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Asymptotic growth of the spectral radii of collocation matrices approximating elliptic boundary value problems

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ABSTRACT

Throughout this paper we consider the Poisson equations in two dimensions. By the collocation methods based on radial basis functions and by exploiting some tools in literature: Perron-Frobenius theory and Weyl Tyrtyshnikov equal distribution, we prove under suitable assumptions on the shape parameter appearing in the radial basis functions that, the spectral radii of the collocation matrices grow as the size of the matrices, that is, $\lim_{n\to\infty}\frac{\rho_n}{d_n} = constant$ where ρ_n and d_n are respectively the spectral radius and the size of the collocation matrix.

Keywords: radial basis functions; elliptic boundary value problems; collocation matrices; Block Toeplitz matrices; Perron-Fobenius theory; Weyl Tyrtyshnikov equal distribution; spectral radii; asymptotic growth.

1. Introduction

The purpose of this paper is to provide a deep study of the spectral radii of collocation matrices approximating elliptic boundary value problems in two dimensional case

$$\begin{array}{ll} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f & \text{if } (x,y) \in \Omega = (0,1)^2 \\ u(x,y) = g(x,y) & \text{for } (x,y) \in \partial\Omega \end{array} \tag{1.1}$$

The chosen method for approximating is based on the radial basis functions. These types of approximations are often very useful for obtaining a numerical solution of certain PDE_s. Under certain conditions, the convergence is very fast (exponential in the number of grid points) when compared with Finite Difference or Finite Elements. The price that is paid is often an extreme ill-conditioning of the resulting structured matrices. A main role for approximation space is played a radial function and this space is made by translating a standard radial function with zero as its center. Some of the most commonly used radial basis functions are:

- Direct Multiquadric (MQ): $\phi(t) = (t^2 + c^2)^{\frac{1}{2}}$
- Inverse Multiquadric (IMQ): $\phi(t) = (t^2 + c^2)^{\frac{-1}{2}}$
- Gaussian: $\phi(t) = e^{-\frac{t^2}{c^2}}$

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where the parameter c is often called **shape parameter** and whose the role is to determine the accuracy and the stability. Denoting by Ω an opened domain of model problem (1.1), $\partial\Omega$ its boundary and $\widehat{\Omega} = \partial\Omega \cup \Omega$ an artificial domain greater than real domain Ω, $h = \frac{1}{n+1}$ the maximal step size and, $\{(x_i, y_j) = (ih, jh)\}_{i,j=0}^{n+1}$ the selected points that are chosen out of the real domain Ω and are in the artificial domain, we present an interesting method using the nodes that most of them are selected out of the real domain and the others in the domain. Hence, the real collocation matrix is the sum of a symmetric block Toeplitz matrices with symmetric Toeplitz blocks of size $d_n = (n+2)^2$ generated by unbounded function over Ω and a matrix of rank at most $4n+4$, which collects the boundary conditions. More in detail we are interested in fast solution methods, and especially in the asymptotic growth of the spectral radii of the resulting matrices and in the global distribution results. For the latter point, we need to think not to a single linear system but to a sequence of linear systems of increasing dimensions d_n^2 related to a finesse parameter, as usually occurs in the approximation of PDE_s. Such kind of spectral knowledge is then employed for suggesting appropriate $O(d_n^2 logd_n)$ preconditioners for krylov subspace methods. A first important step for understanding the spectral behavior of the considered matrices done in [1], where the link with Toeplitz sequences generated by a symbol was exploited. One of the advantages of meshless methods based on radial functions with respect to others, is high decreased of computational volume that arises when changing multi- dimensions to one dimension. Kansa [4] is the first researcher that applied an approximation by radial basis functions (Pseudo interpolation) to PDE_s. The use of the globally supported radial functions, reaches to the large linear systems, poorly condition number and full matrices.

The paper is organized as follows. In section 2, we recall some useful results due to the Perron-Frobenius theory and to the Weyl Tyrtyshnikov equal distribution. Section 3 deals with the approximation of Poisson equation (1.1) by the collocation systems. The approximation of collocation sequences by the sequences of block Toeplitz matrices and the asymptotic behavior of the generating function of Toeplitz sequences are established in section 4. Section 5 is reserved to the asymptotic growth of the spectral radii of collocation matrices while section 6 deals with the general conclusions and future works.

2. Some Definitions and main results

The aim of this section is to recall some definitions and main results of the linear algebra which are useful for the study of the collocation matrices.

2.1 *Perron Frobenius theory*

Throughout this subsection, we recall the Perron Frobenius theory.

Definition 2.1. *Let $A = (a_{jk})$ and $B = (b_{jk})$ be two $n \times r$ matrices. Then, $A \geq B$ ($A > B$) if $a_{jk} \geq b_{jk}$ ($a_{jk} > b_{jk}$) for all $j = 1, 2, ..., n$ and $k = 1, 2, ..., r$.*

Definition 2.2. *$A \in \mathbb{R}^{n\times r}$ is said to be nonnegative (positive) matrix if $A \geq 0$ ($A > 0$).*

Definition 2.3. *A matrix $A \in \mathbb{R}^{n\times n}$ possesses the Perron-Frobenius property if its dominant eigenvalue λ_1 is positive and the corresponding eigenvector $x^{(1)}$ is nonnegative.*

Definition 2.4. *A matrix $A \in \mathbb{R}^{n\times n}$ possesses the strong Perron-Frobenius property if its dominant eigenvalue λ_1 is positive, simple ($\lambda_1 > |\lambda_j|, j = 2, 3, ..., n$) and the corresponding eigenvector $x^{(1)}$ is positive.*

Definition 2.5. *A matrix $A \in \mathbb{R}^{n\times n}$ is said to be eventually positive (eventually nonnegative) if there exists a positive integer k_0 such that $A^k > 0$ ($A^k \geq 0$) for all $k \geq k_0$.*

Theorem 2.1. *[5, 7, 13]. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, the following properties are equivalent:*

(i) *A possesses the strong Perron-Frobenius property.*

(ii) *A is an eventually positive matrix.*

Proof. $(i \Rightarrow ii): \lambda_1 = \rho(A) > |\lambda_2| \geq \lambda_3| \geq ... \geq \lambda_n|$, where λ_1 is a simple eigenvalue with the eigenvector $x^{(1)} \in \mathbb{R}^n$ being positive. Choose the i-th column $a^{(i)} \in \mathbb{R}^n$ of A.
Expand $a^{(i)}$: $a^{(i)} = \sum_{j=1}^{n} c_j x^{(j)}$ (where $\{x^{(1)}, x^{(2)}, ..., x^{(n)}\}$ is an orthonormal basis of $\mathbb{R}^n$).
$c_j = (a^{(i)}, x^{(j)})$, $j = 1, 2, ..., n$. So, $c_1 = (a^{(i)}, x^{(1)}) = \lambda_1 x_i^{(1)} > 0$. Apply power method: $\lim_{k\to\infty} A^k a^{(i)} > 0 \Rightarrow A^k a^{(i)} > 0\ \forall k > m$. Choose $m_0 = \min\{m :\ A^k a^{(i)} > 0\ \forall k \geq m\}$, then, $A^k > 0\ \forall k \geq k_0 = m_0 + 1$. So, A is an eventually positive matrix.
$(ii \Rightarrow i)$: From the Perron-Frobenius theory of nonnegative matrices, the assumption $A^k > 0$ means that the dominant eigenvalue of A^k is positive and the only one in the circle while the corresponding eigenvector is positive. It is well known that the matrix A has as eigenvalues the k-th roots of those of A^k with the same eigenvectors. Since this happens $\forall k \geq k_0$, A possesses the strong Perron-Frobenius property. □

Theorem 2.2. *[5, 7, 13]. For a matrix $A \in \mathbb{R}^{n\times n}$ the following properties are equivalent:*

i. *Both matrices A and A^T possess the strong Perron-Frobenius property.*

ii. *A is an eventually positive matrix.*

iii. *A^T is an eventually positive matrix.*

Proof. $(i \Rightarrow ii)$: Let $A = XDX^{-1}$ be the Jordan canonical form of the matrix A. We assume that the eigenvalue $\lambda_1 = \rho(A)$ is the first diagonal entry of D. So the Jacobi canonical form can be written as

$$A = [x^{(1)}|X_{n,n-1}] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D_{n-1,n-1} \end{array}\right] \left[\begin{array}{c} {y^{(1)}}^T \\ \hline Y_{n-1,n} \end{array}\right] \tag{2.1}$$

where ${y^{(1)}}^T$ and $Y_{n-1,n}$ are the first row and the matrix formed by the last $n-1$ rows of X^{-1}, respectively. Since A possesses the strong Perron-Frobenius property, the eigenvector $x^{(1)}$ is positive. From (2.1), the block form of A^T is

$$A^T = [y^{(1)}|Y^T_{n,n-1}] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D^T_{n-1,n-1} \end{array}\right] \left[\begin{array}{c} {x^{(1)}}^T \\ \hline X^T_{n-1,n} \end{array}\right] \tag{2.2}$$

