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Structure of generalized bivariate Lomax distribution based on dependence and information measures

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ABSTRACT

In this paper, we study the dependence structure of the generalized bivariate Lomax family. We also obtain some association measures and the local dependence function due to Kotz and Nadarajah [25]. In addition, we derive entropy, mutual information and quadratic mutual information measures for this family and discuss about them. Furthermore, we compare local dependence function and Pearson's ρ via a numerical study.

Keywords: Copula function; Dependence measures; Generalized bivariate Lomax distribution; Entropy; Mutual information; Quadratic mutual information; Local dependence function; Heavy tailed distributions.

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1. Introduction and Preliminaries

In recent years, a number of studies in economics, finance, networking and some other sciences like these, have focused on dependence measuring, modeling, information measuring and heavy tailed distributions. In addition, the bivariate measures of dependence and the copula based approaches to dependence modeling are two interrelated parts of the study of dependence structure of bivariate distributions in mathematical statistics and probability theory. Many authors studied the dependence structures of some bivariate distributions, among them, Shaked [37] has presented some concepts of dependence for bivariate distributions, Schweizer and Wolff [36] obtained nonparametric measures of dependence for random variables. Apparently, some authors wrote useful papers in the field of dependence via computing well-known dependence measures for some bivariate distributions. For example, Bairamove and Kotz [2] studied the dependence structure of Farlie-Gumbel-Morgenstern distributions and their extensions, Nadarajah et al. [28] determined the local dependence function for extreme value distributions in view of Kotz and Nadarajah's [25] Local dependence function. A new measure of linear local dependence has been obtained by Bairamov et al. [3], Tavangar and Asadi [39] extended of the linear local dependence due to Bairamov et al. [2] and studied some of its properties, Sankaran and Gupta [35] studied the properties of the local dependence function that introduced by Holland and Wang [19]. Also, Xie et al. [40] discussed some association measures and their collapsibility. Moreover, they presented characterizations for bivariate Lomax distribution, bivariate Dirichlet distribution, and bivariate normal distribution using local function and regression functions. Asadian et al. [1] investigated aspects of dependence in Lomax family. Cuadras and Auge [12] introduced the family of bivariate distributions and investigated dependence structure of it, Cuadras [11] derived dependence measures, Kendall's tau and Spearman's rho in Cuadras-Auge family and expanded this bivariate distribution in terms of

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Fréchet-Hoeffding lower and upper bounds. Ruiz-Rivas and Cuadras [34] studied some geometrical, probabilistic and statistical properties of the Cuadras-Ague family and Bolbolian et al. [7] studied dependence structure of Cuadras-Auge family via some measure of association and tail dependence coefficients. Genest [17] considered the dependence structure of Frank's family of bivariate distribution, also suggested three nonparametric estimator of the association parameter and compared their small-sample behavior of the Maximum likelihood estimator for this family. The other way of determining the measure of dependence between two random variables is using the information theory. Some measures such as entropy, mutual information and quadratic mutual information play important role in dependence measuring of bivariate distributions and some papers have written in this subject. For example, Joe [22] has presented the relative entropy measures of multivariate dependence and Bell [5] has used mutual information as a measure of dependence. Xu and Principe [41] discussed a novel algorithm to train nonlinear mappers with information theoretic criteria (entropy and mutual information) directly from a training set. Also, the heavy tailed random variables and their asymptotic behaviors and applications have been extensively investigated in half past century by many authors. The heavy tailed random variables have very considerable role in some sciences like finance, insurance and economics and study the structures of the distributions for these random variables is one of the interesting topics for statisticians. In particular, study the dependence structure of tail of distribution of random variables is considerable. Frahm et al. [15] derived some properties of estimating the tail dependence coefficient. Caillault and Guegan [8] have introduced non-parametric estimators for upper and lower tail dependence and confidence intervals are obtained with a bootstrap method. Dobric and Schmid [13] estimated the lower tail dependence in bivariate copulas by nonparametric approach. Many other authors presented some papers in this case, for example, Juri and Wüthrich [24], Charpentier and Segers [9], Peng [32] and Zi-sheng et al. [42].

In view of these themes, we want to study the dependence structure of a family of bivariate distributions which contains the distributions with heavy tailed marginal distributions via some dependence structure and information measures.

Let (X, Y) be a random vector with the following survival distribution function:

$$\overline{F}(x,y) = (1 + a_1 x^{a_2} + b_1 y^{b_2})^{-p}, \quad x, y \geq 0, \tag{1.1}$$

where a_1, a_2, b_1, b_2 and p are positive real numbers. This family of bivariate distributions is an extended version of the bivariate Lomax distribution, that is called "generalized bivariate Lomax (GBL)". This class contains two main categories of the bivariate distributions such as:

- Bivariate Lomax distributions for $a_1 = a_2 = 1$, which is widely used in reliability theory (see for details, Nayak, [29], Nadarajah, [28] and Barlow and Proschan [4]).
- Bivariate Burr distributions for $b_1 = b_2 = 1$, that belongs to the class of heavy tailed distribution. It has many applications in finance, insurance and networking (see, Resnik, [33]).
- In general case, if Z has a gamma distribution with unit scale parameter and shape parameter $p > 0$, and V_1 and V_2 independent of Z are independent, exponentially distributed with mean $\frac{1}{a_1}$ and $\frac{1}{b_1}$ respectively, then $(X^{\frac{1}{a_2}}, Y^{\frac{1}{b_2}})$ has the survival function given in equation (1.1) and this distribution is known as the bivariate Pareto of the fourth kind. (Kotz et al. [26]), Where $X = Z^{-1}V_1$ and $Y = Z^{-1}V_2$ for all $a_i, b_i > 0, i = 1, 2$.

The purpose of this paper is to examine the dependence structure of GBL family of distributions. In Section 2 some concepts of dependence for GBL family have presented. Some association measures such as tail dependence coefficients and extremal dependence coefficients are derived for this family in Section 3, and also we compare these coefficients by numerical approach in this section. In Section 4, we discuss the behavior of the local dependence via Bairamov and Kotz local dependence function and Clayton-Oakes association measure in GBL family. Furthermore, we draw graphs of this local dependence measure. In section 5, we compute three information measures in GBL family and evaluate these measures of information for this family. In some cases, we have used copula function, instead of distribution function, since computations are simple and the copula is independent from marginal distributions. We will also investigate dependence structure of this family via computing measures of information and dependency.

2. Some Concepts of Dependence

Let (X, Y) be a random vector with joint density function $f(x, y)$, distribution function $F(x, y)$ and marginals $F_1(x)$ and $F_2(y)$. Then, the following quantities are defined:

1. The real function $h(x, y)$ is totally positive of order two (TP_2) if $h(x, y) \geq 0$, and
$$h(x, y)h(x', y') \geq h(x, y')h(x', y), \quad for\ \ all\ \ x < x', y < y'.$$
2. The random vector (X, Y) is said to be positive likelihood ratio dependent ($PLRD(X, Y)$) if $f(x, y)$ is TP_2.
3. The random vector (X, Y) or its distribution function is said to be right corner set increasing ($RCSI(X, Y)$) if $P(X > x, Y > y|X > x', Y > y')$ is increasing in x' and y' for all x and y.
4. The random vector (X, Y) or its distribution function is said to be left corner set decreasing ($LCSD(X, Y)$) if $P(X \leq x, Y \leq y|X \leq x', Y \leq y')$ is decreasing in x' and y' for all x and y.
5. The random variable X is said to be stochastically increasing in Y ($SI(X|Y)$) if $P(X > x|Y = y)$ is increasing in y for all x.
6. The random variable X is said to be right tail increasing in Y ($RTI(X|Y)$) if $P(X > x|Y > y)$ is non-decreasing in y for all x.
7. The random variable X is said to be left tail decreasing in Y ($LTD(X|Y)$) if $P(X \leq x|Y \leq y)$ is non-increasing in y for all x.
8. The random variables X and Y are said to be positively quadrant dependent ($PQD(X, Y)$) if,
$$P(X > x, Y > y) \geq P(X > x)P(Y > y), \quad \forall x, y.$$
9. The copula function $C(u, v)$ is a bivariate distribution function with uniform marginals on $[0, 1]$, such that
$$F(x, y) = C_F(F_1(x), F_2(y))$$
By Sklar's Theorem (Sklar, [38]), this copula exists and is unique if F_1 and F_2 are continuous. Thus we can construct bivariate distributions $F(x, y) = C_F(F_1(x), F_2(y))$ with given univariate marginals F_1 and F_2 by using the copula C_F,(Nelsen, [30]).

