International Journal of Applied Mathematics and Computation
ISSN: 0974 - 4665 (Print) 0974 - 4673 (Online)
Journal homepage: www.darbose.in/ijamc
Volume 4(2) 2012 170–182

The harvested enemy and a cover protected ammensal species pair with limited resources

K.V.L.N. Acharyulu[a, *], **N.Ch. Pattabhi Ramacharyulu**[b]

[a]Faculty of Mathematics, Department of Mathematics, Bapatla Engineering College, Bapatla-522101
[b]Former Faculty, Department of Mathematics & Humanities, National Institute of Technology, Warangal – 506004, India

ABSTRACT

The paper elucidates a mathematical model of harvested enemy species and a cover protected Ammensal pair with limited resources. A cover proportionate to the population of Ammensal is provided for Ammensal species to protect from the attacks of the enemy species. More over the enemy with limited resources is harvested at a constant rate. The model is characterized by a couple of first order non-linear ordinary differential equations. All six equilibrium points for this model are identified and their stability criteria are discussed.

Keywords: Equilibrium points; Normal Steady state; stability; harvesting.

1. Introduction

Harvesting plays a major role in the population growth of Ammensal-enemy species.Mathematical modeling of ecosystems was initiated in 1925 by Lotka [13]. The general concepts of modeling have been presented in the treatises of Meyer [14], Kapur [9, 10]. N.C. Srinivas [16] studied the competitive ecosystems of two species and three species with limited and unlimited resources. Lakshminarayan and Pattabhiramacharyulu [11, 12] investigated Prey-predator Ecological models with a partial cover for the prey and alternate food for the predator. Recently, stability analysis of competitive species was carried out by Archana Reddy, Pattabhi Ramachryulu and Gandhi [7] and by Bhaskara Rama Sarma and Pattabhiramacharyulu [8], while the mutualism between to species was examined by Ravindra Reddy [15]. The present authors Acharyulu [1–6] and Pattabhi Ramacharyulu investigated some remarkable results on the stability of an Ammensal- enemy species pair with various resources.

The present paper investigates on an analytical study of a two species Ammensal – enemy model. The enemy is harvested at a constant rate with limited resources. In addition to it, a cover proportionate to the population of Ammensal is provided for Ammensal species to protect from the attacks of the enemy species. The model is characterized by a couple of first order non-linear ordinary differential equations. All six equilibrium points for the model are obtained and their stability criteria are discussed. The linearised perturbed equations are solved and the trajectories are derived.

1.1 *Notation Adopted*

N_1 and N_2 are the populations of the Ammensal (S_1) and enemy (S_2) species with natural growth rates a_1 and a_2 respectively.

*kvlna@yahoo.com

a_{11} is rate of decrease of the Ammensal due to insufficient food.

a_{12} is rate of increase of the Ammensal due to inhibition by the enemy.

a_{22} is rate of decrease of the enemy due to insufficient food.

$h_2 = a_{22}H_2$ is rate of harvest of the enemy.

$K_i = a_i/a_{ii}$ are the carrying capacities of $N_i,\ i = 1, 2, \ldots$

$\alpha = a_{12}/a_{11}$ is the coefficient of Ammensalism.

$b =$ The constant characterized by the cover which is provided for the Ammensal species $(0 < b < 1)$

The state variables N_1 and N_2 as well as the model parameters a_1, a_2, a_{11}, a_{22}, K_1, K_2,α,b , h_2 are assumed to be non-negative constants.

2. Basic Equations

An Ammensal-harvested enemy model with limited resources where a cover proportionate to the population of Ammensal is provided for Ammensal species to protect from the attacks of the enemy species characterized by a pair of coupled non-linear ordinary differential equations.

(I) Equation for the Growth Rate of Ammensal Species (N_1):

$$\frac{dN_1}{dt} = a_{11}(K_1N_1 - N_1^2 - \alpha(1-b)N_1N_2) \tag{2.1}$$

(II) Equation for the Growth Rate of enemy Species (N_2):

$$\frac{dN_2}{dt} = a_{22}(K_2N_2 - N_2^2 - H_2) \tag{2.2}$$

3. Equilibrium points

The system under this investigation has six equilibrium states given by $\frac{dN_i}{dt} = 0$ where $i = 1, 2, \ldots$. Out of these, three are the Ammensal washed out states and the reaming are co-existence states. Here the enemy always survives. These states are described here under.

(E1) Ammensal washed out state I

$$N_1 = 0; \quad N_2 = \frac{K_2}{2} \tag{3.1}$$

This would arise only when $H_2 = \frac{K_2^2}{4}$

(E2) Ammensal washed out state II

$$N_1 = 0; \quad N_2 = K_2 - \frac{H_2}{K_2} \tag{3.2}$$

This would happen only when $H_2 < \frac{K_2^2}{4}$

(E3) Ammensal washed out state III

$$N_1 = 0; \;\; N_2 = \frac{H_2}{K_2} \tag{3.3}$$

This would exist only when $H_2 < \frac{K_2^2}{4}$

(E4) Coexistence state I

$$N_1 = K_1 - \frac{\alpha(1-b)\; K_2}{2}\; ; N_2 = \frac{K_2}{2} \tag{3.4}$$

This would happen only when $K_1 > \frac{\alpha(1-b)\; K_2}{2}$ and $H_2 = \frac{K_2^2}{4}$

