International Journal of Applied Mathematics and Computation
ISSN: 0974 - 4665 (Print) 0974 - 4673 (Online)
Journal homepage: www.darbose.in/ijamc
Volume 4(1) 2012 64–76

Stability analysis of an SEIRS model for the spread of malaria

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ABSTRACT

In this paper, an SEIRS mathematical model for the spread of malaria that incorporates recruitment of human population through constant immigration is considered. Susceptible humans can be infected when they are bitten by an infectious mosquito. They then progress through the exposed, infectious, and recovered class, before reentering the susceptible class. Susceptible mosquitoes can become infected when they bite infectious or recovered humans, and once infected they move in infectious class. The growth rate of mosquito population density is taken to be logistic. We define a reproductive number, H_0, for the number of secondary cases that one infected individual will cause through the duration of the infectious period. We find that the disease-free equilibrium is locally asymptotically stable when $H_0 < 1$ and unstable when $H_0 > 1$.We prove the existence and local asymptotic stability of endemic equilibrium point for $H_0 > 1$. By stability analysis of ordinary differential equation, the conditions for global stability of endemic equilibrium are obtained. Numerical simulations are also carried out to investigate the influence of certain parameters on the spread of disease, to support the analytical results and illustrate possible behavioral scenario of the model. The analysis of the model shows that if the growth rate of human population and vector mosquito population increase, the spread of Malaria increases and the disease becomes more endemic due to human immigration.

Keywords: Susceptible; infective; recovered; reproductive number; stability analysis; mosquito population; malaria; numerical simulation.

1. Introduction

Malaria disease is a major public health problem in the world. These diseases continue to afflict the poor nations. It is one of the top ten killer diseases in the world. In each year, there are estimated between 300 to 500 million clinical episodes of malaria and 1.5 to 2.7 million deaths worldwide, 90% of which occur in tropical Sahara. Outsides Africa, some two-thirds of remaining cases occur in just three countries; Brazil, India and Sri Lanka. However, it exists in some 100 countries [1]. It is an infectious disease caused by the parasite genus Plasmodium. There are four species of this parasite causing Malaria, namely, Plasmodium Vivax, Plasmodium falciparum, Plasmodium ovale and Plasmodium malariae [2]. Malaria is transmitted through the vectors, Anopheles mosquitoes and not directly from human to human. The disease infects humans of all ages and can be lethal. According to the world Health organizations in year 2007, about 40% of the world population, mostly those living in the poorest countries, are at risk of malaria of the billion people at risk, more than 500 million become severely ill with malaria every year and more than 1 million die from the effects of the disease.

There has been a rural-urban movement in search for jobs: this currently is a major phenomenon and a major determinant of disease spread in Africa. This has been resulted into unregulated urban growth

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that leads to an increase in malaria transmission because of poor housing and sanitation, improper drainage of surface water and use of unprotected water reservoirs that increase human-mosquito contact and mosquito breeding. This has been compounded by wars, natural disasters such as droughts and floods that have resulted in displacement of communities into environmental refugees [3].

Mathematical models have long provided important insights into disease dynamics and control. General epidemiological compartmental models that include infective immigrants but with no age structure have been studied in [4–6]. However most of the host-vector mathematical models used have focused on the dynamics of the disease in the absence of infective immigrants. They consider the immigration of susceptible humans (see [7], for a mass action model and [8], for standard incidence) for the horizontal transmission of malaria. The Chitnis et al. [8] model does not include immigration of infectious humans, as they assume that sick people do not travel. However, some of the immigrants from malaria-endemic areas are infected with malaria parasites and can be a source of malaria when they move to a malaria-free zone that has uninfected mosquitoes. Thus, the role of infective migrants cannot be ignored in the spread of malaria. In Tumwiine et al.,[9] a model based on susceptible-infectious-recovered-susceptible $SIRS$ pattern for humans and susceptible-infectious SI pattern for mosquitoes was considered. It was established that recoveries and temporary immunity keep the populations at oscillation patterns and eventually converge to a steady state. In particular Shukla et al. [10] studied the spread of malaria with environmental effects. We have modified this model by considering the growth rate of mosquito population as an increasing function of human populations and also, introduced an exposed class of human population to incorporate the time lag between the events of susceptible population entering into the infective population. This paper is organized as follows: in section 2, we introduce mathematical model. Section 3 focuses on the analysis of the model which includes the study of region of attraction. Section 4 deals with the equilibria of the system. In sections 5 and 6 local and global stability of the equilibrium points are established. Section 7 highlights the results of our analysis using numerical simulation. Section 8 presents a short discussion.