Let the random vector (X, Y) has joint distribution function $F(x, y)$ be with marginals $F_1(x)$ and $F_2(y)$ respectively, then the following properties for copula functions given in Nelsen, [30]:

- The copula C_F is given by
$$C_F(u, v) = F(F_1^{-1}(u), F_2^{-1}(v)), \ \ \forall u, v \in [0, 1],$$
where, F_1^{-1} and F_2^{-1} are quasi-inverses of F_1 and F_2 respectively.
- The partial derivatives $\frac{\partial C_F(u,v)}{\partial u}$ and $\frac{\partial C_F(u,v)}{\partial v}$ exist and $c(u, v) = \frac{\partial^2 C_F(u,v)}{\partial u \partial v}$ is density function of $C_F(u, v)$.

- When the random variables X and Y are continuous, the concepts of dependence considered above are properties of the copula $C(u,v)$.

Noting that, the concept of PLRD is differs from other dependence concepts, because it is defined in terms of the joint density function. Moreover, PLRD is the strongest, implying all of the dependence concepts mentioned above. In terms of the copula of continuous random variables X and Y, we have The random vector (X,Y) is $PLRD$ if and only if

$$PLRD(X,Y) \Leftrightarrow \frac{c(u',v)}{c(u,v)} \ is\ increasing\ in\ v\ for\ all\ u<u'. \tag{2.1}$$

Remark 2.1. (i)- Let (X,Y) be a random vector with GBL distribution function, then, using relation (1.1), it is easy to see that,

$$C(u,v) = \left[(1-u)^{-\frac{1}{p}} + (1-v)^{-\frac{1}{p}} - 1\right]^{-p} + u + v - 1, \quad p>0. \tag{2.2}$$

(ii)- We observe that the copula function of GBL family is independent of the parameters $a_i, b_i, i=1,2$, furthermore, it is equal to copula of bivariate Lomax family that has been studied by Asadian et al. [1].

Proposition 2.2. Let (X,Y) be a random vector with GBL distribution function, then (X,Y) is PLRD.

Proof. Using relation (2.2), we get,

$$\begin{aligned} c(u,v) &= \frac{\partial^2 C(u,v)}{\partial v \partial u} \\ &= \frac{p+1}{p}(1-v)^{-\frac{1}{p}-1}(1-u)^{-\frac{1}{p}-1}\left[(1-u)^{-\frac{1}{p}} + (1-v)^{-\frac{1}{p}} - 1\right]^{-p-2}, \end{aligned}$$

So, we have

$$\frac{c(u',v)}{c(u,v)} = \left(\frac{1-u}{1-u'}\right)^{\frac{p+1}{p}} \left[1 - \frac{(1-u')^{\frac{-1}{p}} - (1-u)^{\frac{-1}{p}}}{(1-u')^{\frac{-1}{p}} + (1-v)^{\frac{-1}{p}} - 1}\right]^{p+2}, \quad \forall\ u<u',\ v\in(0,1),$$

which is increasing function in v and this completes the proof. □

Remark 2.3. Let (X,Y) be a random vector with GBL distribution function, then by Theorem 5.2.19 in Nels, we have

$$PQD(X,Y),\ LTD(Y|X),\ RTI(Y|X),\ SI(Y|X),\ LCSD(X,Y),\ RCSI(X,Y).$$

3. Some measures of association

In this section, we compute measures of association, tail dependence coefficients and extremal dependence coefficients for GBL family.

3.1 *Kendall's τ and Spearman's ρ_s*

The Spearman's ρ_s is connected with PQD concept and formulated by copula function C as follows:

$$\begin{aligned} \rho_s &= 12\int_0^1\int_0^1 [C(u,v) - uv]dudv \\ &= 12\int_0^1\int_0^1 C(u,v)dudv - 3. \end{aligned}$$

The Kendall's τ is connected with PLRD concepts and formulated by copula function C as:

$$\tau = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1 = 1 - 4\int_0^1\int_0^1 \left[\frac{\partial C}{\partial u}.\frac{\partial C}{\partial v}\right] dudv.$$

These measures are obtained for bivariate Lomax family by Asadian et al. [1], and since the copula of GBL family is equal to copula of bivariate Lomax family, so, the following proposition is valid for GBL family.

Proposition 3.1. Let (X,Y) be a random vector with GBL distribution function, then,

(i). $\tau = \frac{1}{2p+1}$,

(ii). $\rho_s = \sum_{k=0}^{\infty} \frac{12p^2(k+1)}{2p+k} B(3p, k+1) - 3.$

Where, $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the Beta function and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$.

Remark 3.2. Let (X,Y) be a random vector with GBL distribution function, then, Proposition 2.3, Theorem 5.1 in [16] and [20] [Exercise, 5.38 in Nelsen [30]] imply that

$$0 \le \tau \le \rho_s \le \frac{3}{2}\tau.$$

3.2 *The Blomqvist medial coefficient*

Blomqvist, [6] introduced this coefficient as evaluating the dependence at the center of a distribution. If X and Y are random variables with copula function $C(u,v)$, then the coefficient of Blomqvist is defined as:

$$\beta = 4C(\frac{1}{2},\frac{1}{2}) - 1.$$

Corollary 3.1. *Let (X,Y) be a random vector with GBL distribution function, then, $\beta = 4(2^{(p+1)/p} - 1)^{-p} - 1$. It is obvious that $0 \le \beta \le 1$ for all $p > 0$.*

3.3 **Schweizer-Wolff's index of dependence**

An index closely related to Spearmans ρ_s is the index σ_{XY} introduced by Schweizer and Wolff [36]. Instead of considering the difference $C(u,v) - uv$, they use its absolute value to define:

$$\sigma_{XY} = 12\int_0^1\int_0^1 |C(u,v) - uv| dudv,$$

which is a measure of the volume between the surfaces $C(u,v)$ and uv.

Corollary 3.2. *Let (X,Y) be a random vector with GBL distribution function, then Proposition 2.2 implies that $C(u,v) \ge uv$, consequently $\sigma_{XY} = \rho_s$.*

3.4 **Gini's gamma coefficient**

The Gini's γ coefficient is defined as

$$\gamma_C = 2\int_0^1\int_0^1 (|u+v-1| - |u-v|)dC(u,v).$$

Another form of Gini's γ is given by

$$\gamma_C = 4\left\{\int_0^1 C(u,1-u)du - \int_0^1 [u - C(u,u)]du\right\}.$$

(For more details see, Nelsen, [30]).

Corollary 3.3. Let (X, Y) be a random vector with GBL distribution function, then,

$$\gamma_C = 4\left[\int_0^1 \left([(1-u)^{\frac{-1}{p}} + u^{\frac{-1}{p}} - 1]^{-p} + [2(1-u)^{\frac{-1}{p}} - 1]^{-p}\right) du\right] - 2,$$

for all $p > 0$.

Remark 3.4. Since there is not closed form for the Gini's gamma coefficient in GBL family, we study and analysis of it with numerical approach. Table 3.1 presents Gini's gamma coefficient and Spearman's ρ (or σ_{XY}) for GBL family for some values of p. It is easy to see that $\rho_s = \sigma_{XY} \geq \gamma_C$ for all values of p.

Table 1: γ_C and ρ_s for some values of p in GBL family.

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
γ_C	.877	.776	.692	.622	.565	.516	.475	.439	.409	.382
$\rho_s = \sigma_{XY}$	.952	.885	.811	.743	.683	.631	.585	.545	.510	.479
p	2	3	4	5	6	7	8	9	10	20
γ_C	.230	.164	.128	.104	.088	.076	.067	.060	.055	.027
$\rho_s = \sigma_{XY}$	.295	.212	.166	.136	.115	.099	.088	.079	.071	.037
p	30	40	50	60	70	80	90	100	...	1000
γ_C	.018	.014	.011	.009	.008	.007	.006	.005	...	.0006
$\rho_s = \sigma_{XY}$	.025	.018	.015	.012	.011	.009	.008	.007	...	.0007

3.5 *Tail dependence coefficients*

Inspect of other dependence measures the tail dependence coefficients explain dependence between the random variables in the upper right quadrant and in the lower left quadrant of $[0, 1] \times [0, 1]$. Let (X, Y) be a random vector with joint distribution function $F(x, y)$ and marginals $F_1(x)$ and $F_2(y)$, respectively. The quantity $\lambda_u = \lim\limits_{t\to 1^-} P\left(F_1(X) > t|F_2(Y) > t\right)$ is called the upper tail dependence coefficient (UTDC) provided the limit exists. We say that (X, Y) has upper tail dependence if $\lambda_u > 0$ and upper tail independent if $\lambda_u = 0$. Similarly, we define the lower tail dependence coefficient (LTDC) by $\lambda_l = \lim\limits_{t\to 0^+} P\left(F_1(X) \leq t|F_2(Y) \leq t\right)$. The upper tail dependence coefficient (or lower tail dependence coefficient) can also be defined via the notion of copula as:

$$\lambda_u = \lim_{t\to 1^-} \frac{1 - 2u + C(u, u)}{1 - u} \quad and \quad \lambda_l = \lim_{t\to 0^+} \frac{C(u, u)}{u}.$$

For more details, [30].

