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Analytical solution of one-dimensional advection-diffusion equation with Interval Parameters

T. Nirmala [a], D. Datta [b], H. S. Kushwaha [b], K. Ganesan(✉)[a]

[a]Department of Mathematics, Faculty of Engineering and Technology, SRM University, Kattankulathur,Chennai - 603203, India.
[b]Health Safety and Environment Group, Bhabha Atomic Research Centre, Trombay, Mumbai-400085, India.

ABSTRACT

The purpose of this paper is to incorporate imprecise information into solute transport process modeling using interval arithmetic. In this paper, we propose a new method for an analytical solution of one-dimensional advection-diffusion equation involving interval parameters of longitudinal dispersion in porous media without converting them to classical models.

Keywords: Advection; diffusion; dispersion; aquifer; groundwater; analytical solution; uncertainty; interval number.

1. Introduction

A pollutant or leachate may enter the groundwater zone directly to a landfill site from an industrial site such as Nuclear power plants, Chemical industries, Construction industries etc by dumping of toxic waste, agriculture and waste disposal. Radioactive waste is disposed under the ground and gets released into groundwater because of advection, dispersion and diffusion.

In recent year's groundwater flow and contaminant transport model play an important role in many projects dealing with groundwater exploitation, protection and remediation. A groundwater flow model is essential for the development of a contaminant transport model. Groundwater flow models are used to calculate the rate and direction of movement of groundwater through aquifers and confining units in the subsurface. Scientists facing lot of problems in assessing extend of groundwater pollution and effectiveness remedial action.

In developing a groundwater flow or solute transport model the analyst begins by preparing a conceptual model consisting of a list of physical and chemical processes of the system under study (solute transport by advection, dispersion and diffusion).The next step is to translate the conceptual model into mathematical model consisting of one or more partial differential equations and set of initial and boundary conditions. Advection-diffusion equation describes the solute transport due to combined effect of diffusion and convection in a medium. Environmentalists, hydrologists, civil engineers and mathematical modelers concentrate more on the advection-diffusion equations to ensure the safe hydro-environment. The analytical and numerical solution of the advection-diffusion equations are useful to assess the time and position at which the concentration level of the pollutants will start affecting the health of the habitats in the polluted water eco-system. Also such solutions help estimate and examine the rehabilitation process and management of a polluted water body after elimination of the pollution. The analytical

✉ Corresponding author.
Email addresses: nirmalaselvan_20@rediffmail.com (T. Nirmala), ddatta@barc.gov.in, dbbrt datta@yahoo.com (D. Datta), kushwaha@barc.gov.in (H. S. Kushwaha), ganesan.k@ktr.srmuniv.ac.in, gansan_k@yahoo.com (K. Ganesan)

solution of dispersion problems in ideal conditions, is obtained by reducing the advection-diffusion equation into a diffusion equation by eliminating the convective term(s).

The analytical solutions of advection-dispersion equation have been discussed by many researchers like Bastian and Lapidus [6], Ogata and Banks [22], Banks and Ali [5], Ogata [**?**], Lai and Jurinak [17], Marino [18], Al-Niami and Rushton [2], Atul Kumar et al. [4], Dilip Kumar et al [8] etc.

It is assumed that the porous medium is homogeneous and isotropic and that no mass transfer occurs between the solid and liquid phases. Also it is assumed that the flow in the medium is unidirectional and the average velocity is taken to be constant throughout the length of the flow field. Further it is assumed that the solute transport, across any fixed plane, due to microscopic velocity variations in the flow tubes, may be quantitatively expressed as the product of a dispersion coefficient and the concentration gradient.

For finding the analytical or numerical solutions of dispersion problems, the decision parameters of the model must have crisp values. But in groundwater hydrology, uncertainties are arise due to insufficient data or imprecise information in the identification and classification of aquifer properties. Also in real-world applications certainty, reliability and precision of data is often not possible and it involves high information costs. These uncertainties are due to insufficient data or imprecise information. In this paper, we model the uncertain parameters as interval numbers.

The rest of this paper is organized as follows. In section 2, we recall the basic concepts of interval numbers and interval arithmetic and related results for our future discussions. In section 3, we introduce a new method for analytical solution of one dimensional advection-diffusion equation involving interval parameters. Finally, in section 4, numerical examples are given to illustrate the method developed in this paper.

2. Preliminaries

The aim of this section is to present some notations, notions and results which are of useful in our further considerations.

Let $\mathbb{IR} = \{\tilde{x} = [x_1, x_2] : x_1 \leq x_2 \text{ and } x_1, x_2 \in \mathbb{R}\}$ be the set of all proper intervals and $\overline{\mathbb{IR}} = \{\tilde{x} = [x_1, x_2] : x_1 > x_2 \text{ and } x_1, x_2 \in \mathbb{R}\}$ be the set of all improper intervals on the real line $\mathbb{R}$. If $x_1 = x_2 = x$, then $\tilde{x} = [x, x]$ is a real number (or a degenerate interval). We shall use the terms "interval" and "interval number" interchangeably. The mid-point and width (or half-width) of an interval number $\tilde{x} = [x_1, x_2]$ are defined as

$$m(\tilde{x}) = \left(\frac{x_1 + x_2}{2}\right) \text{ and } w(\tilde{x}) = \left(\frac{x_2 - x_1}{2}\right)$$

Algebraic properties of classical interval arithmetic defined on $\mathbb{IR}$ (Moore, 1995) are often insufficient if we want to deal with real world problems involving interval parameters. Because, intervals with nonzero width do not have inverses in $\mathbb{IR}$ with respect to the classical interval arithmetical operations. This "incompleteness" stimulated attempts to create a more convenient interval arithmetic extending that based on $\mathbb{IR}$. In this direction, several extensions of the classical interval arithmetic have been proposed. Kaucher [14–16] proposed a new method, in which the set of proper intervals is extended by improper intervals and the interval arithmetic operations and functions are extended correspondingly. We denote the set of generalized intervals (proper and improper) by $\mathbb{D} = \mathbb{IR} \cup \overline{\mathbb{IR}} = \{[x_1, x_2] : x_1, x_2 \in \mathbb{R}\}$. The set of generalized intervals $\mathbb{D}$ is a group with respect to addition and multiplication operations of zero free intervals, while maintaining the inclusion monotonicity. The algebraic properties of the generalized interval arithmetic create a suitable environment for solving algebraic problems involving interval numbers. However, the efficient solution of some interval algebraic problems is hampered by the lack of well studied distributive relations between generalized intervals. Ganesan and Veeramani [10] proposed a new interval arithmetic which satisfies the distributive relations between intervals.

2.1 *Comparing Interval Numbers*

We extend the interval ranking method proposed by Sengupta and Pal [3] on $\mathbb{IR}$ to the set of all generalized intervals $\mathbb{D}$.

Definition 2.1. Let $\preceq$ be an extended order relation between the interval numbers $\tilde{x} = [x_1, x_2]$ and $\tilde{y} = [y_1, y_2]$ in $\mathbb{D}$, then for $m(\tilde{x}) < m(\tilde{y})$, we construct a premise $(\tilde{x} \preceq \tilde{y})$ which implies that $\tilde{x}$ is inferior to $\tilde{y}$ (or $\tilde{y}$ is superior to $\tilde{x}$). Here, the term 'inferior to' ('superior to') is analogous to 'less than' ('greater than').