2. The Mathematical Model

We consider here that the total human population density $N_1(t)$is divided into four classes, the susceptible $S_1(t)$,the exposed $E_1(t)$,the infective $I_1(t)$,and the removed class $R_1(t)$. Let mosquito population density $N_2(t)$is divided into two classes, viz. the susceptible class $S_2(t)$and the infective class $I_2(t)$. Now by considering the criss-cross interaction, $SEIRS$ model can be written as,

$$
\begin{aligned}
\frac{dS_1}{dt} &= A - \mu_1 S_1 - \beta_1 S_1 I_2 + \delta_1 R_1 \ , \\
\frac{dE_1}{dt} &= \beta_1 S_1 I_2 - \nu_1 E_1 - \mu_1 E_1 \ , \\
\frac{dI_1}{dt} &= \nu_1 E_1 - \gamma_0 I_1 - \alpha_1 I_1 - \mu_1 I_1 \ , \\
\frac{dR_1}{dt} &= \gamma_0 I_1 - \mu_1 R_1 - \delta_1 R_1 \ , \\
\frac{dN_1}{dt} &= A - \mu_1 N_1 - \alpha_1 I_1 \ , \\
\frac{dS_2}{dt} &= s(N_1)N_2 - \frac{s_0 N_2^2}{L(N_1)} - \beta_2 S_2 I_1 - \mu_2 S_2 - \alpha_2 S_2 \ , \\
\frac{dI_2}{dt} &= \beta_2 S_2 I_1 - \mu_2 I_2 - \alpha_2 I_2 \ , \\
\frac{dN_2}{dt} &= s(N_1)N_2 - \frac{s_0 N_2^2}{L(N_1)} - \mu_2 N_2 - \alpha_2 N_2 \ ,
\end{aligned} \tag{2.1}
$$

with $S_1 + E_1 + I_1 + R_1 = N_1$,$S_2 + I_2 = N_2$,$s(N_1) = s_0 + S_1 N_1$,$S_1(0) \geq 0$,$E_1(0) \geq 0$,$I_1(0) \geq 0$,$R_1(0) \geq 0$,$S_2(0) \geq 0$,$I_2(0) \geq 0$. Also,

$$L(N_1) = L_0 + L_1 N_1, \quad L_0 > 0, \quad L_1 > 0 \tag{2.2}$$

Now using $S_1 + E_1 + I_1 + R_1 = N_1$ and $S_2 + I_2 = N_2$, the model (2.1) are reduced in the variables $E_1, I_1, R_1, N_1, I_2, N_2$ as follows;

$$\begin{aligned}
\frac{dE_1}{dt} &= \beta_1(N_1 - I_1 - E_1 - R_1)I_2 - \nu_1 E_1 - \mu_1 E_1 \ , \\
\frac{dI_1}{dt} &= \nu_1 E_1 - \gamma_0 I_1 - \alpha_1 I_1 - \mu_1 I_1 \ , \\
\frac{dR_1}{dt} &= \gamma_0 I_1 - \mu_1 R_1 - \delta_1 R_1 \ , \\
\frac{dN_1}{dt} &= A - \mu_1 N_1 - \alpha_1 I_1, \\
\frac{dI_2}{dt} &= \beta_2(N_2 - I_2)I_1 - \mu_2 I_2 - \alpha_2 I_2 \ , \\
\frac{dN_2}{dt} &= s(N_1)N_2 - \frac{s_0 N_2^2}{L(N_1)} - \mu_2 N_2 - \alpha_2 N_2 \ .
\end{aligned} \tag{2.3}$$

In the above model (2.3), A is the constant immigration rate of human population; $\mu_1$the natural death rate constant of human population; β_1 the interaction coefficient of susceptible human with infective mosquito population; γ_0 the recovery rate coefficient of the human population; $\alpha_1$the disease related death rate constant of human population; $\delta_1$the rate coefficient at which individuals in the removed class again become susceptible; $\mu_2$the natural death rate constant of mosquito population; $\alpha_2$the death rate of mosquito due to control measure; $\beta_2$is the interaction coefficient of susceptible mosquito with infective human class; $s(N_1)$is the growth rate per capita of the mosquito population density under the assumption that growth rate increases as total human population increases, so that $s(0) = s_0 > 0$ and $s'(N_1) \geq 0$ where s_0 is the value of $s(N_1)$ when $N_1 = 0$. s_0 is the intrinsic growth rate of mosquito population; $L(N_1)$ is the carrying capacity of the mosquito population and it is taken as to be increasing function of human population density and its value is $L(N_1)\left\{s(N) - (\mu_2 + \alpha_2)\right\}/s_0$.