Corollary 3.5. Let (X, Y) be a random vector with GBL distribution function. Then, $\lambda_u = 2^{-p}$. and $\lambda_l = 0$.

3.6 *Extremal dependence coefficients*

Extremal dependence coefficients were introduced by Frahm [14] for studying the asymptotic dependence structure of the minimum and the maximum of a random vector. Let $(X_1, X_2, ..., X_n)$ be a random vector with joint distribution function $F(x_1, x_2, ..., x_n)$ and marginal distribution functions $F_1, ..., F_n$.The lower extremal dependence coefficient (LEDC) and upper extremal dependence coefficient (UEDC) of $(X_1, X_2, ..., X_n)$ are defined as

$$E_l = \lim_{t\to 0^+} P(F_{max} \leq t|F_{min} \leq t),$$

and

$$E_u = \lim_{t\to 1^-} P(F_{min} > t|F_{max} > t),$$

provided the corresponding limits exist, where, $F_{\min} = \min\{F_1(X_1), ..., F_n(X_n)\}$ and $F_{\max} = \max\{F_1(X_1), ..., F_n(X_n)\}$.

Remark 3.1. By Proposition 1 in Frahm (2006), we can derive E_l and E_u via the quantities λ_l and λ_u as: $E_l = \frac{\lambda_l}{2-\lambda_l}$ *and* $E_u = \frac{\lambda_u}{2-\lambda_u}$. Therefore, if (X, Y) has the GBL distribution, then, the Corollary 3.5 implies that

$$E_l = 0, \quad and \qquad E_u = \frac{1}{2^{p+1}+1} > 0.$$

This means that (X, Y) has UED but not LED.

Remark 3.2. Since we have closed forms expression for τ, β, λ_u and E_u for the GBL family, it is possible to compare analytically these measures. Figure 1 contains graphs of these functions with comparing them for some values of p:

(i). If $0 < p < 1$, then, $E_u < \tau < \beta < \lambda_u$.

(ii). If $1 \leq p < 2.6598$, then, $E_u < \beta \leq \tau < \lambda_u$.

(iii). If $2.6598 < p < 2.7211$, then, $E_u < \beta < \lambda_u < \tau$.

(iv). If $p > 2.7211$, then, $E_u < \lambda_u < \beta < \tau$.

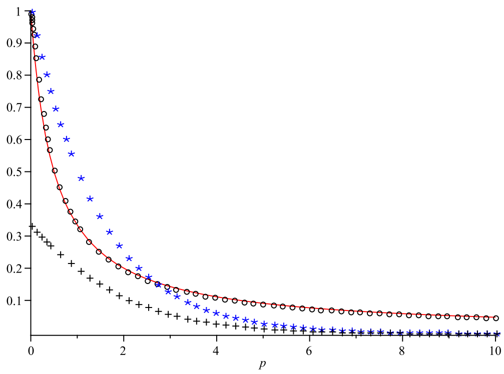

Figure 1: Comparing the dependence coefficients for some values of p, where λ_u shows by " $*$ ", E_u by " $\times$ ", β by " $\circ$ " and τ by line.

4. Local Dependence

We compute the Clayton-Oakes association measure (denoted by $\Theta(x, y)$) and local dependence function due to Kotz and Nadarajah [25] (denoted by $H(x, y)$) in GBL distribution function. Furthermore, we investigate the behavior of these measures drawing their graphs.

4.1 *Clayton-Oakes Association Measure*

Clayton [10] and Oakes [31] defined the association measure as:

$$\theta(x, y) = \frac{\bar{F}(x, y) D_{12}\bar{F}(x, y)}{D_1\bar{F}(x, y) D_2\bar{F}(x, y)},$$

where,

$$D_{12}\bar{F}(x, y) = \frac{\partial^2 \bar{F}(x, y)}{\partial x \partial y}; \; D_1\bar{F}(x, y) = \frac{\partial \bar{F}(x, y)}{\partial x}; \; D_2\bar{F}(x, y) = \frac{\partial \bar{F}(x, y)}{\partial y}.$$

If $\theta(x,y) > (<)1$, we say X and Y are positive dependent(negative dependent). Consequently, if $\theta(x,y) = 1$, X and Y being independent. Gupta [18] proved that $\theta(x,y) = r(x|Y=y)/r(x|Y>y)$, where $r(x|Y=y) = \frac{f(x|y)}{\bar{F}(x|y)}$ and $r(x|Y>y)$ is the hazard rate of the conditional distribution of X given $Y > y$. Using this result, we obtain the following Proposition:

Proposition 4.1. Let (X,Y) be a random vector with GBL distribution function, then,

$$\theta(x,y) = \frac{p+1}{p}.$$

Proof. Via the arguments in Gupta [18], we have,

$$\begin{aligned} r(x|Y=y) &= -\frac{D_{12}\bar{F}(x,y)}{D_2\bar{F}(x,y)} \\ &= a_1a_2(p+1)x^{a_2-1}(1+a_1x_{a_2}+b_1y^{b_2})^{-1}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} r(x|Y>y) &= -\frac{D_1\bar{F}(x,y)}{\bar{F}(x,y)} \\ &= a_1a_2px^{a_2-1}(1+a_1x_{a_2}+b_1y^{b_2})^{-1}, \end{aligned} \tag{4.2}$$

for every real value of x and y. So,

$$\theta(x,y) = \frac{r(x|Y=y)}{r(x|Y>y)} = \frac{p+1}{p},$$

□

4.2 *Local dependence function*

Kotz and Nadarajah [25] have introduced a local dependence function (denoted by $H(x,y)$), which provides a local point of view on dependence at a point (x,y) and defined

$$H(x,y) = \frac{E\{(X-E(X|Y=y))(Y-E(Y|X=x))\}}{\sqrt{E(X-E(X|Y=y))^2}\sqrt{E(Y-E(Y|X=x))^2}},$$

which is obtained from the expression of the Pearson correlation coefficient by replacing mathematical expectations $E(X)$ and $E(Y)$ by conditional expectations $E(X|Y=y)$ and $E(Y|X=x)$, respectively. So, we have,

$$H(x,y) = \frac{\rho+\phi_X(y)\phi_Y(x)}{\sqrt{1+\phi_X^2(y)}\sqrt{1+\phi_Y^2(x)}}, \tag{4.3}$$

where,

$$\rho = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}};\ \phi_X(y) = \frac{E(X)-E(X|Y)}{\sqrt{var(X)}};\ and\ \phi_Y(x) = \frac{E(Y)-E(Y|X)}{\sqrt{var(Y)}}.$$

Proposition 4.2. Let (X,Y) be a random vector with GBL distribution function, then,

$$\phi_X(y) = \frac{B(p-\frac{1}{a_2},1+\frac{1}{a_2})\left[p-(p-\frac{1}{a_2})(1+b_1y^{b_2})^{\frac{1}{a_2}}\right]}{\sqrt{p}\left[B(p-\frac{2}{a_2},1+\frac{2}{a_2})-pB^2(p-\frac{1}{a_2},1+\frac{1}{a_2})\right]^{\frac{1}{2}}},$$

Table 2: Computes of (x^*, y^*)

p	ρ	x^*	y^*	$\Theta(x^*, y^*)$
1	1.000	∞	∞	2.000
2	0.500	1.000	1.000	1.500
3	0.333	0.500	0.500	1.333
4	0.250	0.333	0.333	1.250
5	0.200	0.250	0.250	1.200
6	0.167	0.200	0.200	1.167
7	0.143	0.167	0.167	1.143
8	0.125	0.143	0.143	1.125
9	0.111	0.125	0.125	1.111
10	0.100	0.111	0.111	1.100

$$\phi_Y(x) = \frac{B(p-\frac{1}{b_2}, 1+\frac{1}{b_2})\left[p-(p-\frac{1}{b_2})(1+a_1x^{a_2})^{\frac{1}{b_2}}\right]}{\sqrt{p}\left[B(p-\frac{2}{b_2}, 1+\frac{2}{b_2}) - pB^2(p-\frac{1}{b_2}, 1+\frac{1}{b_2})\right]^{\frac{1}{2}}},$$

and

$$\rho = \frac{\left[(p+1)B(p-\frac{1}{a_2}-\frac{1}{b_2}, 1+\frac{1}{a_2}, 1+\frac{1}{b_2}) - pB(p-\frac{1}{a_2}, 1+\frac{1}{a_2})B(p-\frac{1}{b_2}, 1+\frac{1}{b_2})\right]}{\left[\left(B(p-\frac{2}{a_2}, 1+\frac{2}{a_2}) - pB^2(p-\frac{1}{a_2}, 1+\frac{1}{a_2})\right)\left(B(p-\frac{2}{b_2}, 1+\frac{2}{b_2}) - pB^2(p-\frac{1}{b_2}, 1+\frac{1}{b_2})\right)\right]^{\frac{1}{2}}}.$$

By substituting theses relations in (4.3), $H(x, y)$ can be obtained, on noting that $B(a, b, c) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$. and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$.