The matrix $D^T_{n-1,n-1}$ is the block diagonal matrix formed by the transpose of the Jordan blocks except λ_1. It is obvious that there exists a permutation matrix $P \in \mathbb{R}^{(n-1)\times(n-1)}$ such that the associated permutation transformation on the matrix $D^T_{n-1,n-1}$ transposes all the Jordan blocks.
Thus, $D_{n-1,n-1} = P^T D^T_{n-1,n-1} P$ and relation (2.2) takes the form:

$$\begin{aligned} A^T &= [y^{(1)}|Y^T_{n,n-1}] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^T \end{array}\right] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D^T_{n-1,n-1} \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array}\right] \\ &\times \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^T \end{array}\right] \left[\begin{array}{c} {x^{(1)}}^T \\ \hline X^T_{n-1,n} \end{array}\right] = [y^{(1)}|Y'^T_{n-1,n}] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D_{n-1,n-1} \end{array}\right] \left[\begin{array}{c} {x^{(1)}}^T \\ \hline X'^T_{n,n-1} \end{array}\right] \end{aligned}$$

where $Y'^T_{n-1,n} = Y^T_{n,n-1}P$ and $X'^T_{n,n-1} = P^T X^T_{n-1,n}$. The last relation is the Jordan canonical form of A^T which means that $y^{(1)}$ is the eigenvector corresponding to the dominant eigenvalue λ_1. Since A^T possesses the strong Perron-Frobenius property, $y^{(1)}$ is a positive vector or a negative one. Since ${y^{(1)}}^T$ is the first row of X^{-1}, we have that $(y^{(1)}, x^{(1)}) = 1$ implying that $y^{(1)}$ is a positive vector.
We return now to the Jordan canonical form (2.1) of A and form the power A^k.

$$A^k = [x^{(1)}|X_{n,n-1}] \left[\begin{array}{c|c} \lambda_1^k & 0 \\ \hline 0 & D^k_{n-1,n-1} \end{array}\right] \left[\begin{array}{c} {y^{(1)}}^T \\ \hline Y_{n-1,n} \end{array}\right]$$

then

$$\frac{1}{\lambda_1^k} A^k = [x^{(1)}|X_{n,n-1}] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \frac{1}{\lambda_1^k} D^k_{n-1,n-1} \end{array}\right] \left[\begin{array}{c} {y^{(1)}}^T \\ \hline Y_{n-1,n} \end{array}\right]$$

Since λ_1 is the dominant eigenvalue, the only one of modulus λ_1, we get that $\lim\limits_{k\to\infty} \frac{1}{\lambda_1^k} D^k_{n-1,n-1} = 0$. Thus

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k} A^k = x^{(1)} {y^{(1)}}^T > 0.$$

The last relation means that there exists an integer $k_0 > 0$ such that $A^k > 0$ for all $k \geq k_0$. So, A is an eventually positive matrix and the first part of theorem is proved.
$(ii \Leftrightarrow iii)$: Obvious from Definition 2.5.
$(ii \Rightarrow i)$: The proof is the same as that of Theorem 2.1, by considering that A and A^T are both eventually positive matrices. □

Theorem 2.3. *[2, 13]. Let $A \in \mathbb{R}^{n\times n}$ be an eventually nonnegative matrix. Then, both matrices A and A^T possess the Perron-Frobenius property.*

Proof. Analogous to the proof of the part $(ii \Rightarrow i)$ of Theorem 2.2. □

Theorem 2.4. *[5, 7, 13]. If $A^T \in \mathbb{R}^{n\times n}$ possesses the Perron-Frobenius property, then either*

$$\sum_{j=1}^{n} a_{ij} = \rho(A) \quad \forall i = 1, 2, ..., n, \tag{2.3}$$

or

$$\min_{1\leq i\leq n} \left(\sum_{j=1}^{n} a_{ij} \right) \leq \rho(A) \leq \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} a_{ij} \right) \tag{2.4}$$

Moreover, if A^T possesses the strong Perron-Frobenius property, then both inequalities in (2.4) are strict.

Proof. Let $(\rho(A), y)$ be the Perron-Frobenius eigenpair of the matrix A^T and e be the vector of ones. Then,

$$y^T A e = y^T \begin{pmatrix} \sum_{j=1}^{n} a_{1j} \\ \sum_{j=1}^{n} a_{2j} \\ \vdots \\ \sum_{j=1}^{n} a_{nj} \end{pmatrix} = \sum_{i=1}^{n} \left(y_i \sum_{j=1}^{n} a_{ij} \right) \leq \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} a_{ij} \right) \sum_{i=1}^{n} y_i,$$

$$y^T A e = \sum_{i=1}^{n}\left(y_i \sum_{j=1}^{n} a_{ij}\right) \geq \min_{1\leq i\leq n}\left(\sum_{j=1}^{n} a_{ij}\right)\sum_{i=1}^{n} y_i.$$

On the other hand, we get

$$y^T A e = e^T A^T y = \rho(A) e^T y = \rho(A)\sum_{j=1}^{n} y_j.$$

Combining the relations above, we get our result. Obviously, equality holds if the row sums are equal. If A^T possesses the strong Perron-Frobenius property, then $y > 0$ and the inequalities become strict. □

Corollary 2.5. *[5, 7, 13]. If $A \in \mathbb{R}^{n\times n}$ possesses the Perron-Frobenius property, then either*

$$\sum_{i=1}^{n} a_{ij} = \rho(A) \quad \forall j = 1, 2, ..., n, \tag{2.5}$$

or

$$\min_{1\leq j\leq n}\left(\sum_{i=1}^{n} a_{ij}\right) \leq \rho(A) \leq \max_{1\leq j\leq n}\left(\sum_{i=1}^{n} a_{ij}\right). \tag{2.6}$$

Moreover, if A possesses the strong Perron-Frobenius property, then both inequalities in (2.6) are strict.

Theorem 2.6. *[5, 7, 13]. If the matrices $A, B \in \mathbb{R}^{n\times n}$ are such that $A \leq B$, and both A and B^T possess the Perron-Frobenius property (or both A^T and B possess the Perron-Frobenius property), then*

$$\rho(A) \leq \rho(B). \tag{2.7}$$

Moreover, if the above matrices possess the strong Perron-Frobenius property and $A \neq B$ then, the inequality (2.7) becomes strict.

Proof. Let $x, y \geq 0$ be the Perron right and left eigenvectors of A and B associated with the dominant eigenvalues λ_A and λ_B, respectively. Then the following equalities hold

$$y^T A x = \lambda_A y^T x, \;\; y^T B x = \lambda_B y^T x.$$

Since $A \leq B$, $B = A + C$, where $C \geq 0$. So,

$$\lambda_B y^T x = y^T B x = y^T (A + C) x = y^T A x + y^T C x \geq y^T A x = \lambda_A y^T x.$$

Assuming that $y^T x > 0$, the above relations imply that $\lambda_B \geq \lambda_A$. The case where $y^T x = 0$ is covered by using a continuity argument and perturbation technique. It is also obvious that the inequality becomes strict in the case where the associated Perron-Frobenius properties are strong. □

2.2 *Weyl Tyrtyshnikov equal distribution*

This part recalls some definitions on the distribution of matrix sequences. Furthermore, some tools to evaluate the strength of the equal distribution and equal localization that are based upon estimes of the singular values and involve the Frobenius norm. We denote by $\mathcal{M}_s(\mathbb{C})$ the linear space of all the square complex matrices of dimension $s \times s$, and we equippe this linear space by the Frobenius norm defined by:

$$\|A\|_F = \left[\sum_{j=1}^{s} \sigma_j(A)^2\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{s}\sum_{j=1}^{s} |a_{ij}|^2\right]^{\frac{1}{2}}$$

where $A = [a_{ij}]_{i,j=1}^{s} \in \mathcal{M}_s(\mathbb{C})$ and $\sigma_j(A)$ denotes the j-th singular value of A. The first motivation is "practical" in the sense that, in the approximation of matrix sequences of increasing dimension in the simpler space of matrices, this is the only Shatten p-norm whose calculation is computationally not expensive. The second motivation is theoretical: actually the Frobenius norm is the only Shatten p-norm induced by an inner product which makes the space $\mathcal{M}_s(\mathbb{C})$ into a Hilbert space. More specifically, setting $< A, B >=$trace$(A^\star B)$, we deduce that $\|A\|_F =< A, A >^{\frac{1}{2}}$.

