \tag{2.4}$$

Proof. For proving the above result, we consider the following cases. **Case 1**. If $p = 2r$ with $r > 2$. In this case, C_p is a bipartite graph. By taking all r-vertices of one color class and any one from the other color class, we get a minimum neighbor coloring set of C_p. Hence $\chi_\eta(C_p) = r + 1$ and (2.3) follows.
Case 2. If $p = 2r + 1$ with $r > 2$. In this case,$\chi(C_p) = 3$ and hence C_p has 3-color classes say V_1, V_2 and V_3. Let V_3 contains exactly one vertex. By removing this vertex in V_3 from C_p, we get a path of $2r$-vertices whose neighbor chromatic number is r, since every neighbor coloring set is a vertex coloring set of a graph G. Adding the removed vertex to those 'r' vertices, we get a minimum neighbor coloring set of C_p. Since $\lceil p/2 \rceil \geq p/2$, hence $\chi_\eta(C_p) = r + 1$ and (2.4) follows. □

Proposition 2.4. For any wheel graph W_p with $p \geq 5$ vertices,

$$\chi_\eta(W_p) = (p + 3)/2 if p = 2r + 1; r \geq 2 \tag{2.5}$$

$$\chi_\eta(W_p) = (p/2) + 1 if p = 2r; r \geq 3. \tag{2.6}$$

Proof. By proposition 2.3 and $\chi_\eta(K_1) = 1$ with the fact that $W_p = K_1 + C_{p-1}$, (2.5) and (2.6) follows. □

Proposition 2.5. For any complete multipartite graph $G = K_{r_1,r_2,...,r_k}$ with $r_1 \leq r_2 \leq ... \leq r_k$ vertices,

$$\chi_\eta(K_{r,s}) = r_1 + k - 1 \tag{2.7}$$

Proof. Clearly, $K_{r,s}$ is a complete bipartite graph with partite set V_1 and V_2. Hence $\chi(K_{r,s}) = 2$. By selecting all r-vertices from the vertex color class of minimum cardinality say V_1, those r-vertices cover all edges of the graph $K_{r,s}$. Adding any vertex of the other color class, we get a minimum neighbor coloring set of $K_{r,s}$. Thus (2.7) follows. □

3. BOUNDS ON NEIGHBOR CHROMATIC NUMBER

Theorem 3.1. *For any graph G with no isolated vertices, $\chi_\eta(G) = p$ if and only if G is isomorphic with K_p.*

Proof. Let $\chi_\eta(G) = p$. To show that G is isomorphic with K_p. By contrary, suppose that G is not isomorphic to K_p. Then there exist a vertex $v \in V$ such that $deg(v) < p - 1$. So, the vertex v is adjacent with at most $(p - 2)$-vertices. There fore at least one vertex, say u is not adjacent to v and whence v and u belongs to the same color class. By taking the remaining $(p - 2)$ - vertices together with u, we form a vertex cover coloring set of G with $(p - 1)$ - vertices. This implies that $\chi_\eta(G) < p - 1$. This contradict the assumption that $\chi_\eta(G) = p$. There fore $\chi_\eta(G) = p - 1$ implies G is isomorphic with K_p.

Conversely, suppose that G is isomorphic with K_p. To show that $\chi_\eta(G) = p$. Obviously, G is isomorphic with K_p implies $\chi(G) = \chi(K_p) = p$. Also, this implies $\chi(G) = p \leq \chi(K_p) \leq p$. Hence $\chi_\eta(G) = p$. There fore G is isomorphic with K_p implies $\chi_\eta(G) = p$. □

The following two theorems are easy to see, hence we omit their proofs.