(E5) Co-existence state II

$$N_1 = K_1 - \frac{\alpha(1-b)H_2}{K_2}; N_2 = \frac{H_2}{K_2} \tag{3.5}$$

This would arise only when $K_1K_2 > \alpha(1-b)\; H_2$ and $H_2 < \frac{K_2^2}{4}$

(E6) Co-existence state III

$$N_1 = K_1 - \alpha(1-b)\left(K_2 - \frac{H_2}{K_2}\right); N_2 = K_2 - \tfrac{H_2}{K_2} \tag{3.6}$$

This would exist only when $K_1 > \alpha(1-b)\;\left(K_2 - \frac{H_2}{K_2}\right)$ and $H_2 < \frac{K_2^2}{4}$

Note: There would be no fully washed out state and also enemy washed-out state. However there are three equilibrium states in which the Ammensal is washed out and three (conditionally) coexistence states.

4. Stability of the Equilibrium States

After linearization, we get

$$\frac{dU}{dt} = AU \tag{4.1}$$

where

$$A = \begin{bmatrix} a_{11}(K_1 - 2\bar{N}_1 - \alpha(1-b)\bar{N}_2) & -a_{11}\alpha(1-b)\bar{N}_1 \\ 0 & a_{22}(K_2 - 2\bar{N}_2) \end{bmatrix} \tag{4.2}$$

The corresponding characteristic equation for the system is

$$\det[A - \lambda I] = 0 \tag{4.3}$$

The equilibrium state is stable only when the roots of the equation (4.3) are negative when they are real or have negative real parts in case they are complex.

4.1 *Stability of the Equilibrium State E_1:*

In this case

$$A = \begin{bmatrix} -\,a_{11}\left(K_1 - \frac{\alpha\,(1-b)\,K_2}{2}\right) & -\,\alpha\,(1-b)\,a_{11}\left(K_1 - \frac{\alpha\,(1-b)\,K_2}{2}\right) \\ 0 & 0 \end{bmatrix} \tag{4.4}$$

The characteristic roots of A are $-a_{11}\left(K_1 - \frac{\alpha\,(1-b)K_2}{2}\right)$, 0. Hence the Equilibrium state is **unstable**. The equation (9) yields the solution curves.

$$U_1 = U_{10}\; e^{a_{11}\left(K_1 - \frac{\alpha\,(1-b)\,K_2}{2}\right)\,t},\; U_2 \;=\; U_{20} \tag{4.5}$$

and the solution curves are illustrated as follows.

Case (i) $U_{10} > U_{20}$ i.e. initially the Ammensal species dominates the enemy species. Figure 1 shows, the Ammensal (S_1) dominates over the enemy (S_2) in its natural growth rate as well as in its initial population strength. In this case the Ammensal always outnumbers the enemy. Further the enemy species is noted to be at a constant distance from the equilibrium point in the course of time, while the Ammensal species is going away from the equilibrium point.

Case (ii) $U_{10} < U_{20}$ i.e initially the enemy species dominates the Ammensal species. In this case, $U_1(t) = U_2(t)$ is possible at time

$$t^* = \frac{1}{a_{11}\left(K_1 - \frac{\alpha\,(1-b)\,K_2}{2}\right)} \log\left(\frac{U_{20}}{U_{10}}\right)$$

Initially the enemy outnumbers the Ammensal and this continues up to $t = t^*$, after which the Ammensal out-numbers the enemy illustrated as in Figure 2.

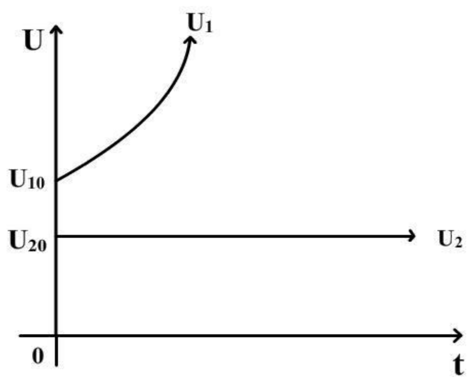

Figure 1

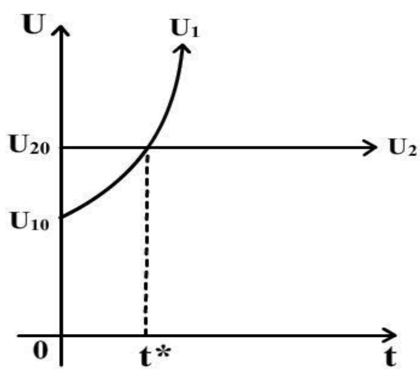

Figure 2

4.1.1 *Trajectories of the Perturbed species:*

The trajectories obtained by solving (13) in $U_1 - U_2$ plane can be given as shown in Figure 3.