3. Region of attraction

Theorem 3.1. *The region of attraction for the system* (2.3) *is given by;*

$$\Omega = \left\{(E_1, I_1, R_1, N_1, I_2, N_2) : \ \frac{A}{\mu_1 + \alpha_1} \leq N_1 \leq \frac{A}{\mu_1}, \ 0 \leq I_2 \leq N_2 \leq \bar{N}_m\right\}$$

which attracts all solutions initiating in the positive orthant, where

$$\bar{N}_m = L(N_1)\left\{s(N_1) - (\mu_2 + \alpha_2)\right\}/s_0.$$

Proof. From fourth equation of model (2.3) the total population $N(t)$ satisfies,

$$\frac{dN_1}{dt} = A - \mu_1 N_1 - \alpha_1 I_1.$$

It follows from $I(t) < N(t)$ that$A - (\mu_1 + \alpha_1)N(t) \leq \frac{dN}{dt} \leq A - \mu_1 N(t)$.

This implies that, $\frac{A}{\mu_1+\alpha_1} \leq N_1 \leq \frac{A}{\mu_1}$.

From sixth equation of model (2.3) we can write,

$$\frac{dN_2}{dt} \le s(N_1)N_2 - \frac{s_0 N_2^2}{L(N_1)} - (\mu_2 + \alpha_2)N_2.$$

This implies that, $0 \le N_2 \le L(N_1)\left\{s(N_1) - (\mu_2 + \alpha_2)\right\}/s_0$ or $0 \le S_2 + I_2 \le \bar{N}_m$. □

4. Equilibrium Analysis

The system (2.3) has three non negative equilibria
$M_1(0,0,0,\bar{N}_1,0,0)$, $M_2(0,0,0,\bar{\bar{N}}_1,0,\bar{\bar{N}}_2)$ and $M_3(\hat{E}_1,\hat{I}_1,\hat{R}_1,\hat{N}_1,\hat{I}_2,\hat{N}_2)$.

4.1 *Existence of* $M_1(0,0,0,\bar{N}_1,0,0)$

Here $\bar{N}_1$is the solution of the following equation

$$A - \mu_1 N_1 = 0.$$

Clearly, $\bar{N}_1 = \frac{A}{\mu_1} > 0$. So the equilibrium point $M_1(0,0,0,\bar{N}_1,0,0)$ exists.

4.2 *Existence of* $M_2(0,0,0,\bar{\bar{N}}_1,0,\bar{\bar{N}}_2)$

Here $\bar{\bar{N}}_1$ and $\bar{\bar{N}}_2$ are given by the solution of the following equations;

$$A - \mu_1 \bar{\bar{N}}_1 = 0$$

and

$$s(\bar{\bar{N}}_1)\bar{\bar{N}}_2 - \frac{s_0 N_2^2}{L(\bar{\bar{N}}_1)} - (\mu_2 + \alpha_2)\bar{\bar{N}}_2 = 0$$

Clearly,

$$\bar{\bar{N}}_1 = \frac{A}{\mu_1} > 0$$

and

$$\bar{\bar{N}}_2 = \frac{L(\bar{\bar{N}}_1)\left\{s(\bar{\bar{N}}_1) - (\mu_2 + \alpha_2)\right\}}{s_0} > 0$$

if

$$s(\bar{\bar{N}}_1) > (\mu_2 + \alpha_2)$$

So the equilibrium point $M_2(0,0,0,\bar{\bar{N}}_1,0,\bar{\bar{N}}_2)$ exists if $s(\bar{\bar{N}}_1) > (\mu_2 + \alpha_2)$.

4.3 *Existence of* $M_3(\hat{E}_1, \hat{I}_1, \hat{R}_1, \hat{N}_1, \hat{I}_2, \hat{N}_2)$

The non trivial interior equilibrium point M_3 is the positive solution of the following algebraic equations

$$\beta_1(N_1 - I_1 - E_1 - R_1)I_2 - \nu_1 E_1 - \mu_1 E_1 = 0. \tag{4.1}$$

$$\nu_1 E_1 - \gamma_0 I_1 - \alpha_1 I_1 - \mu_1 I_1 = 0. \tag{4.2}$$

$$\gamma_0 I_1 - \mu_1 R_1 - \delta_1 R_1 = 0. \tag{4.3}$$

$$A - \mu_1 N_1 - \alpha_1 I_1 = 0. \tag{4.4}$$

$$\beta_2(N_2 - I_2)I_1 - \mu_2 I_2 - \alpha_2 I_2 = 0. \tag{4.5}$$

$$s(N_1)N_2 - \frac{s_0 N_2^2}{L(N_1)} - (\mu_2 + \alpha_2)N_2 = 0. \tag{4.6}$$

Now from equation (4.4) and (4.3) we get,

$$I_1 = \frac{(A - \mu_1 N_1)}{\alpha_1}$$

and

$$R_1 = \frac{\gamma_0(A - \mu_1 N_1)}{\alpha_1(\mu_1 + \delta_1)} = g_1(N_1)(\text{say})$$