Definition 2.6. *Two real sequences $\{a_i^{(n)}\}_{i\le d_n}$, $\{b_i^{(n)}\}_{i\le d_n}$ $(d_n < d_{n+1})$ are equally distributed (ED) if and only if, for any real-valued continuous function F with bounded support, the following relation holds:*

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} \left(F(a_i^{(n)}) - F(b_i^{(n)}) \right) = 0. \tag{2.8}$$

When the previous limit goes to zero as $O(d_n^{-1})$ and F is Lipschitz continuous, we say that there is ***strong equal distribution*** *(SED). The same definition applies to the case of sequences of matrices $\{A_n\}_n$ and $\{B_n\}_n$ of dimension $d_n \times d_n$: in this case $\{a_i^{(n)}\}_{i\le d_n}$ and $\{b_i^{(n)}\}_{i\le d_n}$ are the sets of their singular values (or eigenvalues if the involved matrices are Hermitian).*

Notation $\{A_n\}_n \simeq_D \{B_n\}_n$ means that the matrix sequences $\{A_n\}_n$ and $\{B_n\}_n$ are equally distributed.

Definition 2.7. *Two real sequences $\{a_i^{(n)}\}_{i\le d_n}$, $\{b_i^{(n)}\}_{i\le d_n}$ $(d_n < d_{n+1})$ are equally localized (EL) if and only if, for any nontrivial interval $[\alpha, \beta]$ $(\alpha < \beta)$, the following relation holds:*

$$\lim_{n\to\infty} \frac{1}{d_n} \left(card\{i : a_i^{(n)} \in [\alpha, \beta]\} - card\{i : b_i^{(n)} \in [\alpha, \beta]\} \right) = 0. \tag{2.9}$$

When the previous limit goes to zero as $O(d_n^{-1})$, we say that there is ***strong equal localization*** *(SEL). The same definition applies to the case of matrix sequences $\{A_n\}_n$ and $\{B_n\}_n$ of dimension $d_n \times d_n$: in this case $\{a_i^{(n)}\}_{i\le d_n}$ and $\{b_i^{(n)}\}_{i\le d_n}$ are the sets of their singular values (or eigenvalues if the involved matrices are Hermitian)*

Notation $\{A_n\}_n \simeq_L \{B_n\}_n$ means that the matrix sequences $\{A_n\}_n$ and $\{B_n\}_n$ are equally localized.

Proposition 2.7. *[9, 11, 12]. Let $\{A_n\}_n$ and $\{B_n\}_n$ be two sequences of $d_n \times d_n$ matrices.*

1. *Assume that $rank(A_n - B_n) = o(d_n)$. Then the sequences $\{A_n\}_n$ and $\{B_n\}_n$ are equally localized (EL) and equally distributed (ED).*

2. *If $rank(A_n - B_n) = O(1)$. Then the sequences $\{A_n\}_n$ and $\{B_n\}_n$ are strongly equally localized (SEL) and strongly equally distributed (SED).*

Proof. 1. Let $r_n = rank(A_n - B_n)$. As a consequence of the Cauchy interlace theorem we have $\sigma_{i-2r_n}(B_n) \ge \sigma_i(A_n) \ge \sigma_{i+2r_n}(B_n)$ for $i = 2r_n + 1, ..., d_n - 2r_n$. Therefore, for any interval $[\alpha, \beta]$ we have

$$card\{i : \sigma_i(A_n) \in [\alpha, \beta]\} = card\{i : \sigma_i(B_n) \in [\alpha, \beta]\} + e_n \quad |e_n| \le 4r_n. \tag{2.10}$$

Consequently $r_n = o(d_n)$ and then the sequences $\{A_n\}_n$ and $\{B_n\}_n$ are equally localized (EL). Hence, the equal distribution (since the equal localization (EL) implies the equal distribution).

2. If $r_n = O(1)$, then there is SEL by (2.10). For the proof of the last part, recall that F is Lipschitz continuous with bounded support contained in $M = [\alpha, \beta]$. Owing to its Lipschitzness, F is of bounded variation ($F \in BV$) too. Therefore it can be expressed as the sum of two monotone functions. By linearity it is enough to focus our attention on the monotone functions restricted to M. Let $S(A_n)$ and $S(B_n)$ be the sets of the singular values ordered nonincreasingly. Let q be an integer number and let $S(B_n, q)$ be such that $(S(B_n, q))_i = (S(B_n))_{i+q}$, $i = 1, 2, ..., d_n$, where $(S(B_n))_j = \min\{\alpha, (S(B_n))_{d_n}\}$ if $j \geq d_n + 1$ and $(S(B_n))_j = \max\{\beta, (S(B_n))_1\}$ if $j \leq 0$. Now, supposing that $r_n = O(1)$ i.e., $r_n \leq k$ for some positive k, we find that $S(B_n, -2k) \geq S(B_n), S(A_n) \geq S(B_n, 2k)$, where " $\geq$ " is intended componentwise. Finally, by monotonicity we deduce that

$$\begin{aligned}
\left|\sum_{i=1}^{d_n}(F(\sigma_i(A_n)) - F(\sigma_i(B_n)))\right| &\leq \left|\sum_{i=1}^{d_n}(F(\sigma_i(S(B_n, -2k))) - F(\sigma_i(S(B_n, 2k))))\right| \\
&= \left|\frac{1}{d_n}\sum_{i=1-2k,...,2k, j=d_n-2k+1,...,d_n+2k}(F(\sigma_i(S(B_n))) - F(\sigma_j(B_n)))\right| \\
&= O(d_n^{-1})
\end{aligned}$$

and the proof is complete. $\square$

Theorem 2.8. *[9, 11, 12]. Let $\{A_n\}_n$ and $\{B_n\}_n$ be two sequences of $d_n \times d_n$ matrices.*

1. *If $\|A_n - B_n - D_n\|_F^2 = o(d_n)$ and $rank(D_n) = o(d_n)$, then the sequences $\{A_n\}_n$ and $\{B_n\}_n$ are equally distributed (ED).*

2. *When $\|A_n - B_n - D_n\|_1 = O(1)$, with $rank(D_n) = o(1)$, then $\{A_n\}_n$ and $\{B_n\}_n$ are strongly equally distributed (SED).*

3. Approximation of Poisson equations (1.1) by collocation systems

In the preceding section we have recalled some definitions and main results of the Perron-Frobenius theory and of the Weyl Tyrtyshnikov equal distribution. Here, by the use of the radial function $\phi : \mathbb{R}^n \to \mathbb{R}^+$, we determine the collocation systems approximating the Poisson equations (1.1).

Discretizing Ω with grid points $z_{jk} = (x_j, y_k) = (hj, hk)$; $j, k = 0, 1, ..., n + 1$ and $h = \frac{1}{n+1}$, we define an approximated solution of the Poisson equations (1.1) by setting

$$v(x, y) = \sum_{j=0}^{n+1}\sum_{k=0}^{n+1} v_{jk}\phi((x, y) - (x_j, y_k)) \tag{3.1}$$

where

$$\phi(x, y) = \begin{cases} \sqrt{x^2 + y^2 + c^2} & \text{Multiquadric (MQ)} \\ \frac{1}{\sqrt{x^2+y^2+c^2}} & \text{Inverse multiquadric (IMQ)} \\ e^{-\frac{x^2+y^2}{c^2}} & \text{Gaussian} \end{cases} \tag{3.2}$$

The use of (1.1) and (3.1) yields

$$\begin{cases} \sum_{j=0}^{n+1}\sum_{k=0}^{n+1} v_{jk}\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right](x-x_j,y-y_k)=f(x,y) & \text{for } (x,y)\in\Omega \\ \sum_{j=0}^{n+1}\sum_{k=0}^{n+1} v_{jk}\phi(-x_j,y-y_k)=g(0,y) & \text{if } y\in[0,1] \\ \sum_{j=0}^{n+1}\sum_{k=0}^{n+1} v_{jk}\phi(1-x_j,y-y_k)=g(1,y) & \text{if } y\in[0,1] \\ \sum_{j=0}^{n+1}\sum_{k=0}^{n+1} v_{jk}\phi(x-x_j,-y_k)=g(x,0) & \text{if } x\in(0,1) \\ \sum_{j=0}^{n+1}\sum_{k=0}^{n+1} v_{jk}\phi(x-x_j,1-y_k)=g(x,1) & \text{if } x\in(0,1) \end{cases} \tag{3.3}$$