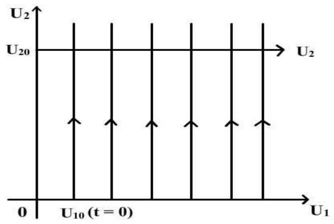

Figure 3

4.2 *Stability of the Equilibrium state (E2)*

In this case

$$A = \begin{bmatrix} a_{11}\left(K_1 - \alpha\,(1-b)\left(K_2 - \frac{H_2}{K_2}\right)\right) & 0 \\ 0 & -a_{22}\left(K_2 - \frac{H_2}{K_2}\right) \end{bmatrix} \tag{4.6}$$

The characteristic roots of A are

$$a_{11}\left(K_1 - \alpha\,(1-b)\left(K_2 - \frac{H_2}{K_2}\right)\right)$$

and

$$-a_{22}\left(K_2 - \frac{H_2}{K_2}\right)$$

since one of these roots is positive, the steady state is **unstable**.

The equation (9) yields the solution curves.

$$U_1 = U_{10}\, e^{a_{11}\left(K_1 - \alpha(1-b)\left(K_2 - \frac{H_2}{K_2}\right)\right) t} \text{ and } U_2 = U_{20}\, e^{-a_{22}\left(K_2 - \frac{H_2}{K_2}\right) t} \tag{4.7}$$

we have divided it in to two cases and the solution curves are illustrated here under.

Case(A) When

$$K_1 > \alpha(1-b)\left(K_2 - \frac{H_2}{K_2}\right) \tag{4.8}$$

Here one of the roots is positive. Hence the state is **stable**. The solution curves are illustrated in Figure 4.

Case (i) $U10 > U20$ i.e. initially the Ammensal (S_1) species dominates over the enemy (S_2) species. In this case the Ammensal dominates the enemy in natural growth rate as well as in its population strength. The Ammensal species is noted to be going away from the equilibrium point as shown in Figure 4, while the enemy species is asymptotic to the equilibrium point.

Case (ii) $U_{10} < U_{20}$ i.e. initially the enemy species dominates the Ammensal species.

The Ammensal (S_1) dominates over the enemy (S_2) in natural growth rate but its initial strength is less than the enemy. In this case the enemy outnumbers the Ammensal till the time instant

$$t^* = \frac{1}{a_{22}\left(K_2 - \frac{H_2}{K_2}\right) + a_{11}\left(K_1 - \alpha(1-b)\left(K_2 - \frac{H_2}{K_2}\right)\right)} \log\left[\frac{U_{20}}{U_{10}}\right]$$

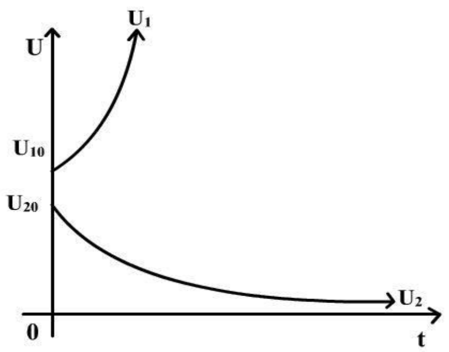

Figure 4

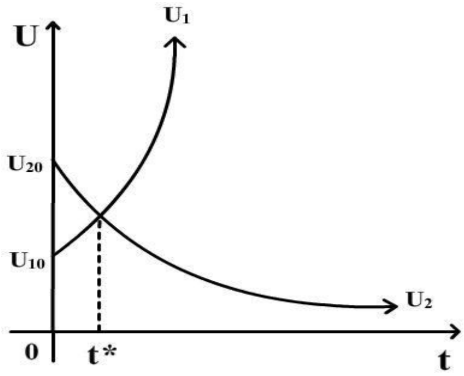

Figure 5

after which the Ammensal goes away from the equilibrium point, while the enemy species is asymptotic to the equilibrium point. Hence the equilibrium point is **unstable**, as shown in Figure 5.

Case(B) When

$$K_1 < \alpha(1-b)\left(K_2 - \frac{H_2}{K_2}\right) \tag{4.9}$$

Here both the roots are negative. Hence the state is **stable**.

The equation (9) yields the solution curves as

$$U_1 = U_{10}\, e^{-\,a_{11}\left[\left(\alpha(1-b)\left(K_2 - \frac{H_2}{K_2}\right) - K_1\right)\right] t} \text{ and } U_2 = U_{20} e^{-\,a_{22}\left(K_2 - \frac{H_2}{K_2}\right)\, t} \tag{4.10}$$

Let $a_{11}\left(\alpha(1-b)\left(K_2 - \frac{H_2}{K_2}\right) - K_1\right) = Q_1$ and $a_{22}\left(K_2 - \frac{H_2}{K_2}\right) = Q_2$

Case(i) When $Q_1 < Q_2$ and $U_{10} > U_{20}$ The enemy (S_2) dominates over the Ammensal (S_1) in natural growth rate but it's initial strength is less than that of Ammensal and Ammensal out numbers the enemy till the time-instant

$$t^* = \frac{1}{a_{22}\left(K_2 - \frac{H_2}{K_2}\right) + a_{11}\left[\left(K_1 - \alpha\,(1-b)\left(K_2 - \frac{H_2}{K_2}\right)\right)\right]} \log\left[\frac{U_{20}}{U_{10}}\right]$$

after that the dominance is reversed,as depicted in Figure 6.

Case (ii) When $Q_1 > Q_2$ and $U_{10} > U_{20}$. The Ammensal (S_1) dominates over the enemy (S_2) in natural growth rate as well as in its initial population strength, as illustrated in Figure 7.