Corollary 4.3. In particular if $a_1 = a_2 = b_1 = b_2 = 1$, $H(x, y)$ in GBL distribution function is,

$$H(x, y) = \frac{p^3 + (p-2)^2[x(1-p)+1][y(1-p)+1]}{\sqrt{[p^4 + (p-2)^2(x(1-p)+1)^2][p^4 + (p-2)^2(y(1-p)+1)^2]}}.$$

As we see, $H(x, y)$ is a decreasing function of p and tends to infinity when p tends to infinity. Figure 2 shows the behavior of $H(x, y)$ for values of $p = 1, 2, 4, 6, 10, 25$, under the conditions of the Corollary 4.4.

Remark 4.4. We compute the saddle point (x^*, y^*) such that $H(x^*, y^*) = \rho$ when $a_1 = a_2 = b_1 = b_2 = 1$ in GBL family. By solving the equation $\phi_X(y^*) = \phi_Y(x^*) = 0$ analytically, we obtain,

$$(x^*, y^*) = \left(\frac{1}{p-1}, \frac{1}{p-1}\right).$$

As we see, $H(x, y)$ is equal to Pearson's correlation coefficient ρ when X and Y are equal to their expectation. It is easy to see that in this case, $H(x^*, y^*) = \rho = \frac{1}{p}$.

In order to compar Pearson's ρ, $\Theta(x, y)$ and $H(x, y)$ for some values of p and (x^*, y^*), Table 2.1 presents ρ, and $\Theta(x^*, y^*)$.

5. Some Information measures

In this section, we derive three information measures for GBL distribution function. Also, we study behavior of these measures via a numerical study:

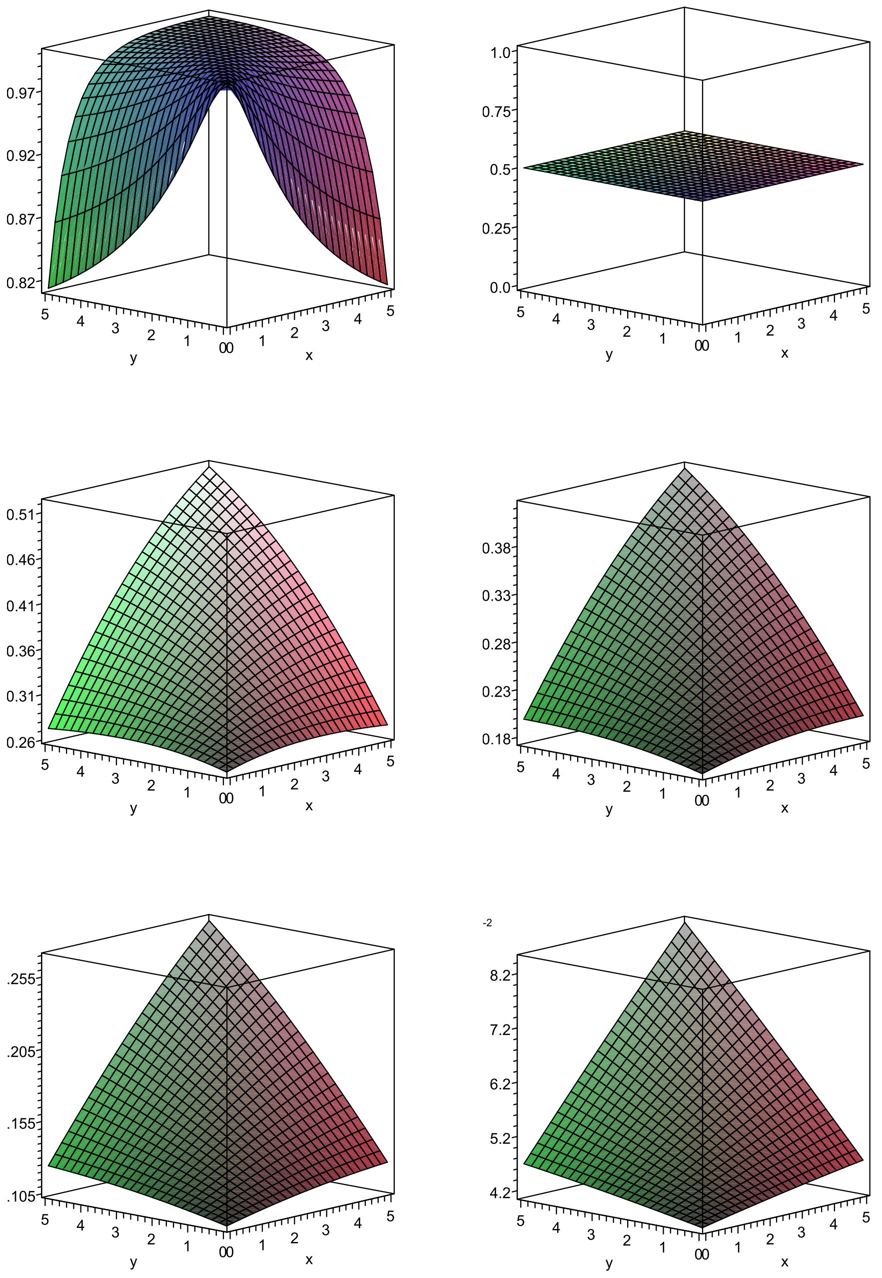

Figure 2: The diagram of $H(X,Y)$ for different values of p. Top left: $p=1$, Top right: $p=2$, Middle left: $p=4$, Middle right: $p=6$, Bottom left: $p=10$ and Bottom right $p=25$.

Table 3: Values of $H_e(X,Y)$ for some values of p

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$H_e(X,Y)$	25.12	14.26	10.38	8.29	6.95	6.00	5.27	4.69	4.21
p	1	2	3	**3.55**	4	5	6	7	8
$H_e(X,Y)$	3.81	1.54	0.43	**0**	−0.30	−0.83	−1.26	−1.61	−1.92
p	9	10	11	15	20	25	30	35	40
$H_e(X,Y)$	−2.18	−3.28	−3.89	−4.36	−4.74	−5.05	−5.33	−5.57	−2.41

5.1 *Entropy*

If (X,Y) is a random vector with the joint density function $f(x,y)$, the joint entropy for two random variables X and Y is

$$H_e(X,Y) = -E[\log(f(X,Y))].$$

This measure is maximum, when X and Y are independent and if X and Y are dependent random variables, then H_e is a real number, (see, Joe, [21–23]).

Proposition 5.1. Let (X,Y) be a random vector with GBL distribution function, then,

$$H_e(X,Y) = \frac{a_2-1}{b_1b_2a_2}[\ln a_1 + c_1(p)] + \frac{b_2-1}{a_1a_2b_2}[\ln b_1 + c_1(p)] - c_2(p) - \ln(A),$$

where, $A = a_1a_2b_1b_2p(p+1)$ and $c_i(p); i = 1,2$ are functions of p. For the forms of $c_i(p), i = 1,2$ see appendices.

Corollary 5.2. If $a_1 = a_2 = b_1 = b_2 = 1$ then, the entropy of GBL distribution function is

$$\begin{aligned} H_e(X,Y) &= \log(p(p+1)) + \sum_{i=0}^{\infty} \frac{p+2}{\Gamma(p)\Gamma(i+1)}[\Gamma(p+i)(\Psi(1+i)-\Psi(p+i)) \\ &\quad +\Gamma(p+i+1)(\Psi(1+i)-\Psi(p+i+1))], \end{aligned} \tag{5.1}$$

where, $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$.