Definition 2.2. An *acceptability function* $\mathcal{A}_{\preceq} : \mathbb{D} \times \mathbb{D} \longrightarrow [0, \infty)$ *is defined as:*

$$\mathcal{A}_{\preceq}(\tilde{x}, \tilde{y}) = \mathcal{A}(\tilde{x} \preceq \tilde{y}) = \frac{(m(\tilde{y}) - m(\tilde{x}))}{(w(\tilde{y}) + w(\tilde{x}))}$$

$where\ w(\tilde{y}) + w(\tilde{x}) \neq 0$. $\mathcal{A}_{\preceq}$ *may be interpreted as the grade of acceptability of the 'first interval number $\tilde{x}$ to be inferior to the second interval number $\tilde{y}$'.*

For any two interval numbers $\tilde{x}$ and $\tilde{y}$ in $\mathbb{D}$, either $\mathcal{A}(\tilde{x} \preceq \tilde{y}) > 0$ or $\mathcal{A}(\tilde{y} \preceq \tilde{x}) > 0$ or $\mathcal{A}(\tilde{x} \preceq \tilde{y}) = \mathcal{A}(\tilde{y} \preceq \tilde{x}) = 0and\mathcal{A}(\tilde{x} \preceq \tilde{y}) + \mathcal{A}(\tilde{y} \preceq \tilde{x}) = 0$. Also the proposed $\mathcal{A}$-index is transitive; for any three interval numbers $\tilde{x}, \tilde{y}$ and $\tilde{z}$ in $\mathbb{D}$, if $\mathcal{A}(\tilde{x} \preceq \tilde{y}) \geq 0$ and $\mathcal{A}(\tilde{y} \preceq \tilde{z}) \geq 0$, then $\mathcal{A}(\tilde{x} \preceq \tilde{z}) \geq 0$. But it does not mean that $\mathcal{A}(\tilde{x} \preceq \tilde{z}) \geq \max\{\mathcal{A}(\tilde{x} \preceq \tilde{y}), \mathcal{A}(\tilde{y} \preceq \tilde{z})\}$.

If $\mathcal{A}(\tilde{x} \preceq \tilde{y}) = 0$ with $w(\tilde{x}) = w(\tilde{y})$ or $w(\tilde{x}) \neq w(\tilde{y})$, then we say that the interval numbers $\tilde{x}$ and $\tilde{y}$ are equivalent (or non-inferior to each other) and we denote it by $\tilde{x} \approx \tilde{y}$. In particular, whenever $\mathcal{A}(\tilde{x} \preceq \tilde{y}) = 0$ and $w(\tilde{x}) = w(\tilde{y})$, then $\tilde{x} = \tilde{y}$. Also if $\mathcal{A}(\tilde{x} \preceq \tilde{y}) \geq 0$, then we say that $\tilde{x} \preceq \tilde{y}$ and if $\mathcal{A}(\tilde{y} \preceq \tilde{x}) \geq 0$, then we say that $\tilde{y} \preceq \tilde{x}$.

Remark 2.1. For any two interval numbers $\tilde{x}, \tilde{y} \in \mathbb{D}$, we have $\mathcal{A}(\tilde{x} \preceq \tilde{y}) + \mathcal{A}(\tilde{y} \preceq \tilde{x}) = 0$.

2.2 *A New Interval Arithmetic*

The "dual" is an important monadic operator that reverses the end-points of the intervals and expresses an element-to-element symmetry between proper and improper intervals in $\mathbb{D}$.

For $\tilde{x} = [x_1, x_2] \in \mathbb{D}$, its dual is defined by $\mathrm{dual}(\tilde{x}) = \mathrm{dual}\,[x_1, x_2] = [x_2, x_1]$. The opposite of an interval $\tilde{x} = [x_1, x_2]$ is $\mathrm{opp}\{[x_1, x_2]\} = [-x_2, -x_1]$ which is the additive inverse of $[x_1, x_2]$ and $\left[\frac{1}{x_1}, \frac{1}{x_2}\right]$ is the multiplicative inverse of $[x_1, x_2]$, provided $0 \notin [x_1, x_2]$. That is,

$$\begin{aligned}
\tilde{x} + (-\mathrm{dual}\ \tilde{x}) &= \tilde{x} - \mathrm{dual}\ (\tilde{x}) \\
&= [x_1, x_2] - \mathrm{dual}\,([x_1, x_2]) \\
&= [x_1, x_2] - [x_2, x_1] \\
&= [x_1 - x_1, x_2 - x_2] = [0, 0]
\end{aligned}$$

and

$$\begin{aligned}
\tilde{x} \times \left(\frac{1}{\mathrm{dual}\ \tilde{x}}\right) &= [x_1, x_2] \times \left(\frac{1}{\mathrm{dual}\,([x_1, x_2])}\right) \\
&= [x_1, x_2] \times \frac{1}{[x_2, x_1]} \\
&= [x_1, x_2] \times \left[\frac{1}{x_1}, \frac{1}{x_2}\right]
\end{aligned}$$

$$= \left[\frac{x_1}{x_1}, \frac{x_2}{x_2}\right] = [1, 1].$$

We incorporate the concept of dual in the interval arithmetic defined by Ganesan and Veeramani [10] on the set of generalized interval numbers $\mathbb{D}$: For $\tilde{x} = [x_1, x_2]$, $\tilde{y} = [y_1, y_2] \in \mathbb{D}$ and for $* \in \{+, -, \cdot, \div\}$, we define $\tilde{x} * \tilde{y} = [m(\tilde{x}) * m(\tilde{y}) - k, m(\tilde{x}) * m(\tilde{y}) + k]$, where $k = \min\{(m(\tilde{x}) * m(\tilde{y})) - \alpha,\ \beta - (m(\tilde{x}) * m(\tilde{y}))\}$, α and β are the end points of the interval $\tilde{x} \odot \tilde{y}$ under the existing interval arithmetic. In particular

(i) Addition:

$$\begin{aligned} \tilde{x} + \tilde{y} &= [x_1, x_2] + [y_1, y_2] \\ &= [(m(\tilde{x}) + m(\tilde{y})) - k, (m(\tilde{x}) + m(\tilde{y})) + k], \end{aligned}$$

where $k = \left\{\frac{(y_2 + x_2) - (y_1 + x_1)}{2}\right\}$.

(ii) Subtraction:

$$\begin{aligned} \tilde{x} - \tilde{y} &= [x_1, x_2] - [y_1, y_2] \\ &= [(m(\tilde{x}) - m(\tilde{y})) - k, (m(\tilde{x}) - m(\tilde{y})) + k], \end{aligned}$$

where $k = \left\{\frac{(y_2 + x_2) - (y_1 + x_1)}{2}\right\}$

Also if $\tilde{x} = \tilde{y}$ i.e. $[x_1, x_2] = [y_1, y_2]$, then

$$\begin{aligned} \tilde{x} - \tilde{y} &= \tilde{x} - \text{dual}(\tilde{x}) = [x_1, x_2] - [x_2, x_1] \\ &= [x_1 - x_1, x_2 - x_2] = [0, 0] = 0 \end{aligned}$$

(If $\tilde{x} = \tilde{y}$, then $\tilde{x} - \tilde{y} = \tilde{0} = 0$, may be taken as dual subtraction)

(iii) Multiplication:

$$\begin{aligned} \tilde{x} \cdot \tilde{y} &= [x_1, x_2] \cdot [y_1, y_2] \\ &= [m(\tilde{x})m(\tilde{y}) - k, m(\tilde{x})m(\tilde{y}) + k], \end{aligned}$$

where $k = \min\{(m(\tilde{x})m(\tilde{y})) - \alpha,\ \beta - (m(\tilde{x})m(\tilde{y}))\}$,
$\alpha = \min(x_1y_1, x_1y_2, x_2y_1, x_2y_2)$ and
$\beta = \max(x_1y_1, x_1y_2, x_2y_1, x_2y_2)$.