Case (iii) when $Q_1 > Q_2$ and $U_{10} < U_{20}$ The Ammensal (S_1) dominates over the enemy (S_2) in natural growth rate but it's initial strength is less than that of an enemy, and the enemy outnumbers the Ammensal till the time-instant

$$t^* = \frac{1}{a_{22}\left(K_2 - \frac{H_2}{K_2}\right) + a_{11}\left[\left(K_1 - \alpha\,(1-b)\left(K_2 - \frac{H_2}{K_2}\right)\right)\right]} \log\left[\frac{U_{20}}{U_{10}}\right]$$

after that the dominance is reversed, as shown in Figure 8.

Case (iv) When $Q_1 < Q_2$ and $U_{10} < U_{20}$. The enemy (S_2) dominates over the Ammensal (S_1) all throughout the strength, as depicted in Figure 9.

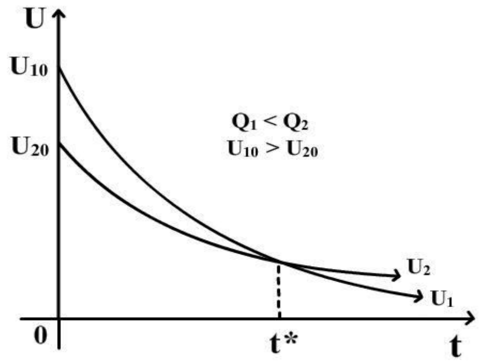

Figure 6

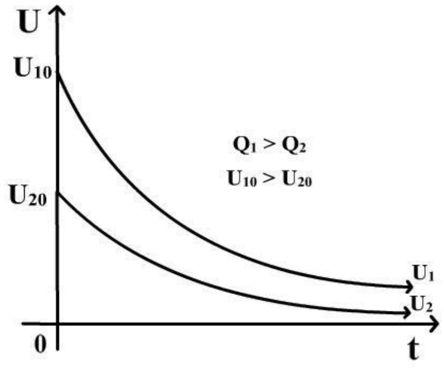

Figure 7

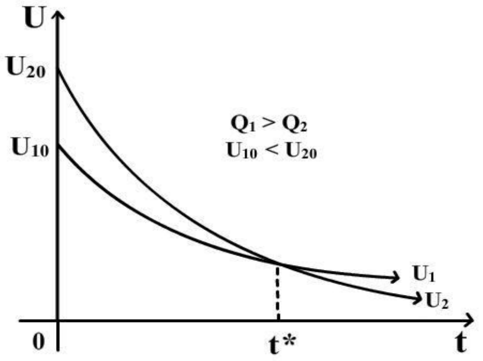

Figure 8

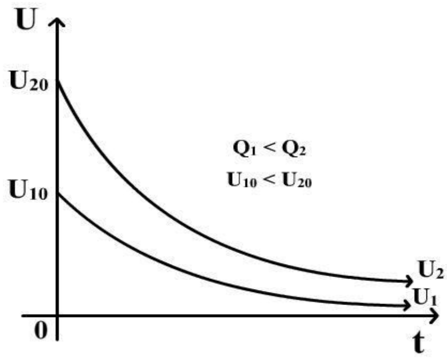

Figure 9

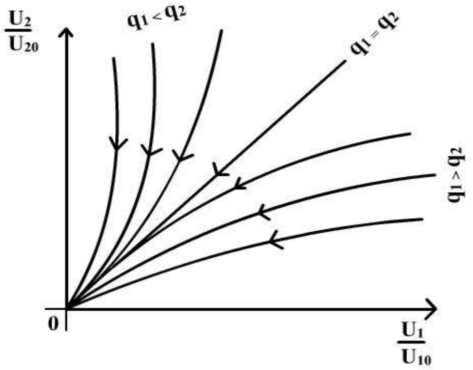

Figure 10

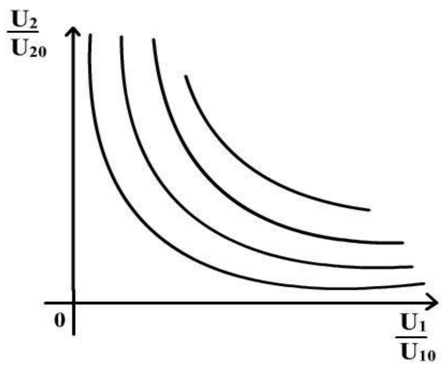

Figure 11

4.2.1 *Trajectories of Perturbed species:*

The trajectories obtained by solving (18) in $U_1 - U_2$ plane are given by

$$\left(\frac{U_1}{U_{10}}\right)^Q \left(\frac{U_2}{U_{20}}\right) = 1 \text{ where } Q = \frac{a_{22}\left(K_2 - \frac{H_2}{K_2}\right)}{a_{11}\left[\left(K_1 - \alpha(1-b)\left(K_2 - \frac{H_2}{K_2}\right)\right)\right]} = \frac{q_1}{q_2} \tag{4.11}$$

which are illustrated in Figure 10 and Figure 11.