Table 2.2 and Figure 3 show the behavior of $H_e(X,Y)$ with respect to p increasing, we observe that H_e is positive for $p < 3.55$, zero for $p \cong 3.55$ and negative for $p < 3.55$.

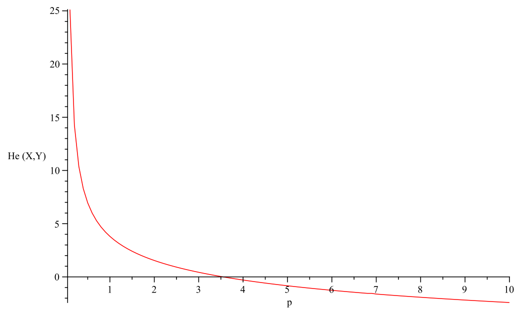

Figure 3: $H_e(X,Y)$ for some values of p.

Table 4: Values of $I(X,Y)$ and $\delta(X,Y)$ for some values of p

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$I(X,Y)$	1.49	.96	.70	.54	.43	.36	.30	.26	.22
$\delta(X,Y)$	.97	.92	.87	.81	.76	.71	.67	.63	.60
p	1	2	3	4	5	6	7	8	9
$I(X,Y)$	.19	.072	.038	.023	.016	.011	.009	.007	.005
$\delta(X,Y)$	.56	.37	.27	.21	.18	.15	.13	.12	.10
p	10	15	20	25	30	35	40	...	
$I(X,Y)$	.004	.002	.001	.0007	.0005	.0004	.0003	...	
$\delta(X,Y)$	.09	.063	.048	.039	.032	.028	.025	...	

5.2 *Mutual Information*

Mutual information measures are the amount of information that can be obtained about one random variable by observing another. It is important in communication where it can be used to maximize the amount of information shared between sent and received signals. The mutual information of X relative to Y with joint density function $f(x,y)$ and marginal density functions $f_1(x)$ and $f_2(y)$, respectively is given by:

$$I(X,Y) = E\left[\log\left(\frac{f(X,Y)}{f_1(X)f_2(Y)}\right)\right] = H_e(X) + H_e(Y) - H_e(X,Y). \tag{5.2}$$

Proposition 5.3. Let (X,Y) be a random vector with GBL distribution function, then,

$$\begin{aligned} I(X,Y) \;=\; & \frac{a_2-1}{a_2}\left[\ln a_1\left(1-\frac{1}{b_1b_2}\right)+\psi(p)+\gamma-\frac{c_1(p)}{b_1b_2}\right] \\ & +\frac{b_2-1}{b_2}\left[\ln b_1\left(1-\frac{1}{a_1a_2}\right)+\psi(p)+\gamma-\frac{c_1(p)}{a_1a_2}\right] \\ & +c_3(p)-\log(a_1a_2)-\log(b_1b_2), \end{aligned}$$

where $c_1(p)$ and $c_3(p)$ are functions of p, see appendices.

Corollary 5.4. If $a_1 = a_2 = b_1 = b_2 = 1$, then,

$$H_e(X) = H_e(Y) = \frac{p+1}{p} - \log(p).$$

Using (5.1), and (5.2), we get

$$\begin{aligned} I(X,Y) \;=\; & \frac{2p-2}{p} - \log(p^3(p+1)) + \sum_{i=0}^{\infty}\frac{p+2}{\Gamma(p)\Gamma(i+1)}[\Gamma(p+i)(\Psi(1+i)-\Psi(p+i)) \\ & +\Gamma(p+i+1)(\Psi(1+i)-\Psi(p+i+1))]. \end{aligned} \tag{5.3}$$

If the components of (X,Y) are independent, then $I(X,Y)$ is zero and conversely, when the dependence is maximal, $I(X,Y)$ tends to infinity. Joe [21] defined $\delta(X,Y) = \sqrt{1-\exp(-2I(X,Y))}$ which is normalizing this index.

The measure of δ is confined to the interval $[0,1]$. If X and Y are independent then, $\delta = 0$ and when the dependence is maximal, δ achieves to one.

Our numerical results in Table 2.3 show that when p is decreasing the measures I and δ are increasing and when p is increasing I and δ decreasing. Note that, we observed in sections 3 and 4 that the measures τ, ρ and θ, increase with decreasing of p. It means that, we have more information and dependence in this case.

5.3 *Quadratic Mutual Information*

Let X and Y be two random variables with marginal density functions $f_1(x)$ and $f_2(y)$ and joint density function $f(x,y)$, the mutual information between two random variables can be estimated by Kullack-Leibler divergence between the joint density function and the factored marginals. By using quadratic forms of density functions, Xu and Principe (1998), proposed the following distance based on the Cauchy-Schwarz inequality:

$$C(X,Y) = \log \frac{\left(\int_0^\infty \int_0^\infty f^2(x,y)dxdy\right)\left(\int_0^\infty \int_0^\infty f^2(x)f^2(y)dxdy\right)}{\left(\int_0^\infty \int_0^\infty f(x,y)f(x)f(y)dxdy\right)^2}.$$

It is obvious that $C(X,Y) \geq 0$ and $C(X,Y) = 0$ if and only if X and Y are independent. So, $C(X,Y)$ is an appropriate measure for the independence of two random variables (minimization of mutual information). Although, it is difficult to prove a strict justification that $C(X,Y)$ is appropriate to measure dependence. We compute this measure for GBL family and then, study the behavior of it via a numerical study and drawing it's graph.

Proposition 5.5. Let (X,Y) be a random vector with GBL distribution function, then,

$$C(X,Y) = \log(C_1) + \log(C_2) - 2\log(C_3),$$

where,

$$C_1 = Dp^2(p+1)^2 B(2p + \frac{1}{a_2} + \frac{1}{b_2}, 2 - \frac{1}{a_2}, 2 - \frac{1}{b_2}),$$

$$C_2 = Dp^4 B(2p + \frac{1}{a_2}, 2 - \frac{1}{a_2}) B(2p + \frac{1}{b_2}, 2 - \frac{1}{b_2}),$$

$$C_3 = Dp^3(p+1)\left[\frac{B(2-\frac{1}{a_2}, p+\frac{1}{a_2})\Gamma(1-p-\frac{1}{a_2})}{\Gamma(p+1)\Gamma(2-\frac{1}{a_2})}\sum_{i=1}^{\infty} C_4(i) + \frac{B(2p+1+\frac{1}{a_2}, -p-\frac{1}{a_2})\Gamma(1+p+\frac{1}{a_2})}{\Gamma(p+2)\Gamma(2p+\frac{1}{a_2}+1)}\sum_{i=1}^{\infty} C_5(i)\right],$$

and

$$C_4(i) = \frac{B(2p+\frac{2}{b_2}+i-1, 2-\frac{1}{b_2})\Gamma(p+1+i)\Gamma(2-\frac{1}{a_2}+i)}{i!\Gamma(1-p-\frac{1}{a_2}+i)},$$

$$C_5(i) = \frac{B(p+i+\frac{1}{b_2}-1, 2-\frac{1}{b_2})\Gamma(p+2+i)\Gamma(2p+1+\frac{1}{a_2}+i)}{i!\Gamma(1+p+\frac{1}{a_2}+i)},$$

$$D = a_2 b_2 a_1^{\frac{1}{a_2}} b_1^{\frac{1}{b_2}}.$$

Corollary 5.6. If $a_1 = a_2 = b_1 = b_2 = 1$, then,

$$C(X,Y) = \log\left(\frac{p^2(p+1)^2}{(2p+3)(2p+2)}\right) + 2\log\left(\frac{p^2}{2p+1}\right) - 2\log\left[\frac{p^3\Gamma(-p)}{\Gamma(p+1)}\sum_{i=0}^{\infty}\frac{\Gamma(p+1+i)}{(2p+i)\Gamma(i-p)} + p^3\Gamma(-p-2)\sum_{i=0}^{\infty}\frac{\Gamma(2p+2+i)}{(p+i+1)i!}\right].$$

Under the assumptions of Corollary 5.6, we computed $C(X,Y)$ for some values of p in Table 2.4, also Figure 4 shows the behavior of $C(X,Y)$ with respect to p. Table 2.4 and Figure 4 show that, increasing p the quadratic mutual information decrease, this agree with behavior of I and δ.