Theorem 3.2. *Let G be graph with no isolated vertices. If S is a minimum neighbor coloring set of G. Then the following statements are equivalent:*

(i) *$V - S$ is not containing a neighbor coloring set of G.*

(ii) *$\langle V - S \rangle$ is an independent set of G.*

(iii) *$\chi_\eta(G) = \eta_m(G) = N(G) = p$ if and only if G is isomorphic with K_p.*

Theorem 3.3. *Let G be a nontrivial graph. Then,*

$$\gamma(G) \leq \eta(G) \leq \chi_\eta(G) \tag{3.1}$$

$$\chi(G) \leq \chi_d(G) \leq \chi_\eta(G). \tag{3.2}$$

Theorem 3.4. *For any graph G, $\chi(G) = \chi_\eta(G)$ if and only if the graph G is isomorphic with K_p or $K_{1,p-1}$ or C_5 or P_4 or T^* (A double star T^* is a tree T with exactly two adjacent vertices of degrees greater than one).*

Proof. Let S be a minimum neighbor coloring set of a graph G. If $\chi(G) = \chi_\eta(G)$, then one of the following cases are hold:
Case 1. G is a complete graph because for a complete graph $S = V$ which contains exactly one vertex from each color class, by (2.1). So, G is isomorphic with K_p.
Case 2. G is a star because for a star, S contains 2 vertices from only 2 existing color classes, by (2.7). So, G is isomorphic with $K_{1,p-1}$.
Case 3. G is a cycle C_5, because C_5 has 3-color classes with $\chi(C_5)$ = 3 and $\eta(C_5)$ = 3, by (2.4). So,G is isomorphic with C_5.
Case 4. G is a path P_2 or P_4 because for a path S contains 2 vertices from only 2 existing color classes, by (2.2). So, G is isomorphic with P_2 or P_4.
Case 5. G is a double star T^*, because T^* is a bipartite graph, then S contains 2 vertices from only 2 existing color classes, by Theorem 3. 1. So, G is isomorphic with T^*. From the above cases proves the $\chi(G) = \chi_\eta(G)$.

Conversely, suppose $\chi(G) = \chi_\eta(G)$ holds. On the contrary, suppose the graph G is not isomorphic with K_p or $K_{1,p-1}$ or C_5 or P_4 or T^*, then there exist at least three vertices u, v and w such that u and v are adjacent and w is adjacent to any one of u and v, which form a path of length two, this implies that $(V - w)$ is a neighbor coloring set of G, which is a contradiction. □

Theorem 3.5. *For any graph G,*

$$\frac{p^2}{(p^2 - 2q)} \leq \chi_\eta(G) \leq \lceil \frac{q}{2} \rceil + 1. \tag{3.3}$$

Proof. The lower bound follows from (i) of Theorem 1.1 and (3.2), and all q-edges may be covered by a half of vertices, because any 2 consecutive edges are incident in exactly one vertex and hence it can cover both at a time. When all edges are covered, one additional vertex is sufficient to cover all colors in case it remains. Thus upper bound follows. □

Theorem 3.6. *For any graph G,*

$$\delta(G) + 1 \leq \chi_\eta(G) \tag{3.4}$$

$$p - \lceil q/\delta(G) \rceil \leq \chi_\eta(G). \tag{3.5}$$

Proof. Let v be a vertex of minimum degree in G with v has exactly $\delta(G)$-neighbors. Let $N(v) = \{u_1, u_2, ..., u_{\delta(G)-1}, u_{\delta(G)}\}$ be an open neighbors of G. Let H be an induced sub-graph of G generated by $N(v)$.
We claim that $\chi(H) > \delta(G)$. Then we consider the following cases:
Case 1. Any pair of vertices in $N(v)$ are mutually non-adjacent. Hence, any neighbor coloring set of H must have at least $\delta(G)$-vertices to cover all edge of H and at least one more vertex to cover all colors. Thus, $\chi_\eta(H) > \delta(G) + 1$.

Case 2. Some vertices from $N(v)$ are adjacent. For those adjacent vertices say u_i, u_j, belong to different color classes which each should have a representative in any vertex-coloring of H. Thus, $\chi_\eta(H) > \delta(G) + 1$. Hence, from the above cases, $\chi_\eta(H) > \delta(G) + 1$, since H is a sub graph of G, $\delta(G) + 1 \leq \chi_\eta(H) \leq \chi_\eta(G)$. Thus (3.4) follow.

Let S be the minimum neighbor coloring set of G. If every vertex in $V - S$ is adjacent with at least $\delta(G)$-vertices in S, then by Observation 1.1 and Theorem 3.2, this implies that $q \geq \mid V - S \mid \delta(G)$. Thus (3.5) follows from the fact $\lceil q/\delta(G) \rceil \geq q/\delta(G)$. $\square$

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