4.3 *Stability of the Equilibrium State E_3*

In this case

$$A = \begin{bmatrix} a_{11}\left(K_1 - \frac{\alpha\,(1-b)\,H_2}{K_2}\right) & 0 \\ 0 & a_{22}\left(K_2 - \frac{2\,H_2}{K_2}\right) \end{bmatrix} \tag{4.12}$$

The characteristic roots at A are $a_{11}\left(K_1 - \frac{\alpha\,(1-b)\,H_2}{K_2}\right)$, $a_{22}\left(K_2 - \frac{2H_2}{K_2}\right)$ and these are both positive, hence the steady state is **unstable**.

The equation (9) yields the solution curves.

$$U_1 = U_{10}\, e^{a_{11}\left(K_1 - \frac{\alpha\,(1-b)\,H_2}{K_2}\right)\,t} \text{ and } U_2 = U_{20}\, e^{a_{22}\left(K_2 - \frac{2\,H_2}{K_2}\right)\,t} \tag{4.13}$$

and these are illustrated as follows in Figure 12 and Figure 13.

Case (i) $U_{10} > U_{20}$ i.e. initially the Ammensal (S_1) species dominates over the enemy (S_2) species. The Ammensal dominates the enemy in natural growth rate as well as in its initial population strength. In this case the Ammensal continuously outnumbers the enemy as shown in Figure 12.

Case (ii) $U_{10} < U_{20}$ i.e initially the enemy (S_2) species dominates over the Ammensal (S_1) species. The enemy dominates the Ammensal in natural growth rate as well as in its initial population strength. In this case the enemy continuously outnumbers the Ammensal as shown in Figure 13.

4.3.1 *Trajectories of Perturbed Species:*

The trajectories obtained by solving (21) in $U_1 - U_2$ plane can be given by

$$\frac{U_1}{U_{10}} = \left(\frac{U_2}{U_{20}}\right)^q \tag{4.14}$$

where $q = \dfrac{a_{11}\left(K_1 - \frac{\alpha(1-b)H_2}{K_2}\right)}{a_{22}\left(K_2 - \frac{2H_2}{K_2}\right)}$ which are illustrated in Figure14.

4.4 *Stability of the Equilibrium State E_4*

In this case

$$A = \begin{bmatrix} -\,a_{11}\left(K_1 - \frac{\alpha(1-b)\,K_2}{2}\right) & -\,\alpha\, a_{11}\left(K_1 - \frac{\alpha(1-b)\,K_2}{2}\right) \\ 0 & 0 \end{bmatrix} \tag{4.15}$$

The characteristic roots of A are $-\,a_{11}\left(K_1 - \frac{\alpha(1-b)\,K_2}{2}\right)$, 0. Hence the equilibrium state is **unstable**.

The equation (9) yields the solution curves

$$U_1 = (U_{10}+L_4)\ e^{-\ a_{11}\left(K_1 - \frac{\alpha(1-b)\ K_2}{2}\right)\ t} - L_4, U_2 = U_{20} \tag{4.16}$$

where $L_4 = \alpha\ (1-b)U_{20}$ which are illustrated here under.

Case (i) $U_{10} > U_{20}$. i.e. initially the Ammensal (S_1) species dominates over the enemy (S_2) species. In this case $U_1(t) = U_2(t)$ is possible at time

$$t^* = \frac{1}{a_{11}\left(K_1 - \frac{\alpha(1-b)\ K_2}{2}\right)} log\left(\frac{U_{10}+L_4}{U_{20}+L_4}\right)$$

The Ammensal out numbers the enemy and this continues up to $t = t^*$, after which the enemy outnumbers the Ammensal as shown in Figure 15.

Case (ii) $U_{10} < U_{20}$. i.e initially the enemy (S_2) species dominates over the Ammensal (S_1) species. The enemy dominates the Ammensal in its natural growth as well as in its initial population strength. In this case the enemy always outnumbers the Ammensal.Further the enemy species is noted to be at a constant distance from the equilibrium point in the course of time, while the Ammensal species is asymptotic to the equilibrium point.

Hence the equilibrium point is neutrally **stable** shown as in Figure 16.

4.4.1 *Trajectories of Perturbed species :*

The trajectories obtained by solving (4.16) in $U_1 - U_2$ plane can be given by Figure 17

4.5 *Stability of the Equilibrium State E_5*

In this case

$$A = \begin{bmatrix} -\ a_{11}\left(K_1 - \frac{\alpha(1-b)H_2}{K_2}\right) & -\ \alpha\,(1-b)\ a_{11}\left(K_1 - \frac{\alpha(1-b)H_2}{K_2}\right) \\ 0 & a_{22}\ \left(K_2 - \frac{2H_2}{K_2}\right) \end{bmatrix} \tag{4.17}$$

The characteristic roots of A are $-\ a_{11}\left(K_1 - \frac{\alpha(1-b)H_2}{K_2}\right)$, $a_{22}\ \left(K_2 - \frac{2H_2}{K_2}\right)$ and one of these roots is positive, hence the steady state is unstable.

The equation (9) yields the solution curves.

$$U_1 = -L_5 e^{a_{22}\left(K_2 - \frac{2H_2}{K_2}\right)t} + (U_{10} + L_5)e^{-a_{11}\left(K_1 - \frac{\alpha(1-b)H_2}{K_2}\right)t},\ U_2 = U_{20}e^{a_{22}\left(K_2 - \frac{2H_2}{K_2}\right)t} \tag{4.18}$$

where $L_5 = \dfrac{\alpha(1-b)\ a_{11}\left(K_1 - \frac{\alpha(1-b)H_2}{K_2}\right)U_{20}}{a_{22}\left(K_1 - \frac{2H_2}{K_2}\right) + a_{11}\left(K_1 - \frac{\alpha(1-b)H_2}{K_2}\right)}$ and these are illustrated as below.