(iv) Division:

$$\begin{aligned} 1 \div \tilde{x} = \frac{1}{\tilde{x}} &= \frac{1}{[x_1, x_2]} \\ &= \left[\frac{1}{m(\tilde{x})} - k, \frac{1}{m(\tilde{x})} + k\right], \end{aligned}$$

where $k = \min\left\{\frac{1}{x_2}\left(\frac{x_2 - x_1}{x_1 + x_2}\right), \frac{1}{x_1}\left(\frac{x_2 - x_1}{x_1 + x_2}\right)\right\}$ and $0 \notin [x_1, x_2]$.

Also if $\tilde{x} = \tilde{y}$ i.e. $[x_1, x_2] = [y_1, y_2]$, then

$$\begin{aligned}\frac{\tilde{x}}{\tilde{y}} = \frac{\tilde{x}}{\tilde{x}} &= \frac{\tilde{x}}{\text{dual}(\tilde{x})} = [x_1, x_2] \cdot \frac{1}{[x_2, x_1]} \\ &= [x_1, x_2] \cdot \left[\frac{1}{x_1}, \frac{1}{x_2}\right] \\ &= \left[\frac{x_1}{x_1}, \frac{x_2}{x_2}\right] = [1, 1] = 1.\end{aligned}$$

(If $\tilde{x} = \tilde{y}$, then $\dfrac{\tilde{x}}{\tilde{y}} = \tilde{1} = 1$, may be taken as dual division)

(v) Exponential:

$$\exp \tilde{x} = \exp [x_1, x_2] = [\exp x_1, \exp x_2]$$

(vi) Square Root: The square root of an interval is given by $\sqrt{\tilde{x}} = \sqrt{[x_1, x_2]} = \left[\sqrt{x_1}, \sqrt{x_2}\right]$, for $x_1 \geq 0$.

(vii) Complementary Error Function:

The complementary error function of an interval is given by

$$\mathrm{erfc}(\tilde{\mathrm{x}}) = \mathrm{erfc}\,[\mathrm{x}_1, \mathrm{x}_2] = \{\min[\mathrm{erfc}(\mathrm{x}_1), \mathrm{erfc}(\mathrm{x}_2)], \max[\mathrm{erfc}(\mathrm{x}_1), \mathrm{erfc}(\mathrm{x}_2)]\}$$

From (iii), it is clear that

$$\lambda\tilde{x} = \begin{cases} [\lambda x_1, \lambda x_2], \text{ for } \lambda \geq 0 \\ [\lambda x_2, \lambda x_1], \text{ for } \lambda < 0. \end{cases}$$

It is to be noted that $\tilde{x} * \tilde{y} \subseteq \tilde{x} \circledast \tilde{y} = \{x \circledast y / x \in \tilde{x}, y \in \tilde{y}\}$, where $\circledast$ is the existing interval arithmetic. For example if $\tilde{x} = [-1, 2]$ and $\tilde{y} = [3, 5]$, then $\tilde{x} \circledast \tilde{y} = [-1, 2] \circledast [3, 5] = [\min(-3, -5, 6, 10), \max(-3, -5, 6, 10)] = [-5, 10]$ and $\tilde{x} * \tilde{y} = [-1, 2] * [3, 5] = [-5, 9]$ so that $\tilde{x} * \tilde{y} \subseteq \tilde{x} \circledast \tilde{y}$.

It is also important to note that by using this modified interval arithmetic we are able to prove the distributive law for interval numbers.

Example 2.1. Let $\tilde{c} = [2, 3], \tilde{x} = [-1, 2]$ and $\tilde{y} = [2, 5]$. By existing arithmetic operations for interval numbers, we have $\tilde{x} + \tilde{y} = [1, 7] \Rightarrow \tilde{c}(\tilde{x} + \tilde{y}) = [2, 21]$. So $m(\tilde{c}(\tilde{x} + \tilde{y})) = 11.5$ and $w(\tilde{c}(\tilde{x} + \tilde{y})) = 9.5$. Now $\tilde{c}\tilde{x} = [2, 3][-1, 2] = [-3, 6]$ and $\tilde{c}\tilde{y} = [2, 3][2, 5] = [4, 15]$ and therefore $(\tilde{c}\tilde{x} + \tilde{c}\tilde{y}) = [1, 21]$. So that $m(\tilde{c}\tilde{x} + \tilde{c}\tilde{y}) = 11$ and $w(\tilde{c}\tilde{x} + \tilde{c}\tilde{y}) = 10$. We see that $m(\tilde{c}(\tilde{x} + \tilde{y})) \neq m(\tilde{c}\tilde{x} + \tilde{c}\tilde{y})$ and $w(\tilde{c}(\tilde{x} + \tilde{y})) \neq w(\tilde{c}\tilde{x} + \tilde{c}\tilde{y})$. Hence, $\tilde{c}(\tilde{x} + \tilde{y}) \not\approx \tilde{c}\tilde{x} + \tilde{c}\tilde{y}$. But according to the modified interval arithmetic, we have $\tilde{c}(\tilde{x} + \tilde{y}) = [2, 18]$, $\tilde{c}\tilde{x} = \left[-3, \frac{11}{2}\right]$, $\tilde{c}\tilde{y} = \left[4, \frac{27}{2}\right]$, $\tilde{c}\tilde{x} + \tilde{c}\tilde{y} = [1, 19]$. So that $m(\tilde{c}(\tilde{x} + \tilde{y}))$=10 and $w(\tilde{c}(\tilde{x} + \tilde{y})) = 8$. We see that $m(\tilde{c}(\tilde{x} + \tilde{y})) = m(\tilde{c}\tilde{x} + \tilde{c}\tilde{y})$ and $w(\tilde{c}(\tilde{x} + \tilde{y})) + w(\tilde{c}\tilde{x} + \tilde{c}\tilde{y}) \neq 0$. Hence, $\tilde{c}(\tilde{x} + \tilde{y}) \approx (\tilde{c}\tilde{x} + \tilde{c}\tilde{y})$