Also from equation (4.2), (4.6) and (4.2) we get,

$$E_1 = \frac{(\gamma_0 + \alpha_1 + \mu_1)(A - \mu_1 N_1)}{\nu_1 \alpha_1}, N_2 = \frac{L(N_1)\left\{s(N_1 - (\mu_2 + \alpha_2)\right\}}{s_0}$$

and

$$I_2 = \frac{\beta_2 N_2 I_1}{(\beta_2 I_1 + \mu_2 + \alpha_2)} = \frac{\beta_2 L(N_1)\left\{s(N_1) - (\mu_2 + \alpha_2)\right\}(A - \mu_1 N_1)}{s_0 \alpha_1 \left[\frac{\beta_2(A - \mu_1 N_1)}{\alpha_1} + (\mu_2 + \alpha_2)\right]}.$$

Now putting the value of I_1, E_1, R_1, I_2 and N_2in equation (4.1) then whole equation reduces to N_1. So we can write,

$$F(N_1) = \frac{\beta_1 \beta_2 L(N_1)\left\{s(N_1) - (\mu_2 + \alpha_2)\right\}\left[N_1 - \frac{(A - \mu_1 N_1)}{\alpha_1} - \frac{(\gamma_0 + \alpha_1 + \mu_1)(A - \mu_1 N_1)}{\nu_1 \alpha_1} - g_1(N_1)\right]}{s_0 \left[\frac{\beta_2(A - \mu_1 N_1)}{\alpha_1} + (\mu_2 + \alpha_2)\right]}$$

$$-\frac{(\nu_1+\mu_1)(\gamma_0+\alpha_1+\mu_1)}{\nu_1} \tag{4.7}$$

It is clear from equation (4.7) that,

$$F\left(\frac{A}{\mu_1+\alpha_1}\right)=\frac{-\beta_1\beta_2 L\left(\frac{A}{\mu_1+\alpha_1}\right)\left\{s\left(\frac{A}{\mu_1+\alpha_1}\right)-(\mu_2+\alpha_2)\right\}\left[\frac{(\gamma_0+\alpha_1+\mu_1)A}{\nu_1(\mu_1+\alpha_1)}+\frac{\gamma_0 A}{(\mu_1+\delta_1)(\mu_1+\alpha_1)}\right]}{s_0\left[\frac{\beta_2 A}{(\mu_1+\alpha_1)}+(\mu_2+\alpha_2)\right]} -\frac{(\nu_1+\mu_1)(\gamma_0+\alpha_1+\mu_1)}{\nu_1}<0. \tag{4.8}$$

and

$$F\left(\frac{A}{\mu_1}\right)=\frac{\beta_1\beta_2 L(A/\mu_1)\left\{s(A/\mu_1)-(\mu_2+\alpha_2)\right\}A}{\mu_1 s_0(\mu_2+\alpha_2)}-\frac{(\nu_1+\mu_1)(\gamma_0+\alpha_1+\mu_1)}{\nu_1}>0.$$

If

$$H_0=\frac{\nu_1\beta_1\beta_2 L(A/\mu_1)\left\{s(N_1)-(\mu_2+\alpha_2)\right\}A}{\mu_1 s_0(\mu_2+\alpha_2)(\nu_1+\mu_1)(\gamma_0+\alpha_1+\mu_1)}>1$$

It would be sufficient if we show that $F(N)=0$has one and only one root. From equation (4.7), we note that $F\left(\frac{A}{\mu_1+\alpha_1}\right)<0$and $F\left(\frac{A}{\mu_1}\right)>0$.This implies that there exists a root $\hat{N}_1$of $F(N_1)=0$in$\frac{A}{\mu_1+\alpha_1}<N_1<\frac{A}{\mu_1}$.

Also,$F'(N_1)>0$,provided$\left[N_1-\frac{(A-\mu_1N_1)}{\alpha_1}-\frac{(\gamma_0+\alpha_1+\mu_1)(A-\mu_1N_1)}{\nu_1\alpha_1}-\frac{\gamma_0(A-\mu_1N_1)}{\alpha_1(\mu_1+\delta_1)}\right]>0$ in$\frac{A}{\mu_1+\alpha_1}<N_1<\frac{A}{\mu_1}$. Thus, there exists a unique root of $F(N_1)=0$, $(say\ \ \hat{N}_1)$ in$\frac{A}{\mu_1+\alpha_1}<N_1<\frac{A}{\mu_1}$. So the equilibrium point M_3exists provided $H_0>1$.