Case (i) $U_{10} < U_{20}$. i.e. initially the enemy (S_2) species dominates over the Ammensal (S_1) species. In this case the enemy dominates the Ammensal in natural growth rate as well as in its population strength. The enemy species is noted to be going away from the equilibrium point as shown in Figure 18.

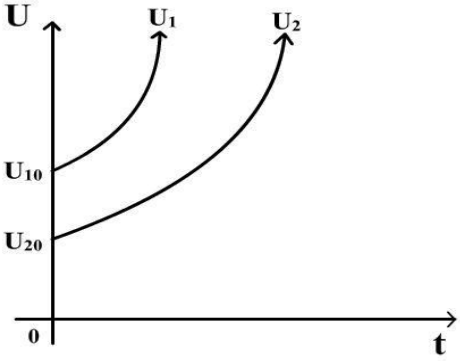

Figure 12

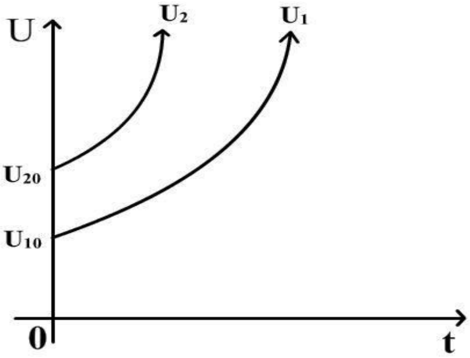

Figure 13

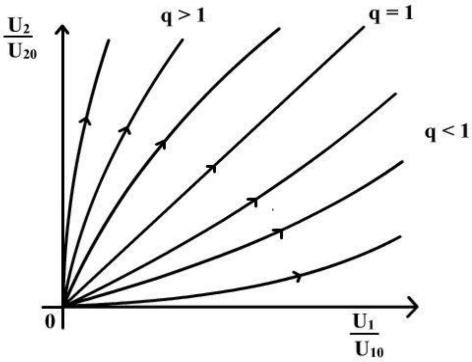

Figure 14

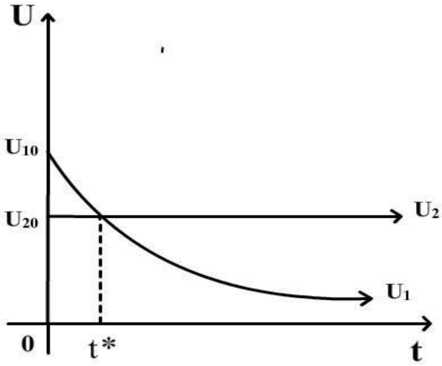

Figure 15

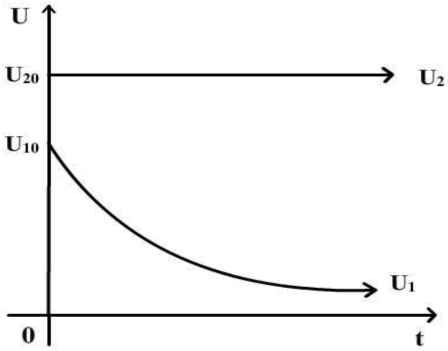

Figure 16

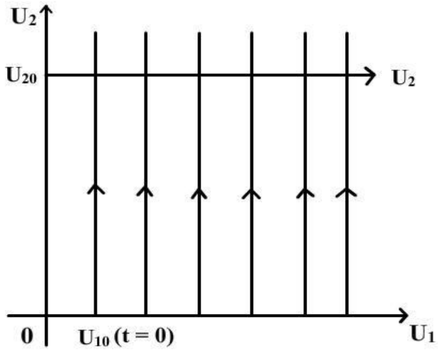

Figure 17

Case (ii) $U_{10} > U_{20}$. i.e. initially the Ammensal species dominates the enemy species. The enemy (S_2) dominates over the Ammensal (S_1) in natural growth rate but its initial strength is less than the