Table 5: Values of $C(X,Y)$ for some values of p

p	1	2	3	4	5	6	7	8	9	10
$C(X,Y)$	.0542	.0227	.0122	.0078	.0044	.0033	.0026	.0020	.0016	.0014

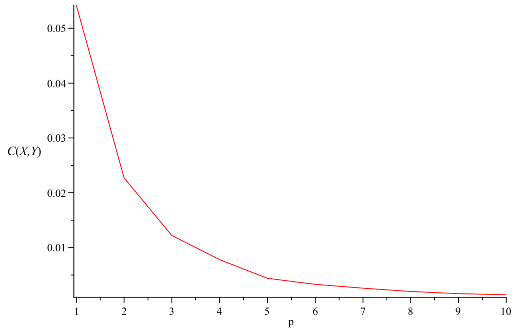

Figure 4: $C(X,Y)$ for some values of p.

6. Conclusion

In this paper, the dependence structure of the GBL family has been studied via dependence coefficients and information coefficients. We show that X and Y in GBL family have positive dependence and this dependency will be weaker as p goes to be larger. In information measure, terms are similar, in fact when p tends to infinity, the measure of information between two random variables tends to be negligible. Also, we find links between the properties of GBL family via information measures and dependency.

7. Appendices

In this section, we present proof of the propositions.

***Proof of Proposition 4.2*:**

$$E(X) = \int_0^\infty a_1 a_2 p x^{a_2}(1+a_1x^{a_2})^{-p-1}dx = pa_1^{\frac{-1}{a_2}} B(p-\frac{1}{a_2}, 1+\frac{1}{a_2}),$$

and,

$$E(X^2) = pa_1^{\frac{-2}{a_2}} B(p-\frac{2}{a_2}, 1+\frac{2}{a_2}).$$

We have,

$$var(X) = pa_1^{\frac{-2}{a_2}}\left[B(p-\frac{2}{a_2}, 1+\frac{2}{a_2}) - pB^2(p+\frac{1}{a_2}, 1+\frac{1}{a_2})\right].$$

Moreover,

$$E(X|Y=y) = \int_0^\infty x.\frac{f(x,y)}{f_1(y)}dx = (p+1)a_1^{\frac{-1}{a_2}}(1+b_1y^{b_2})^{\frac{1}{a_2}} B(p+1-\frac{1}{a_2}, 1+\frac{1}{a_2}).$$

Hence, we get,

$$E(X) - E(X|Y=y) = a_1^{\frac{-1}{a_2}} B(p-\frac{1}{a_2}, 1+\frac{1}{a_2})\left[p-(p-\frac{1}{a_2})(1+b_1y^{b_2})^{\frac{1}{a_2}}\right].$$

So, by some mathematical calculation, we get,

$$\phi_X(y) = \frac{B(p-\frac{1}{a_2}, 1+\frac{1}{a_2})\left[p-(p-\frac{1}{a_2})(1+b_1y^{b_2})^{\frac{1}{a_2}}\right]}{\sqrt{p}\left[B(p-\frac{2}{a_2}, 1+\frac{2}{a_2}) - pB^2(p-\frac{1}{a_2}, 1+\frac{1}{a_2})\right]^{\frac{1}{2}}}.$$

Similarly, we obtain,

$$\phi_Y(x) = \frac{B(p-\frac{1}{b_2}, 1+\frac{1}{b_2})\left[p-(p-\frac{1}{b_2})(1+a_1x^{a_2})^{\frac{1}{b_2}}\right]}{\sqrt{p}\left[B(p-\frac{2}{b_2}, 1+\frac{2}{b_2}) - pB^2(p-\frac{1}{b_2}, 1+\frac{1}{b_2})\right]^{\frac{1}{2}}}.$$

On the other hand, we have

$$\begin{aligned} E(XY) &= \int_0^\infty \int_0^\infty xyf(x,y)dxdy = \frac{p(p+1)\Gamma(p-\frac{1}{a_2}-\frac{1}{b_2})\Gamma(1+\frac{1}{a_2})\Gamma(1+\frac{1}{b_2})}{a_1^{\frac{1}{a_2}}\Gamma(p+2)b_1^{\frac{1}{b_2}}} \\ &= a_1^{\frac{1}{a_2}} b_1^{\frac{1}{b_2}} p(p+1)B(p-\frac{1}{a_2}-\frac{1}{b_2}, 1+\frac{1}{a_2}, 1+\frac{1}{b_2}), \end{aligned}$$

where,

$$B(a,b,c) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}.$$

thus,

$$\begin{aligned} cov(X,Y) = a_1^{\frac{1}{a_2}} b_1^{\frac{1}{b_2}} p \Big[&(p+1)B(p-\frac{1}{a_2}-\frac{1}{b_2}, 1+\frac{1}{a_2}, 1+\frac{1}{b_2}) \\ &-pB(p+1, 1+\frac{1}{a_2})B(p+1, 1+\frac{1}{b_2})\Big], \end{aligned}$$

then,

$$\rho = \frac{\left[(p+1)B(p-\frac{1}{a_2}-\frac{1}{b_2}, 1+\frac{1}{a_2}, 1+\frac{1}{b_2}) - pB(p+1, 1+\frac{1}{a_2})B(p+1, 1+\frac{1}{b_2})\right]}{\left[\left(B(p+1, 1+\frac{2}{a_2}) - pB^2(p+1, 1+\frac{1}{a_2})\right)\left(B(p+1, 1+\frac{2}{b_2}) - pB^2(p+1, 1+\frac{1}{b_2})\right)\right]^{\frac{1}{2}}}.$$

These complete the proof.

Proof of Proposition 5.1: By definition of entropy function, we have,

$$\begin{aligned} H_e(X,Y) &= E\left[\log(f(X,Y))\right] \\ &= -\int_0^\infty \int_0^\infty \log\left(Ax^{a_2-1}y^{b_2-1}(1+a_1x^{a_2}b_1y^{b_2})^{-p-2}\right) \\ &\qquad Ax^{a_2-1}y^{b_2-1}(1+a_1x^{a_2}b_1y^{b_2})^{-p-2}dxdy \\ &= H_1 + H_2 + H_3 - H_4, \end{aligned} \tag{7.1}$$

where,

$$H_1 = -A\int_0^\infty \int_0^\infty \log(A)\, x^{a_2-1}y^{b_2-1}(1+a_1x^{a_2}b_1y^{b_2})^{-p-2}dxdy \tag{7.2}$$

$$= -\log(A). \tag{7.3}$$

For H_2, we get,

$$\begin{aligned} H_2 &= -A\int_0^\infty \int_0^\infty (a_2-1)\log(x)\, x^{a_2-1}y^{b_2-1}(1+a_1x^{a_2}b_1y^{b_2})^{-p-2}dxdy \\ &= -A\int_0^\infty y^{b_2-1}\int_0^\infty (a_2-1)\log(x)\, x^{a_2-1}(1+a_1x^{a_2}b_1y^{b_2})^{-p-2}dxdy \\ &= -A\int_0^\infty y^{b_2-1}H_5(y)dy. \end{aligned}$$

By some algebraic computation we get

$$H_5(y) = \frac{(1-a_2)[p(\ln a_1 - \ln(1+b_1y^{b_2}) + \psi(p)+\gamma)+1]}{a_1a_2^2p(p+1)(1+b_1y^{b_2})^{p+2}},$$

where,

$$\psi(p) = \frac{\partial \ln(\Gamma(p))}{\partial p} = \frac{\Gamma'(p)}{\Gamma(p)}; \quad \gamma = \lim_{x\to\infty}\left(\sum_{i=1}^{n}\frac{1}{i} - \ln n\right) \simeq 0.577.$$

Therfore, we obtain,

$$\begin{aligned} H_2 &= -A\int_0^\infty y^{b_2-1}\frac{(1-a_2)[p(\ln a_1 - \ln(1+b_1y^{b_2}) + \psi(p)+\gamma)+1]}{a_1a_2^2p(p+1)(1+b_1y^{b_2})^{p+2}}dy \\ &= \frac{(a_2-1)}{a_2}\int_0^\infty \frac{y^{b_2-1}}{(1+b_1y^{b_2})^{p+2}}[p(\ln a_1 - \ln(1+b_1y^{b_2}) + \psi(p)+\gamma)+1]dy \\ &= \frac{(a_2-1)}{a_2}\Bigg[[p(\ln a_1 + \psi(p)+\gamma)+1]\int_0^\infty \frac{y^{b_2-1}}{(1+b_1y^{b_2})^{p+2}}dy \\ &\qquad -p\int_0^\infty \frac{y^{b_2-1}\ln(1+b_1y^{b_2})}{(1+b_1y^{b_2})^{p+2}}dy\Bigg]. \\ &= \frac{(a_2-1)}{a_2}\left[[p(\ln a_1 + \psi(p)+\gamma)+1]\left(\frac{\Gamma(p)}{b_1b_2\Gamma(p+1)}\right) - pH_6\right], \end{aligned}$$

where,

$$\begin{aligned} H_6 &= \int_0^\infty \frac{y^{b_2-1}\ln(1+b_1y^{b_2})}{(1+b_1y^{b_2})^{p+2}}dy \\ &= \frac{1}{b_1b_2\Gamma(p+1)}[\Gamma(2)\Gamma(p-1) heypergeom([1,1,2],[2,2-p],1) \\ &\qquad + \frac{\pi^2\csc(\pi(p-1))\Gamma(p+1)}{p\sin(\pi p)\Gamma(0)}]. \end{aligned}$$