By straightforward computations, one has:

$$\frac{\partial\phi}{\partial x}(x,y)=\begin{cases} x(x^2+y^2+c^2)^{-\frac{1}{2}} \\ -x(x^2+y^2+c^2)^{-\frac{3}{2}} \\ \frac{-2x}{c^2}e^{-\frac{x^2+y^2}{c^2}} \end{cases}, \quad \frac{\partial\phi}{\partial y}(x,y)=\begin{cases} y(x^2+y^2+c^2)^{-\frac{1}{2}} \\ -y(x^2+y^2+c^2)^{-\frac{3}{2}} \\ \frac{-2y}{c^2}e^{-\frac{x^2+y^2}{c^2}} \end{cases}$$

then

$$\frac{\partial^2\phi}{\partial x^2}(x,y)=\begin{cases} (x^2+y^2+c^2)^{-\frac{1}{2}}-x^2(x^2+y^2+c^2)^{-\frac{3}{2}} \\ -(x^2+y^2+c^2)^{-\frac{3}{2}}+3x^2(x^2+y^2+c^2)^{-\frac{5}{2}} \\ \frac{-2}{c^2}e^{-\frac{x^2+y^2}{c^2}}+\frac{4x^2}{c^2}e^{-\frac{x^2+y^2}{c^2}} \end{cases}$$

$$\frac{\partial^2\phi}{\partial y^2}(x,y)=\begin{cases} (x^2+y^2+c^2)^{-\frac{1}{2}}-y^2(x^2+y^2+c^2)^{-\frac{3}{2}} \\ -(x^2+y^2+c^2)^{-\frac{3}{2}}+3y^2(x^2+y^2+c^2)^{-\frac{5}{2}} \\ \frac{-2}{c^2}e^{-\frac{x^2+y^2}{c^2}}+\frac{4y^2}{c^2}e^{-\frac{x^2+y^2}{c^2}} \end{cases}$$

then

$$\frac{\partial^2\phi}{\partial x^2}(x,y)+\frac{\partial^2\phi}{\partial y^2}(x,y)=\begin{cases} (x^2+y^2+2c^2)(x^2+y^2+c^2)^{-\frac{3}{2}} & \text{(MQ)} \\ (x^2+y^2-2c^2)(x^2+y^2+c^2)^{-\frac{5}{2}} & \text{(IMQ)} \\ \frac{4}{c^2}((x^2+y^2)-c^2)e^{-\frac{x^2+y^2}{c^2}} & \text{(Gaussian)} \end{cases} \tag{3.4}$$

The linear system associated with (3.3) is given by:

$$\begin{cases} (a) & \sum_{l,p=0}^{n+1} v_{lp}\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right](x_j-x_l,y_k-y_p)=f(x_j,y_k): & j,k=1,...,n \\ (b) & \sum_{l,p=0}^{n+1} v_{lp}\phi(-x_l,y_j-y_p)=g(0,y_j): & j=0,1,...,n+1 \\ (c) & \sum_{l,p=0}^{n+1} v_{lp}\phi(1-x_l,y_j-y_p)=g(1,y_j) & j=0,1,...,n+1 \\ (d) & \sum_{l,p=0}^{n+1} v_{lp}\phi(x_j-x_l,-y_p)=g(x_j,0): & j=1,...,n \\ (e) & \sum_{l,p=0}^{n+1} v_{lp}\phi(x_j-x_l,1-y_p)=g(x_j,1): & j=1,...,n \end{cases} \tag{3.5}$$

Setting $g = \frac{c}{h}$, then

$$
\begin{aligned}
c_{k-p}^{j-l} &= \left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right](x_j - x_l, y_k - y_p) \\
&= \begin{cases} \frac{1}{\sqrt{(x_j-x_l)^2+(y_k-y_p)^2+c^2}} + \frac{c^2}{(x_j-x_l)^2+(y_k-y_p)^2+c^2)^{\frac{3}{2}}} & \text{(MQ)} \\ \frac{(x_j-x_l)^2+(y_k-y_p)^2-2c^2}{((x_j-x_l)^2+(y_k-y_p)^2+c^2)^{\frac{5}{2}}} & \text{(IMQ)} \\ \frac{4}{c^2}((x_j-x_l)^2+(y_k-y_p)^2-c^2)e^{-\frac{(x_j-x_l)^2+(y_k-y_p)^2}{c^2}} & \text{(Gaussian)} \end{cases} \\
&= \begin{cases} \frac{1}{h}\frac{1}{\sqrt{(j-l)^2+(k-p)^2+g^2}} + \frac{1}{h}\frac{g^2}{[(j-l)^2+(k-p)^2+g^2]^{\frac{3}{2}}} & \text{(MQ)} \\ \frac{1}{h^3}\frac{(j-l)^2+(k-p)^2-2g^2}{[(j-l)^2+(k-p)^2+g^2]^{\frac{5}{2}}} & \text{(IMQ)} \\ \frac{4}{h^2g^4}[(j-l)^2+(k-p)^2-g^2]e^{-\frac{(j-l)^2+(k-p)^2}{g^2}} & \text{(Gaussian)} \end{cases}
\end{aligned} \tag{3.6}
$$

and

$$
\phi_{k-p}^{j-l} = \phi(x_j - x_l, y_k - y_p) = \begin{cases} h[(j-l)^2+(k-p)^2+g^2]^{\frac{1}{2}} & \text{(MQ)} \\ \frac{1}{h}((j-l)^2+(k-p)^2+g^2)^{-\frac{1}{2}} & \text{(IMQ)} \\ e^{-\frac{(j-l)^2+(k-p)^2}{g^2}} & \text{(Gaussian)} \end{cases} \tag{3.7}
$$

It follows from (a), (b), (c), (d), and (e) that

$$
\sum_{l,p=0}^{n+1} c_{k-p}^{j-l} v_{lp} = f_{jk} \qquad j,k = 1,2,...,n \tag{3.8}
$$

$$
\sum_{l,p=0}^{n+1} \phi_{j-p}^{l} v_{lp} = g(0, y_j) = g_{0j} \qquad j = 0,1,...,n+1 \tag{3.9}
$$

$$
\sum_{l,p=0}^{n+1} \phi_{j-p}^{n+1-l} v_{lp} = g(1, y_j) = g_{1j} \qquad j = 0,1,...,n+1 \tag{3.10}
$$

$$
\sum_{l,p=0}^{n+1} \phi_{p}^{j-l} v_{lp} = g(x_j, 0) = g_{j0} \qquad j = 1,...,n \tag{3.11}
$$

$$
\sum_{l,p=0}^{n+1} \phi_{n+1-p}^{j-l} v_{lp} = g(x_j, 1) = g_{j1} \qquad j = 1,...,n \tag{3.12}
$$

When exploiting the relations (3.8), (3.9), (3.10), (3.11), and (3.12) above we deduce the following linear system

$$\begin{bmatrix} A^{(n+2)}_{0-0} & A^{(n+2)}_{0-1} & \dots & A^{(n+2)}_{0-n} & A^{(n+2)}_{0-n-1} \\ A^{(n+2)}_{1-0} & A^{(n+2)}_{1-1} & \dots & \dots & A^{(n+2)}_{1-n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{(n+2)}_{n-0} & A^{(n+2)}_{n-1} & \dots & A^{(n+2)}_{n-n} & A^{(n+2)}_{n-n-1} \\ A^{(n+2)}_{n+1-0} & A^{(n+2)}_{n+1-1} & \dots & A^{(n+2)}_{n+1-n} & A^{(n+2)}_{n+1-n-1} \end{bmatrix} \begin{bmatrix} v^{(0)} \\ v^{(1)} \\ \vdots \\ v^{(n)} \\ v^{(n+1)} \end{bmatrix} = \begin{bmatrix} f^{(0)} \\ f^{(1)} \\ \vdots \\ f^{(n)} \\ f^{(n+1)} \end{bmatrix}$$

i.e.,

$$A_{d_n} v = \tilde{f} \tag{3.13}$$

where $d_n = (n+2)^2$. For $j = 0, 1, ..., n+1$

$$A^{(n+2)}_{0-j} = [\phi^{j}_{l-p}]^{n+1}_{l,p=0}; \quad A^{(n+2)}_{n+1-j} = [\phi^{(n+1)-j}_{l-p}]^{n+1}_{l,p=0} \tag{3.14}$$

for $j = 1, 2, ..., n$ and $l = 0, 1, ..., n+1$

$$A^{(n+2)}_{j-l} = \begin{bmatrix} \phi^{j-l}_0 & \phi^{j-l}_1 & \dots & \phi^{j-l}_n & \phi^{j-l}_{n+1} \\ c^{j-l}_1 & c^{j-l}_0 & \dots & c^{j-l}_{n-1} & c^{j-l}_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c^{j-l}_n & c^{j-l}_{n-1} & \dots & c^{j-l}_0 & c^{j-l}_1 \\ \phi^{j-l}_{n+1} & \phi^{j-l}_n & \dots & \phi^{j-l}_1 & \phi^{j-l}_0 \end{bmatrix}, \tag{3.15}$$

for $k = 0, 1, ..., n+1$,

$$v^{(k)} = (v_{k,0}, v_{k,1}, ..., v_{k,n+1})^T; \quad f^{(0)} = (g_{0,0}, g_{0,1}, ..., g_{0,n+1})^T; \quad f^{(n+1)} = (g_{n+1,0}, ..., g_{n+1,n+1})^T$$

and for $p = 1, 2, ..., n$,

$$f^{(p)} = (g_{p,0}, f_{p,1}, ..., f_{p,n}, g_{p,n+1})^T.$$

The relation given by (3.13) is called the radial basis functions collocation systems approximating the elliptic boundary value problems (1.1) and the associated two level matrices A_{d_n} are the collocation matrices. These matrices are highly structured, full and near Toeplitz [3].