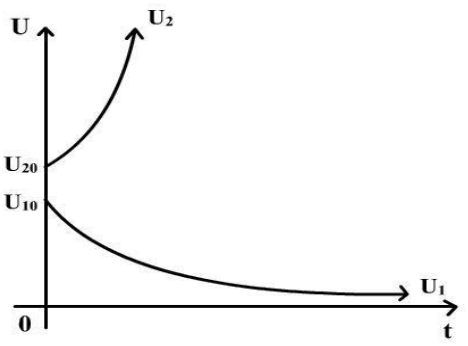

Figure 18

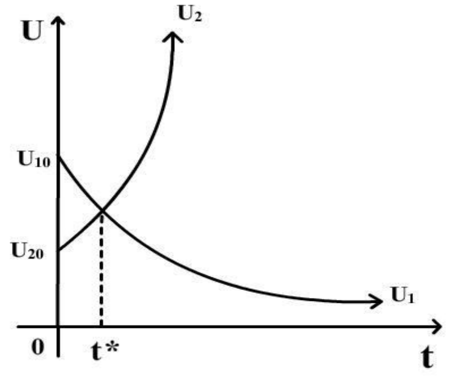

Figure 19

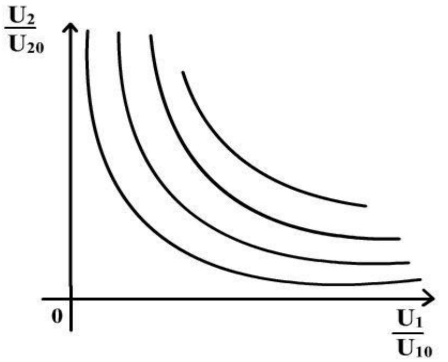

Figure 20

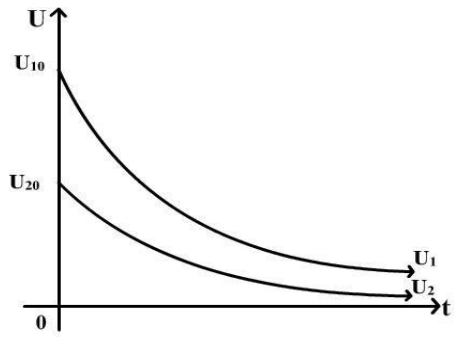

Figure 21

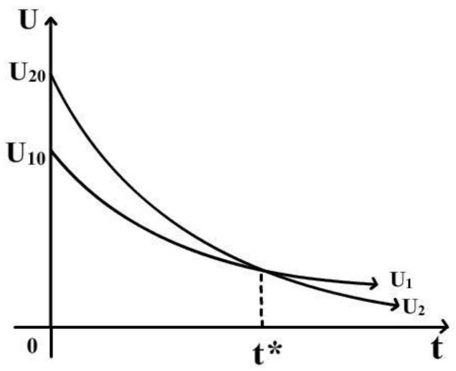

Figure 22

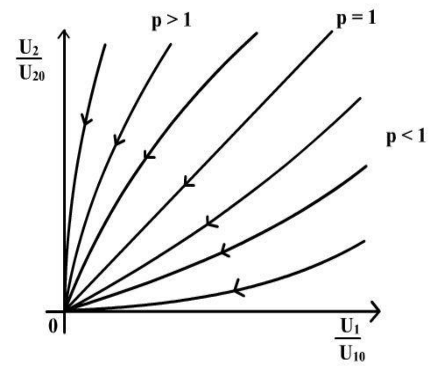

Figure 23

Ammensal. In this case the enemy outnumbers the Ammensal till the time instant

$$t^*=\frac{1}{a_{22}\left(K_2-\frac{2H_2}{K_2}\right)-a_{11}\left(K_1-\alpha(1-b)\left(K_2-\frac{H_2}{K_2}\right)\right)}\log\left[\frac{U_{10}}{U_{20}}\right]$$

After that the enemy is found to be going away from the equilibrium point while the Ammensal species is asymptotic to the equilibrium point. Hence the equilibrium point is unstable, as shown in Figure 19.

4.5.1 *Trajectories of Perturbed Species:*

The trajectories obtained by solving (4.18) in $U_1 - U_2$ plane can be given by

$$\frac{U_1}{U_{10}}+\frac{L_5}{U_{10}}\left(\frac{U_2}{U_{20}}\right)=\left(1+\frac{L_5}{U_{10}}\right)\left(\frac{U_2}{U_{20}}\right)^{-\gamma} \tag{4.19}$$

Where $\gamma = \frac{a_{11}\left(K_1-\frac{\alpha(1-b)H_2}{K_2}\right)}{a_{22}\left(K_2-\frac{2H_2}{K_2}\right)}$ and these are illustrated in Fig .20.

4.6 *Stability of the Equilibrium State E_6*

In this case

$$A=\begin{bmatrix} -a_{11}\left(K_1-\alpha(1-b)\left(K_2-\frac{H_2}{K_2}\right)\right) & -\alpha(1-b)\,a_{11}\left((K_1-\alpha(1-b)\ (K_2-\frac{H_2}{K_2})\right) \\ 0 & -a_{22}\left(K_2-\frac{2H_2}{K_2}\right)\end{bmatrix} \tag{4.20}$$

The characteristic roots of A are - $a_{11}\left(K_1-\alpha(1-b)\left(K_2-\frac{H_2}{K_2}\right)\right)$, -$a_{22}\left(K_2-\frac{2H_2}{K_2}\right)$ and these are both negative. Hence the steady state is stable.

The equation (9) yields the solution curves:

$$\begin{aligned} U_1 &= L_6 e^{-a_{22}\left(K_2-\frac{2H_2}{K_2}\right)t}+(U_{10}+L_6)e^{-a_{11}\left[K_1-\alpha(1-b)\left(K_2-\frac{2H_2}{K_2}\right)\right]t}, \\ U_2 &= U_{20}e^{-a_{22}\left(K_2-\frac{2H_2}{K_2}\right)t} \end{aligned} \tag{4.21}$$

where $L_6 = \frac{\alpha(1-b)\ a_{11}\ \overline{N_1}\ U_{20}}{a_{22}\ \overline{N_1}\ -\ a_{22}\ \left(K_2-\frac{2H_2}{K_2}\right)}$ and these curves are illustrated here under.