3. Main Results

In this section we shall derive the analytical solution of one-dimensional advection-diffusion equation with interval parameters without converting them to classical forms. By the principle of conservation of mass using Fick's law, the governing interval differential equation of the one-dimensional solute transport problem whose decision parameters are interval numbers is given by

$$\tilde{D}\nabla^2\tilde{C} = \tilde{u}\frac{\partial\tilde{C}}{\partial x} + \frac{\partial\tilde{C}}{\partial t} \tag{3.1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
$\tilde{D}$ = dispersion coefficient(interval number),
$\tilde{C}$ =concentration of solute in the fluid (interval number),
$\tilde{u}$ =average velocity of fluid or superficial velocity/porsity of medium (interval number),
x = coordinate parallel to flow,
y, z =coordinates normal to flow and t=time. In the event that mass transfer takes place between the liquid and solid phase, the differential equation becomes

$$\tilde{D}\nabla^2\tilde{C} = \tilde{u}\frac{\partial \tilde{C}}{\partial x} + \frac{\partial \tilde{C}}{\partial t} + \frac{\partial \tilde{F}}{\partial t} \tag{3.2}$$

where $\tilde{F}$ is the concentration (interval number) of the solute in the solid phase.
If we consider a semi infinite media having plane source at $x = 0$, then equation (3.1) becomes

$$\tilde{D}\frac{\partial^2 \tilde{C}}{\partial x^2} - \tilde{u}\frac{\partial \tilde{C}}{\partial x} = \frac{\partial \tilde{C}}{\partial t} \tag{3.3}$$

which describe the one-dimensional transport and dispersion of a given nonreactive dissolved chemical species in flowing groundwater for a homogeneous and isotropic aquifer. Initially, the solute concentration throughout the aquifer is zero. For times greater than or equal to zero $(t \geq 0)$, the concentration at the left edge of the aquifer becomes a constant C_0. These initial and boundary conditions could simulate the sudden dumping of non-reactive (conservative) constituent into the groundwater flow. The equation (3.3) is the governing equation to be used to predict the change in concentrations with time at different distances x through the aquifer. In mathematical terms, the initial and boundary condition of this problem are

$$\begin{aligned}
\tilde{C}(0,t) &= \tilde{C}_0; \quad t \geq 0\\
\tilde{C}(x,0) &= 0; \quad x \geq 0\\
\tilde{C}(\infty,t) &= 0; \quad t \geq 0.
\end{aligned}$$

Now the problem expresses the concentration as a function of x and t alone.

To modify the equation (3.1) to a more familiar form, let us assume that

$$\tilde{C}(x,t) = \tilde{\Gamma}(x,t)\ \exp\left(\frac{\tilde{u}x}{2\tilde{D}} - \frac{\tilde{u}^2 t}{4\tilde{D}}\right) \tag{3.4}$$

From equation (3.4), we have

$$\frac{\partial \tilde{C}}{\partial x} = \tilde{\Gamma}(x,t)\exp\left(\frac{\tilde{u}x}{2\tilde{D}} - \frac{\tilde{u}^2 t}{4\tilde{D}}\right)\left(\frac{\tilde{u}}{2\tilde{D}}\right) + \frac{\partial \tilde{\Gamma}}{\partial x}\exp\left(\frac{\tilde{u}x}{2\tilde{D}} - \frac{\tilde{u}^2 t}{4\tilde{D}}\right)$$

$$\begin{aligned}
\frac{\partial^2 \tilde{C}}{\partial x^2} &= \tilde{\Gamma}(x,t)\exp\left(\frac{\tilde{u}x}{2\tilde{D}} - \frac{\tilde{u}^2 t}{4\tilde{D}}\right)\left(\frac{\tilde{u}^2}{4\tilde{D}^2}\right)\\
&\quad + \left(\frac{\tilde{u}}{\tilde{D}}\right)\frac{\partial \tilde{\Gamma}}{\partial x}\exp\left(\frac{\tilde{u}x}{2\tilde{D}} - \frac{\tilde{u}^2 t}{4\tilde{D}}\right) + \frac{\partial^2 \tilde{\Gamma}}{\partial x^2}\exp\left(\frac{\tilde{u}x}{2\tilde{D}} - \frac{\tilde{u}^2 t}{4\tilde{D}}\right)
\end{aligned}$$

$$\frac{\partial \tilde{C}}{\partial t} = \tilde{\Gamma}(x,t)\exp\left(\frac{\tilde{u}x}{2\tilde{D}} - \frac{\tilde{u}^2 t}{4\tilde{D}}\right)\left(\frac{-\tilde{u}^2}{4\tilde{D}}\right) + \frac{\partial \tilde{\Gamma}}{\partial t}\exp\left(\frac{\tilde{u}x}{2\tilde{D}} - \frac{\tilde{u}^2 t}{4\tilde{D}}\right)$$

$$\tilde{u}\frac{\partial \tilde{C}}{\partial x} = \tilde{\Gamma}(x,t)\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)\left(\frac{\tilde{u}^2}{2\tilde{D}}\right)+\tilde{u}\frac{\partial \tilde{\Gamma}}{\partial x}\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)$$

$$\begin{aligned}\tilde{D}\frac{\partial^2 \tilde{C}}{\partial x^2} &= \tilde{D}\tilde{\Gamma}(x,t)\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)\left(\frac{\tilde{u}^2}{4\tilde{D}^2}\right)\\ &+ \tilde{D}\left(\frac{\tilde{u}}{\tilde{D}}\right)\frac{\partial \tilde{\Gamma}}{\partial x}\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)+\tilde{D}\frac{\partial^2 \tilde{\Gamma}}{\partial x^2}\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)\end{aligned}$$

Applying the concept of dual division, we have

$$\begin{aligned}\tilde{D}\frac{\partial^2 \tilde{C}}{\partial x^2} &= \tilde{\Gamma}(x,t)\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)\left(\frac{\tilde{u}^2}{4\tilde{D}}\right)\\ &+ \tilde{u}\frac{\partial \tilde{\Gamma}}{\partial x}\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)+\tilde{D}\frac{\partial^2 \tilde{\Gamma}}{\partial x^2}\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)\end{aligned}$$

By the distributive law,

$$\tilde{D}\frac{\partial^2 \tilde{C}}{\partial x^2} = \left[\frac{\tilde{u}^2}{4\tilde{D}}\tilde{\Gamma}(x,t)+\tilde{u}\frac{\partial \tilde{\Gamma}}{\partial x}+\tilde{D}\frac{\partial^2 \tilde{\Gamma}}{\partial x^2}\right]\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)$$

$$\tilde{u}\frac{\partial \tilde{C}}{\partial x} = \left[\frac{\tilde{u}^2}{2\tilde{D}}\tilde{\Gamma}(x,t)+\tilde{u}\frac{\partial \tilde{\Gamma}}{\partial x}\right]\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)$$

$$\frac{\partial \tilde{C}}{\partial t} = \left[\frac{-\tilde{u}^2}{4\tilde{D}}\tilde{\Gamma}(x,t)+\frac{\partial \tilde{\Gamma}}{\partial t}\right]\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)$$

Therefore by dual subtraction, we have

$$\begin{aligned}\tilde{D}\frac{\partial^2 \tilde{C}}{\partial x^2}-\tilde{u}\frac{\partial \tilde{C}}{\partial x} &= \left[\frac{\tilde{u}^2}{4\tilde{D}}\tilde{\Gamma}(x,t)+\tilde{u}\frac{\partial \tilde{\Gamma}}{\partial x}+\tilde{D}\frac{\partial^2 \tilde{\Gamma}}{\partial x^2}\right.\\ &- \left.\frac{\tilde{u}^2}{2\tilde{D}}\tilde{\Gamma}(x,t)-\tilde{u}\frac{\partial \tilde{\Gamma}}{\partial x}\right]\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)\\ &= \left[\tilde{D}\frac{\partial^2 \tilde{\Gamma}}{\partial x^2}-\frac{\tilde{u}^2}{4\tilde{D}}\tilde{\Gamma}(x,t)\right]\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right).\end{aligned}$$