4. Approximation of the collocation matrices by Toeplitz sequences

This section is devoted to the approximation of collocation sequences by Toeplitz and to the asymptotic behavior of the generating function of these Toeplitz sequences.

4.1 *Approximation of collocation sequences*

In [1], the authors provided explicit asymptotic estimates, as function of c/h, c being the shape parameter, h being the step size, to the condition number $\mu(T_n)$ of the Toeplitz matrix T_n related to the approximated one-dimensional model problem

$$\begin{aligned} &u^{''}(x) = f(x) \qquad\qquad x \in (0,1) \\ &u(0) = u_0, \quad u(1) = u_1 \end{aligned} \tag{4.1}$$

with the collocation technique over a grid of equally spaced points and based on the MQ, IMQ, and Gaussian radial functions, respectively. Here we are interested by the approximation of two-dimensional model problem (1.1).

In reference to section 3 and for $l = 0, 1, ..., n + 1$; if we set

$$\Delta_{0-l}^{(n+2)} = [\phi_{k-p}^{l} - c_{k-p}^{l}]_{k,p=0}^{n+1} \quad \text{and} \quad \Delta_{n+1-l}^{(n+2)} = [\phi_{k-p}^{n+1-l} - c_{k-p}^{n+1-l}]_{k,p=0}^{n+1} \tag{4.2}$$

then the matrices $A_{0-l}^{(n+2)}$, $A_{n+1-l}^{(n+2)}$, $\Delta_{0-l}^{(n+2)}$, and $\Delta_{n+1-l}^{(n+2)}$ are symmetric of rank at most $n + 2$ and the matrices $T_{n+1-l}^{(n+2)}$ and $T_{n+1-l}^{(n+2)}$ defined respectively by

$$T_{0-l}^{(n+2)} = A_{0-l}^{(n+2)} - \Delta_{0-l}^{(n+2)} \quad \text{and} \quad T_{n+1-l}^{(n+2)} = A_{n+1-l}^{(n+2)} - \Delta_{n+1-l}^{(n+2)} \tag{4.3}$$

are symmetric Toeplitz matrices generated by the function

$$s(x,y) = c_0^0 + 2\sum_{k=1}^{\infty} c_k^0[\cos(2k\pi x) + \cos(2k\pi y)] + 4\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} c_j^k \cos(2j\pi x)\cos(2k\pi y) \quad \forall (x,y) \in \Omega \tag{4.4}$$

Next, for $j = 1, 2, ..., n$ and $l = 0, 1, ..., n + 1$, if we set

$$\Delta_{j-l}^{(n+2)} = \begin{bmatrix} \phi_0^{j-l} - c_0^{j-l} & \dots & \phi_{n+1}^{j-l} - c_{n+1}^{j-l} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \phi_{n+1}^{j-l} - c_{n+1}^{j-l} & \dots & \phi_0^{j-l} - c_0^{j-l} \end{bmatrix} \quad \text{and} \quad T_{j-l}^{(n+2)} = [c_{k-p}^{j-l}]_{k,p=0}^{n+1} \tag{4.5}$$

we deduce the following splitting

$$A_{j-l}^{(n+2)} = T_{j-l}^{(n+2)} + \Delta_{j-l}^{(n+2)} \tag{4.6}$$

It follows from (4.5) that each matrix $\Delta_{j-l}^{(n+2)}$ is of rank equal to 2 and according to relation (4.6), it is obvious that rank$(A_{j-l}^{(n+2)} - T_{j-l}^{(n+2)}) = o(n + 2)$. Exploiting Proposition 2.7 it follows that the matrix sequences $\{A_{j-l}^{(n+2)}\}_n$ and $\{T_{j-l}^{(n+2)}\}_n$ are equally distributed (ED) and equally localized (EL), i.e.,

$$\{A_{j-l}^{(n+2)}\}_n \simeq_{L.D} \{T_{j-l}^{(n+2)}\}_n \tag{4.7}$$

When constructing the two-level matrices A_{d_n} and Δ_{d_n} of the following way:

$$T_{d_n} = [T_{j-l}^{(n+2)}]_{j,l=0}^{n+1} \quad \text{and} \quad \Delta_{d_n} = [\Delta_{j-l}^{(n+2)}]_{j,l=0}^{n+1} \tag{4.8}$$

we deduce from relations (3.14), (4.2), (4.3), (4.4), (4.5), and (4.8) that

$$A_{d_n} = T_{d_n} + \Delta_{d_n} \tag{4.9}$$

where Δ_{d_n} is a matrix of order $d_n = (n+2)^2$ and whose the rank satisfies the inequality

$$rank(\Delta_{d_n}) \leq 4(n+1) = o(d_n) \tag{4.10}$$

Next, T_{d_n} is a symmetric block Toeplitz matrices with symmetric Toeplitz blocks generated by the function $s(x,y)$ defined in (4.4). Using relations (4.9) and (4.10) we have that

$$\begin{array}{l} A_{d_n} - T_{d_n} = \Delta_{d_n} \\ rank(\Delta_{d_n}) = o(d_n) \end{array} \tag{4.11}$$

Exploiting again Proposition 2.7 it follows that the matrix sequences $\{A_{d_n}\}_n$ and $\{T_{d_n}\}_n$ are equally distributed (ED) and equally localized (EL), i.e.,

$$\{A_{d_n}\}_n \simeq_{L.D} \{T_{d_n}\}_n \tag{4.12}$$

so, the Toeplitz matrices T_{d_n} are good approximations for the collocation matrices A_{d_n} which can also be seen as good preconditioners for A_{d_n}.

4.2 *Asymptotic behavior of the generating function $s(x,y)$*

In this subsection we study the asymptotic behavior of the generating function $s(x,y)$ of two-level block Toeplitz matrices $T_{d_n} := T_{d_n}(s)$ in the Multiquadric, Inverse Multiquadric and Gaussian cases. First, we recall the following main result for the integrable functions.

Proposition 4.1. *[8]. Let $\{S_n\}_n$ be a sequence of quasi-uniformly distributed grid points $x_i^{(n)}$ on $I = [-\pi,\pi]$ Then, for any bounded and Riemann integrable function g, we have*

$$\sum_{i=0}^{n-1} g(x_i^{(n)}) = \frac{n}{2\pi}\int_{-\pi}^{\pi} g + o(n).$$

If the distribution is uniform and if g is bounded and Lipschitz continuous except, at most, for a finite number of discontinuity points, then

$$\sum_{i=0}^{n-1} g(x_i^{(n)}) = \frac{n}{2\pi}\int_{-\pi}^{\pi} g + O(1).$$

Armed with Proposition 4.1, the following Lemma holds true

Lemma 4.2. *The real-valued integrable function $s(x,y)$ is even and unbounded over the compact domain $\widehat{\Omega} = [-1,1]^2$.*

Proof. In this proof, we treat separately the cases: Multiquadric, Inverse Multiquadric and Gaussian. For $j,k = 0,1,...,n+1$; let us recall that: $z_{jk} = (x_j, y_k)$, $h = \frac{1}{n+1}$ and $g = \frac{c}{h}$.