In the last relation $heypergeometric$ function is defined as below,

$$heypergeom([a_1,a_2,...],[b_1,b_2,...],z) = \sum_{i=0}^{\infty}\frac{z^i}{i!}.\frac{\prod_j \Gamma(a_j+i)/\Gamma(a_j)}{\prod_j \Gamma(b_j+i)/\Gamma(b_j)}.$$

So, we get,

$$\begin{aligned} H_2 &= \frac{(a_2-1)}{a_2 b_1 b_2 \Gamma(p+1)}[\Gamma(p)[p(\ln a_1 + \psi(p) + \gamma) + 1] \\ &\quad -\Gamma(p-1)\sum_{i=0}^{\infty}\frac{\Gamma^2(i+1)}{i!\Gamma(2+i-p)} - \frac{\pi^2\csc(\pi(p-1))\Gamma(p+1)}{p\sin(\pi p)}] \\ &= \frac{a_2-1}{a_2 b_1 b_2}[\ln a_1 + c_1(p)], \end{aligned} \tag{7.4}$$

where,

$$c_1(p) = \left[\frac{1}{p} + \psi(p) + \gamma - \frac{\Gamma(p-1)}{\Gamma(p+1)}\sum_{i=0}^{\infty}\frac{\Gamma^2(i+1)}{i!\Gamma(2+i-p)} - \frac{\pi^2\csc(\pi(p-1))\Gamma(p+1)}{p\sin(\pi p)}\right].$$

Similarly for H_3, we obtain

$$H_3 = \frac{b_2-1}{b_2 a_1 a_2}[\ln b_1 + c_1(p)]. \tag{7.5}$$

Now, for H_4, we have,

$$\begin{aligned} H_4 &= A(p+2)\int_0^{\infty} y^{b_2-1}\int_0^{\infty}\log\left(1+a_1 x^{a_2} b_1 y^{b_2}\right) x^{a_2-1}(1+a_1 x^{a_2} b_1 y^{b_2})^{-p-2}dxdy \\ &= A(p+2)\int_0^{\infty} y^{b_2-1}H_7(y)dy, \end{aligned}$$

where,

$$\begin{aligned} H_7 &= \frac{1}{a_2 b_1 b_2 \Gamma(p+2)}[(1+b_1 y^{b_2})^{-p-1}[\log(1+b_1 y^{b_2})\Gamma(p+1) \\ &\quad +\Gamma(p)\sum_{i=0}^{\infty}\frac{\Gamma^2(i+1)}{i!\Gamma(1+i-p)} + \frac{\pi^2\csc(\pi(p-1))\Gamma(p+2)}{(p+1)\sin(\pi p)}]] \\ &= \frac{1}{a_1 a_2 \Gamma(p+2)}[(1+b_1 y^{b_2})^{-p-1}[\ln(1+b_1 y^{b_2})\Gamma(p+1) + c'(p)], \end{aligned}$$

We get,

$$\begin{aligned} H_4(y) &= \frac{A(p+2)}{a_2 b_1 b_2 (p+1)}\int_0^{\infty} y^{b_2-1}(1+b_1 y^{b_2})^{-p-1}\log(1+b_1 y^{b_2})dy \\ &\quad + \frac{A(p+2)}{a_2 b_1 b_2 \Gamma(p+2)}\, c'(p)\int_0^{\infty} y^{b_2-1}(1+b_1 y^{b_2})^{-p-1}dy \\ &= \frac{A(p+2)}{a_2 b_1 b_2 (p+1)}\, H_6 + \frac{A(p+2)}{a_2 b_1 b_2 \Gamma(p+2)}\, c'(p)\frac{p}{b_1 b_2} \\ &= b_1 b_2 (p+2) H_6 + \frac{c'(p)}{\Gamma(p)} \\ &= \frac{p+2}{\Gamma(p+1)}\left[\Gamma(p-1)\sum_{i=0}^{\infty}\frac{\Gamma^2(i+1)}{i!\Gamma(2+i-p)} - \frac{\pi^2\csc(\pi(p-1))\Gamma(p+1)}{p\sin(\pi p)}\right] \end{aligned}$$

$$+\frac{c'(p)}{\Gamma(p)} = c_2(p). \tag{7.6}$$

Now by substituting (7.2),(7.4), (7.5) and (7.6) in (7.1), we obtain,

$$H_e(X,Y) = \frac{a_2-1}{b_1 b_2 a_2}[\ln a_1 + c_1(p)] + \frac{b_2-1}{a_1 a_2 b_2}[\ln b_1 + c_1(p)] - c_2(p) - \ln(A),$$

This complete the proof.

Proof of Proposition 5.3:
For computing the mutual information of GBL distribution, we can write

$$\begin{aligned} H_e(X) &= -E(\log(f(X))) \\ &= -\int_0^\infty \log(a_1 a_2 p x^{a_2-1}(1+a_1 x^{a_2})^{-p-1}) a_1 a_2 p x^{a_2-1}(1+a_1 x^{a_2})^{-p-1} dx \\ &= -(K_1 + K_2 - K_3). \end{aligned} \tag{7.7}$$

where,

$$K_1 = \int_0^\infty \log(a_1 a_2 p) a_1 a_2 p x^{a_2-1}(1+a_1 x^{a_2})^{-p-1} dx = \log(a_1 a_2 p),$$

$$\begin{aligned} K_2 &= \int_0^\infty \log(x^{a_2-1}) a_1 a_2 p x^{a_2-1}(1+a_1 x^{a_2})^{-p-1} dx \\ &= \frac{1-a_2}{a_2}[\ln a_1 + \psi(p) + \gamma], \end{aligned}$$

and

$$\begin{aligned} K_2 &= \int_0^\infty \log((1+a_1 x^{a_2})^{-p-1}) a_1 a_2 p x^{a_2-1}(1+a_1 x^{a_2})^{-p-1} dx \\ &= \frac{p+1}{p-1}\sum_{i=1}^\infty \frac{\Gamma^2(i+1)\Gamma(2-p)}{i!\Gamma(2-i+p)} + \frac{(p+1)\pi^2 \csc(\pi(p-1))}{\sin(\pi p)}. \end{aligned}$$

So, we get,

$$\begin{aligned} H_e(X) &= -\log(p a_1 a_2) - \frac{1-a_2}{a_2}[\ln a_1 + \psi(p) + \gamma] \\ &\quad + \frac{p+1}{p-1}\sum_{i=1}^\infty \frac{\Gamma^2(i+1)\Gamma(2-p)}{i!\Gamma(2-i+p)} + \frac{(p+1)\pi^2 \csc(\pi(p-1))}{\sin(\pi p)} \\ &= c_4(p) - \log(p a_1 a_2) - \frac{1-a_2}{a_2}[\ln a_1 + \psi(p) + \gamma], \end{aligned} \tag{7.8}$$

where,

$$c_4(p) = \frac{p+1}{p-1}\sum_{i=1}^\infty \frac{\Gamma^2(i+1)\Gamma(2-p)}{i!\Gamma(2-i+p)} + \frac{(p+1)\pi^2 \csc(\pi(p-1))}{\sin(\pi p)} - \log(p).$$

Similarly for $H(Y)$, we obtain,

$$H_e(Y) = c_4(p) - \log(p b_1 b_2) - \frac{1-b_2}{b_2}[\ln b_1 + \psi(p) + \gamma]. \tag{7.9}$$

Substituting (7.7), (7.8) and (7.9) in (5.2), implies that

$$\begin{aligned}
I(X,Y) &= c_4(p) - \log(pa_1a_2) - \frac{1-a_2}{a_2}[\ln a_1 + \psi(p) + \gamma] \\
&\quad + c_4(p) - \log(pb_1b_2) - \frac{1-b_2}{b_2}[\ln b_1 + \psi(p) + \gamma] \\
&\quad - \left[\frac{a_2-1}{b_1b_2a_2}[\ln a_1 + c_1(p)] + \frac{b_2-1}{a_1a_2b_2}[\ln b_1 + c_1(p)] - c_2(p) - \ln(A)\right] \\
&= 2c_4(p) + c_3(p) + \frac{1-a_2}{a_2}\left[\ln a_1\left(1 - \frac{1}{b_1b_2}\right) + \psi(p) + \gamma - \frac{c_1(p)}{b_1b_2}\right] \\
&\quad + \frac{1-b_2}{b_2}\left[\ln b_1\left(1 - \frac{1}{a_1a_2}\right) + \psi(p) + \gamma - \frac{c_1(p)}{a_1a_2}\right] - \log(a_1a_2) - \log(b_1b_2).
\end{aligned}$$

This complete the proof.