Again by dual subtraction and dual division, we have

$$\tilde{D}\frac{\partial^2 \tilde{C}}{\partial x^2}-\tilde{u}\frac{\partial \tilde{\Gamma}}{\partial x} = \frac{\partial \tilde{C}}{\partial t}$$

$$\Rightarrow \left[\tilde{D}\frac{\partial^2 \tilde{\Gamma}}{\partial x^2}-\frac{\tilde{u}^2}{4\tilde{D}}\tilde{\Gamma}(x,t)\right]\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right) = \left[\frac{-\tilde{u}^2}{4\tilde{D}}\tilde{\Gamma}(x,t)+\frac{\partial \tilde{\Gamma}}{\partial t}\right]\exp\left(\frac{\tilde{u}x}{2\tilde{D}}-\frac{\tilde{u}^2 t}{4\tilde{D}}\right)$$

$$\Rightarrow \tilde{D}\frac{\partial^2\tilde{\Gamma}}{\partial x^2} - \frac{\tilde{u}^2}{4\tilde{D}}\tilde{\Gamma}(x,t) + \frac{\tilde{u}^2}{4\tilde{D}}\tilde{\Gamma}(x,t) = \frac{\partial\tilde{\Gamma}}{\partial t}$$

Hence,

$$\tilde{D}\frac{\partial^2\tilde{\Gamma}}{\partial x^2} = \frac{\partial\tilde{\Gamma}}{\partial t} \tag{3.5}$$

The boundary conditions are transformed to

$$\begin{aligned}\tilde{\Gamma}(0,t) &= \tilde{C}_0\exp\left(\frac{\tilde{u}^2 t}{4\tilde{D}}\right); \quad t \geq 0\\ \tilde{\Gamma}(x,0) &= 0; \quad x \geq 0\\ \tilde{\Gamma}(\infty,t) &= 0; \quad t \geq 0.\end{aligned}$$

Equation (3.5) can be solved for a time-dependent influx of fluid at $x = 0$ by applying the Duhamel's theorem:

"If $C = F(x, y, z, t)$ is the solution of the diffusion equation for semi-infinite media in which the initial concentration is zero and its surface is maintained at concentration unity, then the solution of the problem in which the surface is maintained at temperature $\phi(t)$ is

$$C = \int_0^t \phi(\lambda)\frac{\partial}{\partial t}F(x,y,z,t-\lambda)d\lambda".$$

We know that the Duhamel's theorem is used for heat conduction problems in general, but Equation (3.5) is suitable to fit this specific case of interest. Hence the above initial and boundary conditions becomes

$$\begin{aligned}\tilde{\Gamma}(0,t) &= 1; t \geq 0\\ \tilde{\Gamma}(x,0) &= 0; x \geq 0\\ \tilde{\Gamma}(\infty,t) &= 0; t \geq 0\end{aligned}$$

The Laplace transform of $\Gamma(x,t)$ is defined as

$$\overline{\Gamma}(x,p) = \int_0^\infty e^{-pt}\Gamma(x,t)dt.$$

Taking Laplace transform on both sides of equation (3.5), we get

$$\frac{d^2\overline{\Gamma}}{dx^2} = \frac{p}{\tilde{D}}\overline{\Gamma} \tag{3.6}$$

and its solution is given by

$$\tilde{\overline{\Gamma}} = Ae^{-\tilde{q}x} + Be^{\tilde{q}x}$$

where $\tilde{q} = \sqrt{p/\tilde{D}}$.
By applying the boundary conditions in the above solution, we get $B = 0$ and $A = 1/p$ and hence the particular solution of the Laplace transformed equation is

$$\tilde{\overline{\Gamma}} = \frac{1}{p}e^{-\tilde{q}x}$$

Taking inverse Laplace transform for the above function, we get

$$\tilde{\Gamma} = 1 - erf\left(\frac{x}{2\sqrt{\tilde{D}t}}\right) = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\tilde{D}t}}}^\infty e^{-\eta^2}d\eta \tag{3.7}$$

Applying Duhamel's theorem, the solution of the problem with initial concentration zero and the time-dependent surface condition at $x = 0$ is

$$\tilde{\Gamma} = \int_0^t \phi(\tau) \frac{\partial}{\partial t} \left[\frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{\tilde{D}(t-\tau)}}}^{\infty} e^{-\eta^2} d\eta \right] d\tau$$

Since $e^{-\eta^2}$ is a continuous function, it is possible to differentiate under the integral, which gives

$$\frac{2}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_{\frac{x}{2\sqrt{\tilde{D}(t-\tau)}}}^{\infty} e^{-\eta^2} d\eta = \frac{x e^{-x^2/4\tilde{D}(t-\tau)}}{2\sqrt{\pi \tilde{D}}(t-\tau)^{3/2}}$$

Thus,

$$\tilde{\Gamma} = \frac{x}{2\sqrt{\pi \tilde{D}}} \int_0^t \phi(\tau) e^{-x^2/4\tilde{D}(t-\tau)} \frac{d\tau}{(t-\tau)^{3/2}}$$

Assume that $\tilde{\lambda} = \frac{x}{2\sqrt{\tilde{D}(t-\tau)}}$, the solution becomes

$$\tilde{\Gamma} = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{\tilde{D}t}}}^{\infty} \phi\left(t - \frac{x^2}{4\tilde{D}\tilde{\lambda}^2}\right) e^{-\tilde{\lambda}^2} d\tilde{\lambda} \tag{3.8}$$

Since $\phi(t) = \tilde{C}_0 \mathrm{exp}\left(\frac{\tilde{\mathrm{u}}^2 \mathrm{t}}{4\tilde{\mathrm{D}}}\right)$, the particular solution of the problem becomes

$$\begin{aligned} \tilde{\Gamma}(x,t) = \tilde{C}_0 \frac{2}{\sqrt{\pi}} e^{\frac{\tilde{u}^2 t}{4\tilde{D}}} &\left\{ \int_0^\infty \exp\left(-\tilde{\lambda}^2 - \frac{\tilde{\epsilon}^2}{\tilde{\lambda}^2}\right) \right\} d\tilde{\lambda} \\ &- \left\{ \int_0^{\tilde{\alpha}} \exp\left(-\tilde{\lambda}^2 - \frac{\tilde{\epsilon}^2}{\tilde{\lambda}^2}\right) \right\} d\tilde{\lambda} \end{aligned} \tag{3.9}$$

where $\tilde{\epsilon} = \frac{\tilde{u}x}{4\tilde{D}}$ and $\tilde{\alpha} = \frac{x}{2\sqrt{\tilde{D}t}}$.