- **Multiquadric case**

$$c_j^k = \frac{1}{h}(j^2+k^2+g^2)^{-\frac{1}{2}} + \frac{g^2}{h}(j^2+k^2+g^2)^{-\frac{3}{2}} = c_{j,k}^{(1)} + c_{j,k}^{(2)}$$

where

$$c_{j,k}^{(1)} = \frac{1}{h}(j^2+k^2+g^2)^{-\frac{1}{2}} \quad \text{and} \quad c_{j,k}^{(2)} = \frac{g^2}{h}(j^2+k^2+g^2)^{-\frac{3}{2}}$$

Since the Fourier coefficients c_k^j are nonnegative, to study the behavior of the series $\sum\limits_{k=1}^{\infty}\sum\limits_{j=1}^{\infty} c_j^k$, we study the behavior of the sequence $\sum\limits_{k=1}^{n+1} c_{0,k}^{(2)}$ and we conclude by exploiting the inequality

$$\sum_{k=1}^{n+1} c_{0,k}^{(2)} \leq c_0^0 + 4\left(\sum_{k=1}^{n+1} c_{0,k}^{(2)} + \sum_{k=1}^{n+1}\sum_{j=1}^{n+1} c_j^k\right) \tag{4.13}$$

Indeed:

$$c_{0,k}^{(2)} = \frac{1}{h}\frac{g^2}{(k^2+g^2)^{\frac{3}{2}}} = \frac{g^3}{c(n+1)^3}\frac{1}{[(\frac{k}{n+1})^2+c^2]^{\frac{3}{2}}} = \frac{c^2}{[c^2+y_k^2]^{\frac{3}{2}}}$$

Since the function $y \mapsto \frac{c^2}{(c^2+y^2)^{\frac{3}{2}}}$ is positive and continuous on the interval $[0,1]$, it is Riemann integrable, so

$$0 < \int_0^1 \frac{c^2}{(c^2+y^2)^{\frac{3}{2}}} dy = \alpha_0 < \infty,$$

and according to Proposition 4.1, we have that

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=1}^{n+1} \frac{c^2}{(c^2+y_k^2)^{\frac{3}{2}}} = \int_0^1 \frac{c^2}{(c^2+y^2)^{\frac{3}{2}}} dy = \alpha_0.$$

Then, for $\epsilon = \alpha_0/2$, $\exists N_\epsilon \in \mathbb{N}$ such that

$$\begin{aligned} n > N_\epsilon \Rightarrow & \left| \frac{1}{n+1}\sum_{k=1}^{n+1} \frac{c^2}{(c^2+y_k^2)^{\frac{3}{2}}} - \alpha_0 \right| < \frac{\alpha_0}{2} \\ & \Rightarrow \frac{\alpha_0}{2}(n+1) < \sum_{k=1}^{n+1} \frac{c^2}{(c^2+y_k^2)^{\frac{3}{2}}} < \frac{3\alpha_0}{2}(n+1) \end{aligned}$$

Then,

$$\sum_{k=1}^{n+1} c_{0,k}^{(2)} \sim n+1$$

so

$$\lim_{n\to\infty}\sum_{k=1}^{n+1} c_{0,k}^{(2)} = \infty \tag{4.14}$$

From (4.4), (4.13), and (4.14) we obtain

$$\lim_{(x,y)\to(0,0)} s(x,y) = \infty, \quad \lim_{(x,y)\to(0,1)} s(x,y) = \infty$$

and

$$\lim_{(x,y)\to(1,0)} s(x,y) = \infty, \quad \lim_{(x,y)\to(1,1)} s(x,y) = \infty$$

Hence, $s(x,y)$ is unbounded over $\widehat{\Omega}$. Since the Toeplitz matrix $T_{d_n}(s)$ is symmetric, we deduce that the function $s(x,y)$ is even. □

- **Inverse Multiquadric case**

$$c_j^k = \frac{1}{h^3}\frac{j^2+k^2-2g^2}{(j^2+k^2+g^2)^{\frac{5}{2}}},$$

then

$$c_k^0 = \frac{1}{h^3}\frac{k^2-2g^2}{(k^2+g^2)^{\frac{5}{2}}} = \frac{-2c^2+(\frac{k}{n+1})^2}{[c^2+(\frac{k}{n+1})^2]^{\frac{5}{2}}} = \frac{-2c^2+y_k^2}{(c^2+y_k^2)^{\frac{5}{2}}}$$

One shows as in Multiquadic case the following relations

$$\sum_{k=1}^{n+1} c_k^0 \sim -(n+1) \quad \text{and} \quad \lim_{n\to\infty}\sum_{k=1}^{n+1} c_k^0 = -\infty \tag{4.15}$$

Since the Fourier coefficients c_j^k of $s(x,y)$ are all nonpositive, we have

$$c_0^0 + 4\left(\sum_{k=1}^{n+1} c_k^0 + \sum_{k=1}^{n+1}\sum_{j=1}^{n+1} c_k^j\right) \le \sum_{k=1}^{n+1} c_k^0. \tag{4.16}$$

From (4.4), (4.15), and (4.16), it follows that

$$\lim_{(x,y)\to(0,0)} s(x,y) = -\infty, \quad \lim_{(x,y)\to(0,1)} s(x,y) = -\infty$$

and

$$\lim_{(x,y)\to(1,0)} s(x,y) = -\infty, \quad \lim_{(x,y)\to(1,1)} s(x,y) = -\infty$$

Hence, $s(x,y)$ is unbounded over the domain $\widehat{\Omega}$ and because $T_{d_n}(s)$ is symmetric, we deduce that the function $s(x,y)$ is even.

• **Gaussian**
Also in this case, one shows as in Inverse Multiquadric case the following limits

$$\lim_{(x,y)\to(0,0)} s(x,y) = -\infty, \quad \lim_{(x,y)\to(0,1)} s(x,y) = -\infty$$

and

$$\lim_{(x,y)\to(1,0)} s(x,y) = -\infty, \quad \lim_{(x,y)\to(1,1)} s(x,y) = -\infty$$

Since the Toeplitz matrix $T_{d_n}(s)$ is symmetric, we conclude that the generating function $s(x,y)$ is even.

Remark 4.1. *The solution of the system of linear equations $T_{d_n}(s)v = \tilde{f}$ provides an approximate solution of the collocation system $A_{d_n}v = \tilde{f}$ which is also an approximate solution of the elliptic boundary value problems (1.1). Furthermore, the matrices $T_{d_n}(s)$ are ill-conditioned for any value of n. More precisely, the Euclidean condition number of $T_{d_n}(s)$, as a function of the dimensions, is unbounded:*

$$\lim_{n\to\infty} k_2(T_{d_n}(s)) = \infty. \tag{4.17}$$

Hence, unless some preconditioning are used, all classic iterative methods are very slow.

5. Asymptotic growth of the spectral radii of the collocation matrices

Throughout this section we prove that the spectral radii $\rho(A_{d_n})$ of the collocation matrices A_{d_n} grow as d_n when the shape parameter "c" is strictly greater than $\sqrt{2}$. The proof of this result will be done in the cases: Multiquadric, Inverse Multiquadric and Gaussian.

Lemma 5.1. *If $c > \sqrt{2}$ then, the inequalities (5.1), (5.2), and (5.3) hold true*

$$\rho(-T_{d_n}) = \rho(T_{d_n}) \leq \rho(A_{d_n}) \leq \rho(T_{d_n}) + \rho(\Delta_{d_n}) \tag{5.1}$$

In Multiquadric case

$$\min_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} c_{k-p}^{j-l} \leq \rho(T_{d_n}) \leq \max_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} c_{k-p}^{j-l} \tag{5.2}$$

and in Inverse Multiquadric and Gaussian cases

$$\min_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} (-c_{k-p}^{j-l}) \leq \rho(-T_{d_n}) \leq \max_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} (-c_{k-p}^{j-l}) \tag{5.3}$$

Proof. According to (4.9)

$$A_{d_n} = T_{d_n} + \Delta_{d_n}$$

then

$$(A_{d_n})^2 = (T_{d_n})^2 + T_{d_n}\Delta_{d_n} + \Delta_{d_n}T_{d_n} + (\Delta_{d_n})^2 \tag{5.4}$$

since

$$T_{d_n} = [T_{j-l}^{(n+2)}]_{j,l=0}^{n+1} \text{ and } \Delta_{d_n} = [\Delta_{j-l}^{(n+2)}]_{j,l=0}^{n+1}$$

then

$$(T_{d_n})^2 = \left[\sum_{s=0}^{n+1} T_{j-s}^{(n+2)} T_{s-l}^{(n+2)}\right]_{j,l=0}^{n+1} ; \ (\Delta_{d_n})^2 = \left[\sum_{s=0}^{n+1} \Delta_{j-s}^{(n+2)} \Delta_{s-l}^{(n+2)}\right]_{j,l=0}^{n+1}$$

$$T_{d_n}\Delta_{d_n} = \left[\sum_{s=0}^{n+1} T_{j-s}^{(n+2)} \Delta_{s-l}^{(n+2)}\right]_{j,l=0}^{n+1} ; \ \Delta_{d_n}T_{d_n} = \left[\sum_{s=0}^{n+1} \Delta_{j-s}^{(n+2)} T_{s-l}^{(n+2)}\right]_{j,l=0}^{n+1}$$

then

$$(A_{d_n})^2 = \left[\sum_{s=0}^{n+1} \left\{ T_{j-s}^{(n+2)} T_{s-l}^{(n+2)} + T_{j-s}^{(n+2)} \Delta_{s-l}^{(n+2)} + \Delta_{j-s}^{(n+2)} T_{s-l}^{(n+2)} + \Delta_{j-s}^{(n+2)} \Delta_{s-l}^{(n+2)} \right\}\right]_{j,l=0}^{n+1} \tag{5.5}$$