5. Local Stability Analysis

Now, we analyze the stability of equilibria M_1, M_2 and M_3. The local stability results of these equilibria are stated in the following theorem

Theorem 5.1. *The disease free equilibrium point M_1 is unstable if M_2 exist and disease free equilibrium point M_2 is stable if $H_0<1$ and unstable if$H_0>1$, where $H_0=\frac{\beta_1\nu_1\bar{\bar{N}}_1\beta_2\bar{\bar{N}}_2}{(\gamma_0+\alpha_1+\mu_1)(\mu_2+\alpha_2)(\nu_1+\mu_1)}$.*

The interior equilibrium point M_3 when exists, is locally asymptotically stable provided the following conditions are satisfied:

$$(\nu_1-\beta_1\hat{I}_2)^2<\left(\frac{1}{4}\right)\left(\beta_1\hat{I}_2+\nu_1+\mu_1\right)\left(\gamma_0+\alpha_1+\mu_1\right),$$

$$(\beta_1\hat{I}_2)^2<\left(\frac{1}{2}\right)\left(\beta_1\hat{I}_2+\nu_1+\mu_1\right)\left(\mu_1+\delta_1\right), \tag{5.1}$$

$$\left[\beta_1(\hat{N}_1-\hat{I}_1-\hat{E}_1-\hat{R}_1)\right]^2<\left(\frac{1}{3}\right)\left(\beta_1\hat{I}_2+\nu_1+\mu_1\right)\left(\beta_2\hat{I}_1+\mu_2+\alpha_2\right),$$

$$(\gamma_0)^2 < \left(\frac{1}{2}\right)\ (\gamma_0+\alpha_1+\mu_1)\ (\mu_1+\delta_1)\ ,$$

$$(\beta_1\hat{I}_2)^2 < \left(\frac{1}{3}\right)\ (\beta_1\hat{I}_2+\nu_1+\mu_1)\ (\mu_1)\ ,$$

$$(\alpha_1)^2 < \left(\frac{1}{4}\right)\ (\gamma_0+\alpha_1+\mu_1)\ (\mu_1)\ ,$$

$$\left[\beta_2(\hat{N}_2-\hat{I}_2)\right]^2 < \left(\frac{1}{3}\right)\ (\gamma_0+\alpha_1+\mu_1)\ (\beta_2\hat{I}_1+\mu_2+\alpha_2)\ ,$$

$$\left[s_1\hat{N}_2 - \frac{s_0\hat{N}_2^2 L_1}{\left[L(\hat{N}_1)\right]^2}\right]^2 < \left(\frac{2}{3}\right)\ \left[\mu_2+\alpha_2-s(\hat{N}_1)+\frac{2s_0\hat{N}_2}{L(\hat{N}_1)}\right]\ (\mu_1)\ ,$$

$$(\beta_2 I_1)^2 < \left(\frac{2}{3}\right)\ (\beta_2\hat{I}_1+\alpha_2+\mu_2)\ \left[\alpha_2+\mu_2-s(\hat{N}_1)+\frac{2s_0\hat{N}_2}{L(\hat{N}_1)}\right]\ .$$

Proof. The variational matrix V_1 at $M_1(0,0,0,\bar{N}_1,0,0)$corresponding to the system of equation (2.3) is given by;

$$V_1 = \begin{bmatrix} -\nu_1-\mu_1 & 0 & 0 & 0 & \beta_1\bar{N}_1 & 0 \\ \nu_1 & -\gamma_0-\alpha_1-\mu_1 & 0 & 0 & 0 & 0 \\ 0 & \gamma_0 & -\mu_1-\delta_1 & 0 & 0 & 0 \\ 0 & -\alpha_1 & 0 & -\mu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_2-\alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & s(\bar{N}_1)-(\mu_2+\alpha_2) \end{bmatrix}$$

Since one eigenvalue $s(\bar{N}_1)-(\mu_2+\alpha_2)$of matrix $V_1$is positive because $s(\bar{N}_1) > (\mu_2+\alpha_2)$.So $M_1$is unstable.

The variational matrix V_2 at $M_2(0,0,0,\bar{\bar{N}}_1,0,\bar{\bar{N}}_2)$ corresponding to the system of equation (2.3) is given by;

$$V_2 = \begin{bmatrix} -\nu_1-\mu_1 & 0 & 0 & 0 & \beta_1\bar{\bar{N}}_1 & 0 \\ \nu_1 & -J & 0 & 0 & 0 & 0 \\ 0 & \gamma_0 & -\mu_1-\delta_1 & 0 & 0 & 0 \\ 0 & -\alpha_1 & 0 & -\mu_1 & 0 & 0 \\ 0 & \beta_2\bar{\bar{N}}_2 & 0 & 0 & -\mu_2-\alpha_2 & 0 \\ 0 & 0 & 0 & s_1\bar{\bar{N}}_2+\frac{s_0\bar{\bar{N}}_2^2}{\left[L(\bar{\bar{N}}_1)\right]^2} & 0 & -K \end{bmatrix}$$

Where $J=\gamma_0+\alpha_1+\mu_1$ and $K=s(\bar{\bar{N}}_1)-(\mu_2+\alpha_2)$.