Consider the first integration in (3.9), we get

$$\int_0^\infty \exp\left(-\tilde{\lambda}^2 - \frac{\tilde{\epsilon}^2}{\tilde{\lambda}^2}\right) \mathrm{d}\tilde{\lambda} = \frac{\sqrt{\pi}}{2} \mathrm{e}^{-2\tilde{\epsilon}}.$$

For convenience the second integral may be expressed in terms of error function, because this function is well tabulated. Let

$$\begin{aligned} -\tilde{\lambda}^2 - \frac{\tilde{\epsilon}^2}{\tilde{\lambda}^2} &= -\left(\tilde{\lambda} + \frac{\tilde{\epsilon}}{\tilde{\lambda}}\right)^2 + 2\tilde{\epsilon} \\ &= -\left(\tilde{\lambda} - \frac{\tilde{\epsilon}}{\tilde{\lambda}}\right)^2 - 2\tilde{\epsilon} \end{aligned}$$

The second integral of (3.9) becomes

$$\begin{aligned} I = \int_0^{\tilde{\alpha}} \exp\left[-\tilde{\lambda}^2 - \frac{\tilde{\epsilon}^2}{\tilde{\lambda}^2}\right] \mathrm{d}\tilde{\lambda} = \frac{1}{2} &\left\{ \mathrm{e}^{2\tilde{\epsilon}} \int_0^{\tilde{\alpha}} \exp\left[-\left(\tilde{\lambda} + \frac{\tilde{\epsilon}}{\tilde{\lambda}}\right)^2\right] \mathrm{d}\tilde{\lambda} \right. \\ &\left. + e^{-2\tilde{\epsilon}} \int_0^{\tilde{\alpha}} \exp\left[-\left(\tilde{\lambda} - \frac{\tilde{\epsilon}}{\tilde{\lambda}}\right)^2\right] \mathrm{d}\tilde{\lambda} \right\} = \frac{1}{2}\{I_1 + I_2\} \end{aligned} \tag{3.10}$$

Now assuming $\tilde{z} = \dfrac{\tilde{\epsilon}}{\tilde{\lambda}}$ and using

$e^{2\tilde{\epsilon}} \int_{\tilde{\epsilon}/\tilde{\alpha}}^{\infty} \exp\left[-\left(\dfrac{\tilde{\epsilon}}{\tilde{z}} + \tilde{z}\right)^2\right] d\tilde{z}$ in I_1 of equation (3.10), we get

$$\begin{aligned} I_1 &= e^{2\tilde{\epsilon}} \int_0^{\tilde{\alpha}} \exp\left[-\left(\tilde{\lambda} + \frac{\tilde{\epsilon}}{\tilde{\lambda}}\right)^2\right] d\tilde{\lambda} \\ &= -e^{2\tilde{\epsilon}} \int_{\tilde{\epsilon}/\tilde{\alpha}}^{\infty} \left(1 - \frac{\tilde{\epsilon}}{\tilde{z}^2}\right) \exp\left[-\left(\frac{\tilde{\epsilon}}{\tilde{z}} + \tilde{z}\right)^2\right] d\tilde{z} \\ &+ e^{2\tilde{\epsilon}} \int_{\tilde{\epsilon}/\tilde{\alpha}}^{\infty} \exp\left[-\left(\frac{\tilde{\epsilon}}{\tilde{z}} + \tilde{z}\right)^2\right] d\tilde{z} \end{aligned}$$

Let, $\tilde{\beta} = \left(\dfrac{\tilde{\epsilon}}{\tilde{z}} + \tilde{z}\right)$ in the first term of the above equation, then

$$I_1 = -e^{2\tilde{\epsilon}} \int_{\tilde{\alpha} + \frac{\tilde{\epsilon}}{\tilde{\alpha}}}^{\infty} e^{-\tilde{\beta}^2} d\tilde{\beta} + e^{2\tilde{\epsilon}} \int_{\tilde{\epsilon}/\tilde{\alpha}}^{\infty} \exp\left[-\left(\frac{\tilde{\epsilon}}{\tilde{z}} + \tilde{z}\right)^2\right] d\tilde{z}$$

Similar evaluation of the integral I_2 of equation (3.10) gives

$$\begin{aligned} I_2 &= e^{-2\tilde{\epsilon}} \int_{\tilde{\epsilon}/\tilde{\alpha}}^{\infty} \left(\frac{\tilde{\epsilon}}{\tilde{z}^2} + 1\right) \exp\left[-\left(\frac{\tilde{\epsilon}}{\tilde{z}} - \tilde{z}\right)^2\right] d\tilde{z} \\ &- e^{-2\tilde{\epsilon}} \int_{\tilde{\epsilon}/\tilde{\alpha}}^{\infty} \exp\left[-\left(\frac{\tilde{\epsilon}}{\tilde{z}} - \tilde{z}\right)^2\right] d\tilde{z}. \end{aligned}$$

Again $-\tilde{\beta} = \left(\dfrac{\tilde{\epsilon}}{\tilde{z}} - \tilde{z}\right)$ into the first term, the result is

$$I_2 = e^{-2\tilde{\epsilon}} \int_{\frac{\tilde{\epsilon}}{\tilde{\alpha}} - \tilde{\alpha}}^{\infty} e^{-\tilde{\beta}^2} d\tilde{\beta} - e^{-2\tilde{\epsilon}} \int_{\tilde{\epsilon}/\tilde{\alpha}}^{\infty} \exp\left[-\left(\frac{\tilde{\epsilon}}{\tilde{z}} - \tilde{z}\right)^2\right] d\tilde{z}$$

Note that

$$\int_{\tilde{\epsilon}/\tilde{\alpha}}^{\infty} \exp\left[-\left(\tilde{z} + \frac{\tilde{\epsilon}}{\tilde{z}}\right)^2 + 2\tilde{\epsilon}\right] d\tilde{z} = \int_{\tilde{\epsilon}/\tilde{\alpha}}^{\infty} \exp\left[-\left(\frac{\tilde{\epsilon}}{\tilde{z}} - \tilde{z}\right)^2 - 2\tilde{\epsilon}\right] d\tilde{z}$$

Substitute in equation (3.10) gives

$$\begin{aligned} I &= \frac{1}{2} \left\{ -e^{2\tilde{\epsilon}} \int_{\tilde{\alpha} + \frac{\tilde{\epsilon}}{\tilde{\alpha}}}^{\infty} e^{-\tilde{\beta}^2} d\tilde{\beta} + e^{2\tilde{\epsilon}} \int_{\tilde{\epsilon}/\tilde{\alpha}}^{\infty} \exp\left[-\left(\frac{\tilde{\epsilon}}{\tilde{z}} + \tilde{z}\right)^2\right] d\tilde{z} \right. \\ &+ \left. e^{-2\tilde{\epsilon}} \int_{\frac{\tilde{\epsilon}}{\tilde{\alpha}} - \tilde{\alpha}}^{\infty} e^{-\tilde{\beta}^2} d\tilde{\beta} - e^{-2\tilde{\epsilon}} \int_{\tilde{\epsilon}/\tilde{\alpha}}^{\infty} \exp\left[-\left(\frac{\tilde{\epsilon}}{\tilde{z}} - \tilde{z}\right)^2\right] d\tilde{z} \right\} \end{aligned}$$

By dual subtraction, we get

$$I = \frac{1}{2} \left\{ e^{-2\tilde{\epsilon}} \int_{\frac{\tilde{\epsilon}}{\tilde{\alpha}} - \tilde{\alpha}}^{\infty} e^{-\tilde{\beta}^2} d\tilde{\beta} - e^{2\tilde{\epsilon}} \int_{\frac{\tilde{\epsilon}}{\tilde{\alpha}} + \tilde{\alpha}}^{\infty} e^{-\tilde{\beta}^2} d\tilde{\beta} \right\}$$