***Proof of Proposition 5.5*:**
For computing the quadratic mutual information we rewrite it as below:

$$C(X,Y) = \log\frac{C_1C_2}{C_3^2} = \log(C_1) + \log(C_2) - 2\log(C_3). \tag{7.10}$$

For the first term we have

$$\begin{aligned}
C_1 &= \int_0^\infty \int_0^\infty f^2(x,y)dxdy \\
&= \int_0^\infty \int_0^\infty A^2 x^{2(a_2-1)} y^{2(b_2-1)} (1 + a_1x^{b_2} + b_1y^{b_2})^{-2(p+2)} dxdy \\
&= A^2 \int_0^\infty y^{2(b_2-1)} \int_0^\infty x^{2(a_2-1)} (1 + a_1x^{b_2} + b_1y^{b_2})^{-2(p+2)} dxdy \\
&= A^2 \int_0^\infty y^{2(b_2-1)} C_4(y)dy,
\end{aligned}$$

where,

$$\begin{aligned}
C_4(y) &= \int_0^\infty x^{2(a_2-1)} (1 + a_1x^{b_2} + b_1y^{b_2})^{-2(p+2)} dx \\
&= \frac{1}{a_1^2a_2\Gamma(2p+4)}\left[\left(1 + b_1y^{b_2}\right)^{-2p-2-\frac{1}{a_2}} a_1^{\frac{1}{a_2}} \Gamma(2p+2+\frac{1}{a_2})\Gamma(2-\frac{1}{a_2})\right] \\
&= a_1^{\frac{1}{a_2}-2} a_2^{-1} \left(1 + b_1y^{b_2}\right)^{-2p-2-\frac{1}{a_2}} B(2p+2+\frac{1}{a_2}, 2-\frac{1}{a_2}).
\end{aligned}$$

We get,

$$\begin{aligned}
C_1 &= A^2 a_1^{\frac{1}{a_2}-2} a_2^{-1} B(2p+2+\frac{1}{a_2}, 2-\frac{1}{a_2}) \int_0^\infty y^{2(b_2-1)} \left(1 + b_1y^{b_2}\right)^{-2p-2-\frac{1}{a_2}} dy, \\
&= \left[\frac{A^2 a_1^{\frac{1}{a_2}} \Gamma(2p+2+\frac{1}{a_2})\Gamma(2-\frac{1}{a_2})}{a_1^2a_2\Gamma(2p+4)}\right]\left[\frac{b_1^{-2+\frac{1}{b_2}}\Gamma(2p+\frac{1}{a_2}+\frac{1}{b_2})\Gamma(2-\frac{1}{b_2})}{b_2\Gamma(2p+2+\frac{1}{a_2})}\right] \\
&= \frac{1}{a_2b_2\Gamma(2p+4)}\left[A^2 a_1^{-2+\frac{1}{a_2}} b_1^{-2+\frac{1}{b_2}} \Gamma(2p+\frac{1}{a_2}+\frac{1}{b_2})\Gamma(2-\frac{1}{a_2})\Gamma(2-\frac{1}{b_2})\right]
\end{aligned}$$

$$= a_1^{\frac{1}{a_2}} b_1^{\frac{1}{b_2}} a_2 b_2 p^2 (p+1)^2 B(2p + \frac{1}{a_2} + \frac{1}{b_2}, 2 - \frac{1}{a_2}, 2 - \frac{1}{b_2}). \tag{7.11}$$

Moreover, for C_2 we can write,

$$C_2 = \int_0^\infty \int_0^\infty f_1^2(x) f_2^2(y) dx dy = \int_0^\infty f_1^2(x) dx \int_0^\infty f_2^2(y) dy = C_5 C_6,$$

where,

$$\begin{aligned} C_5 &= \int_0^\infty \left(a_1 a_2 p x^{a_2 - 1} (1 + a_1 x^{a_2})^{-p-1} \right)^2 dx \\ &= \frac{p^2 a_1^{\frac{1}{a_2}} a_2 \Gamma(2p + \frac{1}{a_2}) \Gamma(2 - \frac{1}{a_2})}{\Gamma(2p+2)} \\ &= p^2 a_1^{\frac{1}{a_2}} a_2 B(2p + \frac{1}{a_2}, 2 - \frac{1}{a_2}). \end{aligned}$$

Similarly for C_6, we get,

$$C_6 = p^2 b_1^{\frac{1}{b_2}} b_2 B(2p + \frac{1}{b_2}, 2 - \frac{1}{b_2}).$$

Then, we have,

$$C_2 = p^4 a_1^{\frac{1}{a_2}} a_2 b_1^{\frac{1}{b_2}} b_2 B(2p + \frac{1}{a_2}, 2 - \frac{1}{a_2}) B(2p + \frac{1}{b_2}, 2 - \frac{1}{b_2}). \tag{7.12}$$

For C_3, we have,

$$\begin{aligned} C_3 &= \int_0^\infty \int_0^\infty f(x,y) f_1(x) f_2(y) dx dy \\ &= K \int_0^\infty y^{2b_2 - 2} (1 + b_1 y^{b_2})^{-p-1} \int_0^\infty x^{2a_2 - 2} (1 + a_1 x^{a_2})^{-p-1} dx dy \\ &= K \int_0^\infty y^{2b_2 - 2} (1 + b_1 y^{b_2})^{-p-1} C_7(y) dy, \end{aligned}$$

where, $K = a_1^2 a_2^2 b_1^2 b_2^2 p^3 (p+1)$ and via some calculations

$$\begin{aligned} C_7(y) = a_1^{\frac{1}{a_2} - 2} a_2^{-1} \Bigg[& B(2 - \frac{1}{a_2}, p + \frac{1}{a_2}) \frac{\Gamma(1 - p - \frac{1}{a_2})}{\Gamma(p+1)\Gamma(2 - \frac{1}{a_2})} \times \\ & \sum_{i=1}^\infty \frac{(1 + b_1 y^{b_2})^{-p - \frac{1}{a_2} + i} \Gamma(p + 1 + i) \Gamma(2 - \frac{1}{a_2} + i)}{i! \Gamma(1 - p - \frac{1}{a_2} + i)} + \\ & B(2 - \frac{1}{a_2}, p + \frac{1}{a_2}) \frac{\Gamma(1 - p - \frac{1}{a_2})}{\Gamma(p+1)\Gamma(2 - \frac{1}{a_2})} \times \\ & \sum_{i=1}^\infty \frac{(1 + b_1 y^{b_2})^{-p - \frac{1}{a_2} + i} \Gamma(p + 1 + i) \Gamma(2 - \frac{1}{a_2} + i)}{i! \Gamma(1 - p - \frac{1}{a_2} + i)} \Bigg]. \end{aligned}$$

Therefore,

$$C_3 = a_1^{\frac{1}{a_2}} a_2 b_1^{\frac{1}{b_2}} b_2 p^3(p+1)\Bigg[B(2-\frac{1}{a_2}, p+\frac{1}{a_2})\frac{\Gamma(1-p-\frac{1}{a_2})}{\Gamma(p+1)\Gamma(2-\frac{1}{a_2})}\times$$
$$\sum_{i=1}^{\infty}\frac{B(2p+\frac{2}{b_2}+i-1, 2-\frac{1}{b_2})\Gamma(p+1+i)\Gamma(2-\frac{1}{a_2}+i)}{i!\Gamma(1-p-\frac{1}{a_2}+i)}+$$
$$B(2p+\frac{1}{b_2}+i-1, p+\frac{1}{a_2})\frac{\Gamma(1-p-\frac{1}{a_2})}{\Gamma(p+1)\Gamma(2-\frac{1}{a_2})}\times$$
$$\sum_{i=1}^{\infty}\frac{B(2p+1+i, 2-\frac{1}{b_2})\Gamma(p+1+i)\Gamma(2-\frac{1}{a_2}+i)}{i!\Gamma(1-p-\frac{1}{a_2}+i)}\Bigg]. \tag{7.13}$$

Now by substituting (7.11), (7.12) and (7.13) in (7.10) the proof is completed.

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