Then the characteristic polynomial of V_2 is given by;

$$\begin{aligned} \mathrm{P}(\lambda) = &\ (-\mu_1-\nu_1-\lambda)\,(-\mu_1-\lambda)\,(-K-\lambda) \\ &\qquad \Big(-(\nu_1+\mu_1+\lambda)\,\{(J+\lambda)(\mu_2+\alpha_2+\lambda)\}+\beta_1\bar{\bar{N}}_1\nu_1\beta_2\bar{\bar{N}}_2\Big) \\ = &\ (\mu_1+\nu_1+\lambda)\ \ (\mu_1+\lambda)\ \ (K+\lambda) \\ &\qquad \Big(\ (\nu_1+\mu_1+\lambda)\ \ \{(J+\lambda)\ (\mu_2+\alpha_2+\lambda)\}-\beta_1\bar{\bar{N}}_1\nu_1\beta_2\bar{\bar{N}}_2\Big) \\ = &\ (\mu_1+\nu_1+\lambda)\ \ (\mu_1+\lambda)\ \ (K+\lambda)\ \ \left(\lambda^3+B\lambda^2+C\lambda+D\right) \end{aligned}$$

therefore the eigenvalues of the matrix V_2 are $-\mu_1 - \nu_1$,$-\mu_1$,$-\left(s(\bar{\bar{N}}_1) - (\mu_2 + \alpha_2)\right)$ and the roots of the polynomial $q(\lambda) = \lambda^3 + B\lambda^2 + C\lambda + D$ where;

$$B = (\nu_1 + \gamma_0 + \alpha_1 + 2\mu_1 + \mu_2 + \alpha_2\ )\ ,$$

$$C = ((\gamma_0 + \alpha_1 + \mu_1 + \mu_2 + \alpha_2)(\nu_1 + \mu_1) + (\gamma_0 + \alpha_1 + \mu_1)(\mu_2 + \alpha_2)) \text{ and}$$

$$D = \left\{\ (\gamma_0 + \alpha_1 + \mu_1)\ (\mu_2 + \alpha_2)\ (\nu_1 + \mu_1) - \beta_1\nu_1\bar{\bar{N}}_1\beta_2\bar{\bar{N}}_2\right\}$$

Here, we note that $B$and C are always positive and D is positive if $H_0 < 1$ where,
$H_0 = \frac{\beta_1\nu_1\bar{\bar{N}}_1\beta_2\bar{\bar{N}}_2}{(\gamma_0+\alpha_1+\mu_1)(\mu_2+\alpha_2)(\nu_1+\mu_1)}$. Thus disease free equilibria is stable if $H_0 < 1$ because all the conditions of Routh-Hurwitz criteria are satisfied. However, if $H_0 > 1$although $B > 0$, $BC - D > 0$ but $D < 0$thus disease free equilibria is unstable. At $H_0 = 1$one of the eigenvalues of the disease free equilibria is zero. So we cannot predict anything about the stability of the system in this case. $H_0 = 1$ corresponds to the stability switch in the disease free equilibria as for $H_0 < 1$it is stable and for $H_0 > 1$ it is unstable.

Now we find the conditions for locally asymptotically stability of interior equilibrium point; For this we first linearize the system (2.3) around the positive equilibrium $M_3$by taking the transformation$E_1 = e_1 + \hat{E}_1$,$I_1 = i_1 + \hat{I}_1$,$R_1 = r_1 + \hat{R}_1$,$N_1 = n_1 + \hat{N}_1$,$I_2 = i_2 + \hat{I}_2$ and$N_2 = n_2 + \hat{N}_2$, where e_1, i_1, r_1, n_1, i_2 and n_2 are small perturbations about $M_{3.}$. We consider the following positive definite function in the linearised system of model (2.3).

$$W_1 = \frac{1}{2}\ e_1^2 + \frac{1}{2}\ i_1^2 + \frac{1}{2}\ r_1^2 + \frac{1}{2}\ n_1^2 + \frac{1}{2}i_2^2 + \frac{1}{2}\ n_2^2.$$