Thus, equation (3.9) may be expressed as

$$\tilde{\Gamma}(x,t) = \frac{2C_0}{\sqrt{\pi}} e^{\frac{\tilde{u}^2 t}{4\tilde{D}}} \left\{ \frac{\sqrt{\pi}}{2} e^{-2\tilde{\epsilon}} - \frac{1}{2} \left[e^{-2\tilde{\epsilon}} \int_{\frac{\tilde{\epsilon}}{\tilde{\alpha}}-\tilde{\alpha}}^{\infty} e^{-\tilde{\beta}^2} d\tilde{\beta} \quad - e^{2\tilde{\epsilon}} \int_{\frac{\tilde{\epsilon}}{\tilde{\alpha}}+\tilde{\alpha}}^{\infty} e^{-\tilde{\beta}^2} d\tilde{\beta} \right] \right\} \tag{3.11}$$

However, by definition $e^{2\tilde{\epsilon}} \int_{\tilde{\alpha}+\frac{\tilde{\epsilon}}{\tilde{\alpha}}}^{\infty} e^{-\tilde{\beta}^2} d\tilde{\beta} = \frac{\sqrt{\pi}}{2} e^{2\tilde{\epsilon}} erfc\left(\tilde{\alpha} + \frac{\tilde{\epsilon}}{\tilde{\alpha}}\right)$. Also,

$$e^{-2\tilde{\epsilon}} \int_{\frac{\tilde{\epsilon}}{\tilde{\alpha}}-\tilde{\alpha}}^{\infty} e^{-\tilde{\beta}^2} d\tilde{\beta} = \frac{\sqrt{\pi}}{2} e^{-2\tilde{\epsilon}} \left[1 + erf\left(\tilde{\alpha} - \frac{\tilde{\epsilon}}{\tilde{\alpha}}\right)\right]$$

Writing equation (3.11) in terms of the error functions

$$\tilde{\Gamma}(x,t) = \frac{\tilde{C}_0}{2} e^{\frac{\tilde{u}^2 t}{4\tilde{D}}} \left[e^{2\tilde{\epsilon}} erfc\left(\tilde{\alpha} + \frac{\tilde{\epsilon}}{\tilde{\alpha}}\right) + e^{-2\tilde{\epsilon}} erfc\left(\tilde{\alpha} - \frac{\tilde{\epsilon}}{\tilde{\alpha}}\right) \right]$$

Then, substituting into equation (3.4) the solution is

$$\begin{aligned} \tilde{C}(x,t) &= \frac{\tilde{C}_0}{2} e^{\frac{\tilde{u}^2 t}{4\tilde{D}}} \left[e^{2\tilde{\epsilon}} erfc\left(\tilde{\alpha} + \frac{\tilde{\epsilon}}{\tilde{\alpha}}\right) \right. \\ &\quad \left. + \; e^{-2\tilde{\epsilon}} erfc\left(\tilde{\alpha} - \frac{\tilde{\epsilon}}{\tilde{\alpha}}\right) \right] \exp\left(\frac{\tilde{u}x}{2\tilde{D}} - \frac{\tilde{u}^2 t}{4\tilde{D}}\right) \end{aligned}$$

By dual substraction,

$$\frac{\tilde{C}}{\tilde{C}_0} = \frac{1}{2} \left[erfc\left(\tilde{\alpha} - \frac{\tilde{\epsilon}}{\tilde{\alpha}}\right) + e^{4\tilde{\epsilon}} erfc\left(\tilde{\alpha} + \frac{\tilde{\epsilon}}{\tilde{\alpha}}\right) \right]$$

Resubstituting for $\tilde{\epsilon}$ and $\tilde{\alpha}$ gives

$$\tilde{C}(x,t) = \frac{\tilde{C}_0}{2} \left[erfc\left(\frac{x}{2\sqrt{\tilde{D}t}} - \frac{\frac{\tilde{u}x}{4\tilde{D}}}{\frac{x}{2\sqrt{\tilde{D}t}}} \right) + e^{4\left(\frac{\tilde{u}x}{4\tilde{D}}\right)} erfc\left(\frac{x}{2\sqrt{\tilde{D}t}} + \frac{\frac{\tilde{u}x}{4\tilde{D}}}{\frac{x}{2\sqrt{\tilde{D}t}}} \right) \right]$$

Hence,

$$\tilde{C}(x,t) = \frac{\tilde{C}_0}{2} \left\{ erfc\left(\frac{x - \tilde{u}t}{2\sqrt{\tilde{D}t}}\right) + e^{\frac{\tilde{u}x}{\tilde{D}}} erfc\left(\frac{x + \tilde{u}t}{2\sqrt{\tilde{D}t}}\right) \right\} \tag{3.12}$$

The vagueness of second term in the above equation is more and can be neglected.

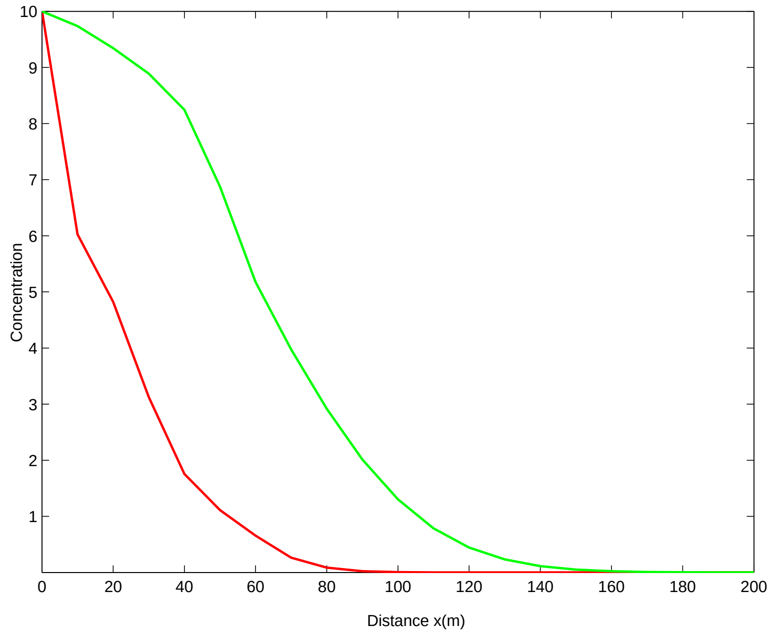

Figure 1: Concentration of the solute in the fluid for fixed distance 200 meters with initial concentration 10 =[10, 10] mg / l

4. Numerical Examples

Example 4.1. Consider a solute transport problem whose input parameters are interval numbers, say $\tilde{u} = [0.05, 0.15]$ m / day, $t = 400$ days, $\tilde{D} = [0.15, 1.5]\text{m}^2/$ day, $x = 200$ m and the initial concentration $C_0 = 10 = [10, 10]$ mg $/l$

We shall compute the concentration of the solute for varying the downward distances at, say around 10, around 20, around 30, ... and so on.

Let us model these uncertain parameters around 10, around 20, around 30, ... as interval numbers: around 10 $= \tilde{10} = [9, 11]$, around $20 = \tilde{20} = [19, 21]$, around $30 = \tilde{30} = [29, 31]$, and so on.