For $k, p = 0, 1, ..., n + 1$

$$(T_{j-s}^{(n+2)} T_{s-l}^{(n+2)})_{k,p} = \sum_{q=0}^{n+1} (T_{j-s}^{(n+2)})_{kq} (T_{s-l}^{(n+2)})_{qp} = \sum_{q=0}^{n+1} c_{k-q}^{j-s} c_{q-p}^{s-l}, \tag{5.6}$$

$$(T_{j-s}^{(n+2)} \Delta_{s-l}^{(n+2)})_{k,p} = \sum_{q=0}^{n+1} (T_{j-s}^{(n+2)})_{kq} (\Delta_{s-l}^{(n+2)})_{qp} = \sum_{q=0}^{n+1} c_{k-q}^{j-s} (\phi_{q-p}^{s-l} - c_{q-p}^{s-l}), \tag{5.7}$$

$$(\Delta_{j-s}^{(n+2)} T_{s-l}^{(n+2)})_{k,p} = \sum_{q=0}^{n+1} (\Delta_{j-s}^{(n+2)})_{kq} (T_{s-l}^{(n+2)})_{qp} = \sum_{q=0}^{n+1} (\phi_{k-q}^{j-s} - c_{k-q}^{j-s}) c_{q-p}^{s-l}, \tag{5.8}$$

$$(\Delta_{j-s}^{(n+2)} \Delta_{s-l}^{(n+2)})_{k,p} = \sum_{q=0}^{n+1} (\Delta_{j-s}^{(n+2)})_{kq} (\Delta_{s-l}^{(n+2)})_{qp} = \sum_{q=0}^{n+1} (\phi_{k-q}^{j-s} - c_{k-q}^{j-s})(\phi_{q-p}^{s-l} - c_{q-p}^{s-l}). \tag{5.9}$$

From (5.5), (5.6), (5.7), (5.8), (5.9), we have that $(A_{d_n})^2 = \left[[a_{k,p}^{j,l}]_{k,p=0}^{n+1}\right]_{j,l=0}^{n+1}$ where for $j, l = 0, 1, ..., n+1$ and $k, p = 0, 1, ..., n + 1$

$$a_{k,p}^{j,l} = \sum_{s=0}^{n+1}\sum_{q=0}^{n+1} \{ c_{k-q}^{j-s} c_{q-p}^{s-l} + C_{k-q}^{j-s} (\phi_{q-p}^{s-l} - c_{q-p}^{s-l}) + (\phi_{k-q}^{j-s} - c_{k-q}^{j-s}) c_{q-p}^{s-l}$$

$$+ (\phi_{k-q}^{j-s} - c_{k-q}^{j-s})(\phi_{q-p}^{s-l} - c_{q-p}^{s-l})\}$$
$$= \sum_{s=0}^{n+1}\sum_{q=0}^{n+1} \phi_{k-q}^{j-s}\phi_{q-p}^{s-l} > 0$$

since $\phi_{n,m}^{i,r} > 0 \quad \forall i, n, m, r = 0, 1, ..., n+1$. Then

$$(A_{d_n})^2 > 0 \tag{5.10}$$

On the other side, $T_{d_n} = \left[[c_{k-p}^{j-l}]_{k,p=0}^{n+1}\right]_{j,l=0}^{n+1} \geq 0$ (in Multiquadric case) and $-T_{d_n} = \left[[-c_{k-p}^{j-l}]_{k,p=0}^{n+1}\right]_{j,l=0}^{n+1} \geq 0$ since $-c_{k-p}^{j-l} \geq 0$ (in fact $c > \sqrt{2}$) $\forall i, j, k, p = 0, 1, ..., n+1$ (Inverse Multiquadric and Gaussian cases). Then $(T_{d_n})^2 = (-T_{d_n})^2 \geq 0$. All the coefficients of the matrices $(A_{d_n})^2$ and $(T_{d_n})^2$ are nonnegative, then both$(A_{d_n})^2$ and $(T_{d_n})^2$ are nonnegative matrices, soA_{d_n} and T_{d_n} are eventually nonnegative matrices. According to Theorem 2.3, the matrices A_{d_n}, $A_{d_n}^T$, T_{d_n} and $T_{d_n}^T$ possess the Perron-Frobenius property. Since $A_{d_n} - T_{d_n} = \Delta_{d_n} \geq 0$ and $\|A_{d_n}\|_2 \leq \|T_{d_n}\|_2 + \|\Delta_{d_n}\|_2$, it follows from Theorem 2.6 that

$$\rho(-T_{d_n}) = \rho(T_{d_n}) \leq \rho(A_{d_n}) \leq \rho(T_{d_n}) + \rho(\Delta_{d_n})$$

and according to Theorem 2.4, we have that

- In Multiquadric case

$$\min_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} c_{k-p}^{j-l} \leq \rho(T_{d_n}) \leq \max_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} c_{k-p}^{j-l}$$

and

- Inverse Multiquadric and Gaussian cases

$$\min_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} (-c_{k-p}^{j-l}) \leq \rho(-T_{d_n}) \leq \max_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} (-c_{k-p}^{j-l})$$

□

Lemma 5.2. *If $c > \sqrt{2}$, then the spectral radius $\rho(T_{d_n})$ of the Toeplitz matrix T_{d_n} grows as $d_n = (n+2)^2$.*

Proof. We treat separately the cases: Multiquadric, Inverse Multiquadric and Gaussian.

- **Multiquadric**

For $j, l, p, k = 0, 1, ..., n+1$

$$c_{k-p}^{j-l} = \frac{1}{h}\frac{1}{\sqrt{(j-l)^2 + (k-p)^2 + g^2}} + \frac{1}{h}\frac{g^2}{[(j-l)^2 + (k-p)^2 + g^2]^{\frac{3}{2}}}$$

Then

$$\sum_{l=0}^{n+1}\sum_{p=0}^{n+1} c_{k-p}^{j-l} = \frac{1}{h}\sum_{l=0}^{n+1}\sum_{p=0}^{n+1}\left(\frac{1}{\sqrt{(j-l)^2 + (k-p)^2 + g^2}} + \frac{g^2}{[(j-l)^2 + (k-p)^2 + g^2]^{\frac{3}{2}}}\right) \tag{5.11}$$

First of all, for $j-l \in [-n-1, n+1]$ fixed, let us study the functions: $f_{n,j-l}(x) = \frac{1}{(g^2+x^2+(j-l)^2)^{\frac{1}{2}}}$ and $g_{n,j-l}(x) = \frac{g^2}{(g^2+x^2+(j-l)^2)^{\frac{3}{2}}}$ over the interval $[-n-1, n+1]$. Since $f_{n,j-l}$ and $g_{n,j-l}$ are even functions, the study of these functions reduces on $[0, n+1]$. Because $f'_{n,j-l}(x) = \frac{-x}{(g^2+x^2+(j-l)^2)^{\frac{3}{2}}} < 0$and $g'_{n,j-l}(x) = \frac{-3g^2x}{(g^2+x^2+(j-l)^2)^{\frac{5}{2}}} < 0$ for $x \in (0, n+1]$, then $f_{n,j-l}$ and $g_{n,j-l}$ are decreasing functions on $[0, n+1]$. So, it follows that,

$$\min_{0\le j,k\le n+1}\sum_{l=0}^{n+1}\sum_{p=0}^{n+1} c_{k-p}^{j-l} = \frac{1}{h}\sum_{l=0}^{n+1}\sum_{p=0}^{n+1}\left(\frac{1}{\sqrt{l^2+p^2+g^2}} + \frac{g^2}{[l^2+p^2+g^2]^{\frac{3}{2}}}\right) \tag{5.12}$$

For $l, p = 0, 1, ..., n+1$,

$$\frac{1}{n+1}\frac{1}{\sqrt{2+c^2}} \le \frac{1}{\sqrt{l^2+p^2+g^2}} \quad \text{and} \quad \frac{1}{n+1}\frac{c^2}{(2+c^2)^{\frac{3}{2}}} \le \frac{g^2}{(l^2+p^2+g^2)^{\frac{3}{2}}}$$

then

$$\left(\frac{1}{\sqrt{2+c^2}} + \frac{c^2}{(2+c^2)^{\frac{3}{2}}}\right)(n+2)^2 \le \frac{1}{h}\sum_{l=0}^{n+1}\sum_{p=0}^{n+1}\left(\frac{1}{\sqrt{l^2+p^2+g^2}} + \frac{g^2}{[l^2+p^2+g^2]^{\frac{3}{2}}}\right) \tag{5.13}$$