Now, differentiating W_1 with respect to time t, we can find $\dot{W}_1$ along the solution of linearised system of (2.3) as follows,

$$\begin{aligned}
\dot{W}_1 = &\left[-\frac{1}{4}\ b_{11}\ e_1^2 + b_{12}\ e_1\ i_1 - \frac{1}{4}\ b_{22}\ i_1^2\right] + \left[-\frac{1}{4}\ b_{11}\ e_1^2 + b_{13}\ e_1\ r_1 - \frac{1}{2}\ \ b_{33}\ r_1^2\right] \\
&+ \left[-\frac{1}{2}\ \ b_{33}\ r_1^2 + b_{23}\ r_1\ i_1 - \frac{1}{4}\ b_{22}\ i_1^2\right] + \left[-\frac{1}{4}\ \ b_{11}\ e_1 + b_{15}\ e_1\ i_2 - \frac{1}{3}\ \ b_{55}\ i_2^2\right] \\
&+ \left[-\frac{1}{3}\ \ b_{55}\ i_2^2 + b_{25}\ i_1\ i_2 - \frac{1}{4}\ b_{22}\ i_1^2\right] + \left[-\frac{1}{4}\ b_{11}\ e_1^2 + b_{14}\ e_1\ n_1 - \frac{1}{3}\ b_{44}\ n_1^2\right] \\
&+ \left[-\frac{1}{4}b_{44}\ n_1^2 + b_{24}\ n_1\ i_1 - \frac{1}{4}b_{22}\ i_1^2\right] + \left[-\frac{1}{3}b_{44}\ n_1^2 + b_{46}\ n_1 n_2 - \frac{1}{2}b_{66}\ n_2^2\right] \\
&+ \left[-\frac{1}{3}\ b_{55}\ i_2^2 + b_{56}\ i_2 n_2 - \frac{1}{2}\ b_{66}\ n_2^2\right].
\end{aligned}$$

Where, $b_{12} = (\nu_1 - \beta_1\hat{I}_2)$, $b_{11} = (\beta_1\hat{I}_2 + \nu_1 + \mu_1)$, $b_{22} = (\gamma_0 + \alpha_1 + \mu_1)$,

$$b_{13} = -\beta_1\hat{I}_2, b_{44} = \mu_1, b_{55} = (\beta_2\hat{I}_1 + \mu_2 + \alpha_2),$$

$$b_{15} = \beta_1(\hat{N}_1 - \hat{I}_1 - \hat{E}_1 - \hat{R}_1), b_{24} = -\alpha_1, b_{14} = \beta_1\hat{I}_2,$$

$$b_{25} = \beta_2(\hat{N}_2 - \hat{I}_2), , b_{23} = \gamma_0,$$

$$b_{46} = \left[s_1\hat{N}_2 - \frac{s_0\hat{N}_2^2L_1}{\left[L(\hat{N}_1)\right]^2}\right], b_{66} = \left[(\mu_2 + \alpha_2) - s(\hat{N}_1) + \frac{2s_0\hat{N}_2}{L(\hat{N}_1)}\right]\ , b_{33} = \mu_1 + \delta_1.$$

Sufficient condition for $\dot{W}_1$ to be negative definite are that the inequalities (5.1) holds. This completes the proof of theorem. □

6. Global Stability

Theorem 6.1. *In addition to the assumption (2.2), let $L(N_1)$satisfy in the regionΩ, $L_m \leq L(N_1) \leq L_0$ and $0 \leq -L'(N_1) \leq p$,for some positive constants $L_m$and p. Let the following inequalities are satisfied;*

$$(\nu_1 - \beta_1 \hat{I}_2)^2 < \left(\frac{1}{4}\right) (\beta_1 \hat{I}_2 + \nu_1 + \mu_1) (\gamma_0 + \alpha_1 + \mu_1) ,$$

$$(\gamma_0)^2 < \left(\frac{1}{2}\right) (\gamma_0 + \alpha_1 + \mu_1) (\mu_1 + \delta_1) ,$$

$$(-\beta_1 \hat{I}_2)^2 < \left(\frac{1}{2}\right) (\beta_1 \hat{I}_2 + \nu_1 + \mu_1) (\mu_1 + \delta_1) ,$$

$$[\beta_1(N_1 - I_1 - E_1 - R_1)]^2 < \left(\frac{1}{3}\right) (\beta_1 \hat{I}_2 + \nu_1 + \mu_1) (\beta_2 \hat{I}_1 + \mu_2 + \alpha_2) , \tag{6.1}$$

$$\left[\beta_2(\hat{N}_2 - \hat{I}_2\right]^2 < \left(\frac{1}{3}\right) (\gamma_0 + \alpha_1 + \mu_1) (\beta_2 \hat{I}_1 + \mu_2 + \alpha_2) ,$$

$$(\beta_1 \hat{I}_2)^2 < \left(\frac{1}{3}\right) (\beta_1 \hat{I}_2 + \nu_1 + \mu_1) (\mu_1) ,$$

$$(\beta_2 \hat{I}_1)^2 < \left(\frac{2}{3}\right) (\beta_2 \hat{I}_1 + \mu_2 + \alpha_2) \left(\frac{\mathrm{s}_0}{\mathrm{L}(\hat{N}_1)}\right) ,$$

$$[s_1 - s_0 N_2 \eta(N_1)]^2 < \left(\frac{2}{3}\right) (\mu_1) \left(\frac{s_0}{L(N_1)}\right) ,$$

$$(-\alpha_1)^2 < \left(\frac{1}{3}\right) (\gamma_0 + \alpha_1 + \mu_1) (\mu_1) ,$$

then $M_3(\hat{E}_1, \hat{I}_1, \hat{R}_1, \hat{N}_1, \hat{I}_2, \hat{N}_2)$ is globally asymptotically stable with respect to the all solution initiating in the positive orthant.