By using the generalized interval arithmetic (a new interval arithmetic introduced in this paper), we get

$$\tilde{u}t = [20, 60], \quad \tilde{D}t = [200, 600], \quad \sqrt{[200, 600]} = [14.1421, 24.4949],$$

$$2[14.1421, 24.4949] = [28.2841, 48.9899] \text{ and } \frac{1}{[28.2841, 48.9899]} = [0.0204, 0.0314].$$

Substituting these values in equation (3.12), we get

$$\tilde{C}(x, t) = 5\,\{\text{erfc}\,(\text{x} - [20, 60][0.0204, 0.0314])\}$$

Example 4.2. In the above example, suppose that the initial concentration is a non degenerate interval number $C_0 = \tilde{10} = [9.5, 10.5]$ mg $/l$ and no other changes in the remaining boundary and initial conditions. Then the problem becomes $\tilde{u} = [0.05, 0.15]$ m / day, $t = 400$ days, $\tilde{D} = [0.15, 1.5]\text{m}^2/$ day, $x = 200$ m and the initial concentration $C_0 = \tilde{10} = [9.5, 10.5]$ mg $/l$

By using the generalized interval arithmetic (a new interval arithmetic introduced in this paper), we get

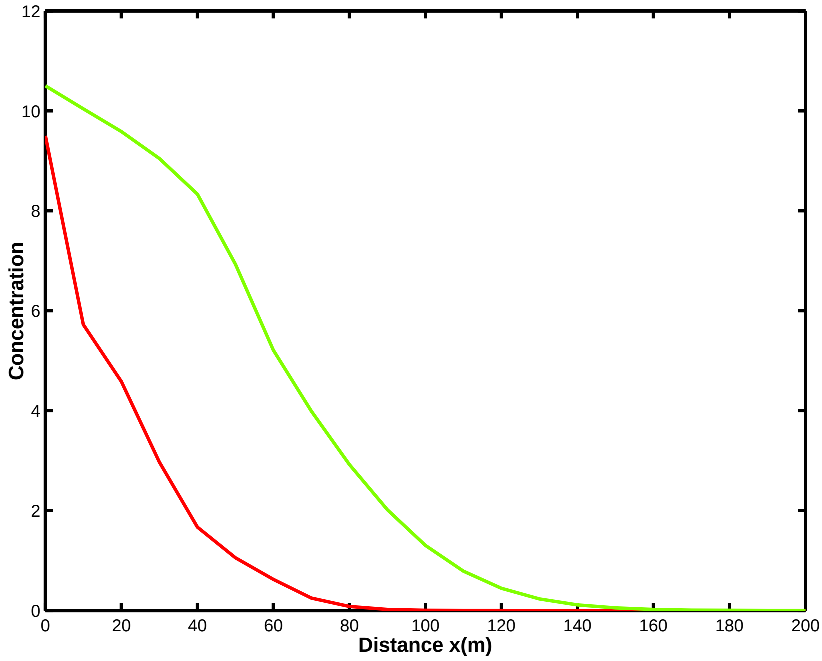

Figure 2: Concentration of the solute in the fluid for fixed distance 200 meters with initial concentration $\tilde{10} = [9.5, 10.5]$ mg / l

$\tilde{u}t = [20, 60]$, $\tilde{D}t = [200, 600]$, $\sqrt{[200, 600]} = [14.1421, 24.4949]$
$2[14.1421, 24.4949] = [28.2841, 48.9899]$, $\dfrac{1}{[28.2841, 48.9899]} = [0.0204, 0.0314]$
and $\dfrac{C_0}{2} = [9.5, 10.5][0.5, 0.5] = [4.7499, 5.2501]$.
Substituting these values in equation (3.12), we get

$$\tilde{C}(x,t) = [4.7499, 5.2501]\ [\text{erfc}[\text{x} - [20, 60][0.0204, 0.0314]]]$$

5. Results and discussion

In this study, we have proposed a new method for the analytical solution of one dimensional solute transport problem involving interval parameters with out converting them to classical models. As the input parameters of the given problems are interval numbers, the concentration of the solute will also be in the form of interval numbers. Figure 1 shows that the Concentration level of the solute in the fluid for fixed distance 200 meters with initial concentration as $10 = [10,\ 10]\ mg/l$ which is a degenerate interval. Figure 2 shows that the Concentration level of the solute in the fluid for fixed distance 200 meters with initial concentration as $[9.5,\ 10.5]\ mg/l$ which is a non-degenerate interval.

Though the initial concentration for the above two graphs different (degenerate and non degenerate), the concentration level of the solute in the fluid remains same. The solution of the above problems may help us to determine the position and time to reach the minimum/maximum or harmless concentration. It may be used as the preliminary predictive tools in groundwater management.

Table 1: Concentration of the solute in the fluid for fixed distance 200 meters with initial concentration [10, 10] mg /*l*

S.No	Distance $X(m) = [x_1,\ x_2]$	Concentration of the Solute $[t_1,\ t_2]$
1	[0, 0]	[10 , 10]
2	[9 ,11]	[6.0243, 9.7370]
3	[19, 21]	[4.8229, 9.3442]
4	[29, 31]	[3.1260, 8.8897]
5	[39, 41]	[1.7553, 8.2447]
6	[49, 51]	[1.1103, 6.8740]
7	[59, 61]	[0.6558, 5.1771]
8	[69, 71]	[0.2630, 3.9757]
9	[79, 81]	[0.0860, 2.9180]
10	[89, 91]	[0.0235, 2.0140]
11	[99, 101]	[0.0053, 1.3027]
12	[109, 111]	[0.0010, 0.7874]
13	[119, 121]	[0.0001, 0.4437]
14	[129, 131]	$[2.0881e^{-005},\ 0.2326]$
15	[139, 141]	$[2.2501e^{-006}, 0.1133]$
16	[149, 151]	$[2.0035e^{-007}, 0.0512]$
17	[159, 161]	$[1.4728e^{-008}, 0.0214]$
18	[169, 171]	$[8.9317e^{-010}, 0.0083]$
19	[179, 181]	$[4.4663e^{-011}, 0.0030]$
20	[189, 191]	$[1.8406e^{-012}, 9.8965e^{-004}]$
21	[199, 201]	$[6.2487e^{-014}, 3.0341e^{-004}]$

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Table 2: Concentration of the solute in the fluid for fixed distance 200 meters with initial concentration [9.5, 10.5] mg /*l*

S.No	Distance $X(m) = [x_1,\ x_2]$	Concentration of the Solute $[t_1,\ t_2]$
1	[0, 0]	[9.5, 10.5]
2	[9, 11]	[5.7229, 10.0383]
3	[19, 21]	[4.5816, 9.5855]
4	[29, 31]	[2.9697, 9.0460]
5	[39, 41]	[1.6675, 8.3325]
6	[49, 51]	[1.0548, 6.9295]
7	[59, 61]	[0.6230, 5.2100]
8	[69, 71]	[0.2499, 3.9889]
9	[79, 81]	[0.0817, 2.9223]
10	[89, 91]	[0.0223, 2.0152]
11	[99, 101]	[0.0051, 1.3029]
12	[109, 111]	$[9.6791e^{-004},\ 0.7874]$
13	[119, 121]	$[1.5226e^{-004},\ 0.4436]$
14	[129, 131]	$[1.9836e^{-005},\ 0.2326]$
15	[139, 141]	$[2.1376e^{-006},\ 0.1133]$
16	[149, 151]	$[1.9033e^{-007},\ 0.0512]$
17	[159, 161]	$[1.3991e^{-008},\ 0.0214]$
18	[169, 171]	$[8.4849e^{-010},\ 0.0083]$
19	[179, 181]	$[4.2429e^{-011},\ 0.0030]$
20	[189, 191]	$[1.7485e^{-012},\ 9.8965e^{-004}]$
21	[199, 201]	$[5.9362e^{-014},\ 3.0341e^{-004}]$

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