On the other side, one easily shows that

$$\begin{aligned}
\max_{0\le j,k\le n+1}\sum_{l=0}^{n+1}\sum_{p=0}^{n+1} c_{k-p}^{j-l} &\le 4\sum_{l=0}^{[\frac{n}{2}]+1}\sum_{p=0}^{[\frac{n}{2}]+1} c_p^l \\
&= 4\left\{c_0^0 + 2\sum_{p=1}^{[\frac{n}{2}]+1} c_p^0 + \sum_{l=1}^{[\frac{n}{2}]+1}\sum_{p=1}^{[\frac{n}{2}]+1} c_p^l\right\} \\
&\underset{(\alpha)}{\le} 4\left[\frac{2}{c} + \frac{2}{c}(n+2) + \frac{1}{2c}(n+2)^2\right] \le \frac{18}{c}(n+2)^2
\end{aligned} \tag{5.14}$$

(α) follows from $c_p^l \le \frac{2}{c}$ $\forall l, p = 0, 1, ..., [\frac{n}{2}]+1$ and $[\frac{n}{2}]+1 \le \frac{n+2}{2}$. It follows from (5.2), (5.12), (5.13), and (5.14) that the spectral radius of the matrix T_{d_n} grows as $d_n = (n+2)^2$.

• **Inverse Multiquadric**

For $j, k = 0, 1, ..., n+1$,

$$\sum_{l=0}^{n+1}\sum_{p=0}^{n+1}(-c_{k-p}^{j-l}) = \frac{1}{h^3}\sum_{l=0}^{n+1}\sum_{p=0}^{n+1}\left(\frac{2g^2-(j-l)^2-(k-p)^2}{[(j-l)^2+(k-p)^2+g^2]^{\frac{5}{2}}}\right) \tag{5.15}$$

For $j-l \in [-n-1, n+1]$ fixed, the function $f_{n,j-l}(x) = \frac{-x^2-(j-l)^2+2g^2}{[x^2+(j-l)^2+g^2]^{\frac{5}{2}}}$ defined on the interval $[-n-1, n+1]$ is even, then the study of $f_{n,j-l}$ reduces on $[0, n+1]$. Because $f'_{n,j-l}(x) = \frac{-x(g^2+x^2+(j-l)^2)^{\frac{3}{2}}\left(12g^2-3x^2-3(j-l)^2\right)}{(g^2+x^2+(j-l)^2)^5} <$

0 for $x \in (0, n+1]$ (since $c > \sqrt{2}$), then $f_{n,j-l}$ is decreasing over [0,n+1]. Hence

$$\min_{0\le j,k\le n+1}\sum_{l=0}^{n+1}\sum_{p=0}^{n+1}(-c_{k-p}^{j-l}) = \min_{0\le j\le n+1}\sum_{l=0}^{n+1}\sum_{p=0}^{n+1}(-c_{p}^{j-l}) \tag{5.16}$$

Also, the function $g_{n,p}(x) = \frac{-x^2-p^2+2g^2}{[x^2+p^2+g^2]^{\frac{5}{2}}}$ is a decreasing function over the interval [0,n+1]. So, we deduce that

$$\min_{0\le j\le n+1}\sum_{l=0}^{n+1}\sum_{p=0}^{n+1}(-c_{p}^{j-l}) = \sum_{l=0}^{n+1}\sum_{p=0}^{n+1}(-c_{p}^{l}) \tag{5.17}$$

According to (5.16), (5.17), it follows that

$$\min_{0\le j,k\le n+1}\sum_{l=0}^{n+1}\sum_{p=0}^{n+1}(-c_{k-p}^{j-l}) = \sum_{l=0}^{n+1}\sum_{p=0}^{n+1}(-c_{p}^{l}) \tag{5.18}$$

Because, $\min_l\{\min_p(-c_p^l)\} = \frac{1}{h^3}\frac{2g^2-2(n+1)^2}{(g^2+2(n+1)^2)^{\frac{5}{2}}} = \frac{1}{h^3}\frac{2}{(n+1)^3}\frac{c^2-1}{(c^2+2)^{\frac{5}{2}}}$, then for $l, p = 0, 1, ..., n+1$,

$$\frac{1}{h^3}\frac{2}{(n+1)^3}\frac{c^2-1}{(c^2+2)^{\frac{5}{2}}} \le \frac{1}{h^3}\frac{2g^2-l^2-p^2}{(g^2+l^2+p^2)^{\frac{5}{2}}} = -c_p^l$$

so,

$$2\left(\frac{c^2-1}{(c^2+2)^{\frac{5}{2}}}\right)(n+2)^2 \le \sum_{l=0}^{n+1}\sum_{p=0}^{n+1}(-c_p^l) \tag{5.19}$$

Next,

$$\begin{aligned}
\max_{0\le j,k\le n+1}\sum_{l=0}^{n+1}\sum_{p=0}^{n+1}(-c_{k-p}^{j-l}) &\le 4\sum_{l=0}^{[\frac{n}{2}]+1}\sum_{p=0}^{[\frac{n}{2}]+1}(-c_p^l) \\
&= 4\left\{-c_0^0 + 2\sum_{p=1}^{[\frac{n}{2}]+1}(-c_p^0) + \sum_{l=1}^{[\frac{n}{2}]+1}\sum_{p=1}^{[\frac{n}{2}]+1}(-c_p^l)\right\} \\
&\underset{(\beta)}{\le} 4\left[\frac{2}{c^3} + \frac{2}{c^3}(n+2) + \frac{1}{2c^3}(n+2)^2\right] \le \frac{18}{c^3}(n+2)^2
\end{aligned} \tag{5.20}$$

(β) follows from $-c_p^l \le \frac{2}{c^3}$ $\forall l, p = 0, 1, ..., [\frac{n}{2}]+1$. It follows from (5.3), (5.18), (5.19), and (5.20) that the spectral radius $\rho(T_{d_n})$ of T_{d_n} grows as $(n+2)^2$.

• Gaussian
One shows as in Inverse Multiquadric case by considering the function $f_{n,j-l}(x) = [g^2 - (j-l)^2 - x^2]e^{-\frac{(j-l)^2+x^2}{g^2}}$ that the spectral radius $\rho(T_{d_n})$ of the collocation matrix T_{d_n} grows as $d_n = (n+2)^2$. $\square$

Lemma 5.3. *If $c > \sqrt{2}$, then the inequalities (5.21) hold true*

$$\rho(\Delta_{d_n}) \leq 2 \max_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} \phi_{k-p}^{j-l} \leq 2\sqrt{2}\, c\, d_n \tag{5.21}$$

Proof. Since $c > \sqrt{2}$ then all the coefficients of Δ_{d_n} are nonnegative, so Δ_{d_n} is an eventually nonnegative matrix, according to Theorem 2.3 both matrices Δ_{d_n} and $(\Delta_{d_n})^T$ possess the Perron-Frobenius property. It follows from Theorem 2.4 that

$$\rho(\Delta_{d_n}) \leq \max_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} (\phi_{k-p}^{j-l} - c_{k-p}^{j-l})$$

Obviously

$$\max_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} (\phi_{k-p}^{j-l} - c_{k-p}^{j-l}) \leq 2 \max_{0\leq j,k\leq n+1} \sum_{l=0}^{n+1}\sum_{p=0}^{n+1} \phi_{k-p}^{j-l} \leq \begin{cases} 2\sqrt{2}\, c\, d_n & \text{(MQ)} \\ \frac{2}{c}\, d_n & \text{(IMQ)} \\ 2d_n & \text{(Gaussian)} \end{cases}$$

□

Lemma 5.4. *If $c > \sqrt{2}$, then the spectral radius $\rho(A_{d_n})$ of the collocation matrix A_{d_n} grows as $d_n = (n+2)^2$.*

Proof. The proof follows from (5.1), Lemmas 5.2 and 5.3. □

6. Conclusion and future works

In this paper we have studied in detail the asymptotic behavior of the generating function of the block Toeplitz matrices which are equally distributed and equally localized as the collocation matrices approximating elliptic boundary value problems (1.1) and we have provided a deep analysis of the asymptotic growth of the spectral radii of collocation matrices. Our future researches will consist to solve the preconditioned collocation systems by the quasi minimal residual (QMR) method with preconditioner chosen in the Tau algebra.

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