Case (i) $U_{10} > U_{20}$. i.e. initially the Ammensal (S_1) species dominates over the enemy (S_2) species and it continues throughout its growth rate. In this case the Ammensal continuously outnumbers the enemy as shown in Figure 21. However both converge asymptotically to the equilibrium point.

Case (ii) $U_{10} < U_{20}$ i.e. initially the enemy (S_2) species dominates the Ammensal (S_1) species. The enemy dominates over the Ammensal up to the time-instant

$$t^*=\frac{1}{a_{22}\left(K_2-\frac{2H_2}{K_2}\right)-a_{11}\left(K_1-\alpha(1-b)\left(K_2-\frac{H_2}{K_2}\right)\right)}\log\left[\frac{U_{20}+L_6}{U_{10}+L_6}\right]$$

there after both Ammensal and the enemy decline further shown as in Figure 22.

4.6.1 *Trajectories of Perturbed species:*

The trajectories obtained by solving (4.21) in $U_1 - U_2$ plane can be given by

$$\frac{U_1}{U_{10}}=\frac{-L_6}{U_{10}}\left(\frac{U_2}{U_{20}}\right)+\left(1+\frac{L_6}{U_{10}}\right)\left(\frac{U_2}{U_{20}}\right)^P, \text{ where } p=\frac{a_{11}\left(K_1-\ \alpha(1-b)\overline{N_2}\right)}{a_{22}\left(K_2-\ \frac{2H_2}{K_2}\right)} \tag{4.22}$$

which are illustrated in Figure 23.

5. Open Problems

i. One can apply the concept of harvesting rate for a four species ecosystem in diffenrent situations.

ii. One can study the model where harvesting rate is proportional to the population sizes of the species with un limited recourses and then construct Liapunov's function to examine the global stability of this model.

References

[1] Acharyulu. K.V.L.N. &Pattabhi Ramacharyulu. N.Ch.;On the stability of an enemy -Ammensal species pair with limited resources, International Journal of Applied Mathematical Analysis and Applications, vol 4, No.2, pp.149-161,July (2009).

[2] Acharyulu. K.V.L.N. &Pattabhi Ramacharyulu. N.Ch.; "An Ammensal-Enemy specie pairwith limited and unlimited resources respectively-A numerical approach", *Int. J. Open Problems Compt. Math(IJOPCM)., Vol. 3, No. 1,pp.73-91., March (2010).*

[3] Acharyulu. K.V.L.N. &Pattabhi Ramacharyulu. N.Ch.; "In view of the reversal time of dominance in an enemy-Ammensal species pair with unlimited and limited resources respectively for stability by numerical technique",International journal of Mathematical Sciences and Engineering Applications(IJMSEA); Vol.4, No. II, June (2010).

[4] Acharyulu. K.V.L.N. &Pattabhi Ramacharyulu. N.Ch.; "On The Stability Of An Enemy – Ammensal Species Pair With Resources Limited For One Species And Unlimited For The Other". International e Journal Of Mathematics And Engineering (I.e.J.M.A.E.) Volume-1, Issue-I,pp.1-14;(2010)

[5] Acharyulu. K.V.L.N. &Pattabhi Ramacharyulu. N.Ch "On The Stability Of An Ammensal - EnemySpecies Pair With Unlimited Resources". International e Journal Of Mathematics And Engineering (I.e.J.M.A.E.) Volume-1,Issue-II,pp-115-124;(2010)

[6] Acharyulu. K.V.L.N. &Pattabhi Ramacharyulu. N.Ch.; "On The Stability of An Ammensal- Enemy Harvested Species Pair With Limited Resources" - *Int. J. Open Problems Compt. Math(IJOPCM)., Vol. 3, No. 2., June(2010).*

[7] Archana Reddy. R; Pattabhi Ramacharyulu N.Ch & Krishna Gandhi. B., "A stability analysis of two competitive interacting species with harvesting of both the species at a constant rate". International journal of scientific computing (1) pp 57-68.January-June (2007).

[8] Bhaskara Rama Sharma & Pattabhi Ramacharyulu N.Ch; "Stability Analysis of two species competitive ecosystem". International Journal of logic based intelligent systems, Vol.2 No.1 January – June (2008).

[9] Kapur J.N., Mathematical modeling in Biology and Medicine, affiliated east west, (1985).

[10] Kapur J.N., Mathematical modeling, wiley, easter, (1985).

[11] Lakshmi Narayan K. "A mathematical study of a prey-predator ecological model with a partial cover for the prey and alternative food for the predator", Ph.D thesis, JNTU., (2005).

[12] Lakshmi Narayanan.K & pattabhi Ramacharyulu N.Ch.,Int.J.Open Problems Compt.Math., Vol. 1, No. 1. June (2008).

[13] Lotka AJ. Elements of physical Biology, Willim & WIlking Baltimore, 1925.

[14] Meyer W.J., Concepts of Mathematical modeling MC. Grawhil, (1985).

[15] Ravindra Reddy "A study on Mathematical models of Ecological metabolism between two interacting species" Ph.D., Thesis OU., (2008).

[16] Srinivas N.C.,"Some Mathematical aspects of modeling in Bio-medical sciences", Ph.D Thesis, Kakatiya University(1991).