Proof. Let us consider the following positive definite function about $M_3(\hat{E}_1, \hat{I}_1, \hat{R}_1.\hat{N}_1, \hat{I}_2, \hat{N}_2)$.

$$V(E_1, I_1, R_1, N_1, I_2, N_2) = \frac{1}{2}\left(E_1 - \hat{E}_1\right)^2 + \frac{1}{2}\left(I_1 - \hat{I}_1\right)^2 + \frac{1}{2}\left(R_1 - \hat{R}_1\right)^2 + \frac{1}{2}\left(N_1 - \hat{N}_1\right)^2$$
$$+\frac{1}{2}\left(I_2 - \hat{I}_2\right)^2 + \left(N_2 - \hat{N}_2 - \hat{N}_2 \log \frac{N_2}{\hat{N}_2}\right) .$$

Now, differentiating above equation with respect to t we get,

$$\frac{dV}{dt} = \left(E_1 - \hat{E}_1\right) \frac{d\hat{E}_1}{dt} + \left(I_1 - \hat{I}_1\right) \frac{d\hat{I}_1}{dt} + \left(R_1 - \hat{R}_1\right) \frac{d\hat{R}_1}{dt} + \left(N_1 - \hat{N}_1\right) \frac{d\hat{N}_1}{dt}$$
$$+ \left(I_2 - \hat{I}_2\right) \frac{d\hat{I}_2}{dt} + \frac{\left(N_2 - \hat{N}_2\right)}{N_2} \frac{d\hat{N}_2}{dt}.$$

After some algebraic manipulation and considering functions,

$$\eta(N_1) = \begin{cases} \frac{\left(\frac{1}{L(N_1)} - \frac{1}{L(\hat{N}_1)}\right)}{(N_1 - \hat{N}_1)}, & N_1 \neq \hat{N}_1 \\ -\frac{L'(N_1)}{L^2(\hat{N}_1)}, & N_1 = \hat{N}_1. \end{cases}$$

Then by using assumptions of the theorem and the mean value theorem we have,

$$|\eta(N_1)| < \frac{p}{L_m^2},$$

derivative of V i.e. $\dot{V}$ can be written as the sum of the quadratics,

$$\begin{aligned}
\frac{dV}{dt} &= -\frac{1}{4}\, a_{11} \left(E_1 - \hat{E}_1\right)^2 + a_{12} \left(E_1 - \hat{E}_1\right) \left(I_1 - \hat{I}_1\right) - \frac{1}{4}\, a_{22} \left(I_1 - \hat{I}_1\right)^2 \\
& -\frac{1}{4}a_{22} \left(I_1 - \hat{I}_1\right)^2 + a_{23} \left(I_1 - \hat{I}_1\right) \left(R_1 - \hat{R}_1\right) - \frac{1}{2}a_{33} \left(R_1 - \hat{R}_1\right)^2 \\
& -\frac{1}{2}a_{33} \left(R_1 - \hat{R}_1\right)^2 + a_{13} \left(E_1 - \hat{E}_1\right) \left(R_1 - \hat{R}_1\right) - \frac{1}{4}a_{11} \left(E_1 - \hat{E}_1\right)^2 \\
& -\frac{1}{4}a_{11} \left(E_1 - \hat{E}_1\right)^2 + a_{14} \left(E_1 - \hat{E}_1\right) \left(N_1 - \hat{N}_1\right) - \frac{1}{3}a_{44} \left(N_1 - \hat{N}_1\right)^2 \\
& -\frac{1}{4}a_{11} \left(E_1 - \hat{E}_1\right)^2 + a_{15} \left(E_1 - \hat{E}_1\right) \left(I_2 - \hat{I}_2\right) - \frac{1}{3}a_{55} \left(I_2 - \hat{I}_2\right)^2 \\
& -\frac{1}{3}a_{55} \left(I_2 - \hat{I}_2\right)^2 + a_{25} \left(I_1 - \hat{I}_1\right) \left(I_2 - \hat{I}_2\right) - \frac{1}{4}a_{22} \left(I_1 - \hat{I}_1\right)^2 \\
& -\frac{1}{3}a_{55} \left(I_2 - \hat{I}_2\right)^2 + a_{56} \left(N_2 - \hat{N}_2\right) \left(I_2 - \hat{I}_2\right) - \frac{1}{2}a_{66} \left(N_2 - \hat{N}_2\right)^2 \\
& -\frac{1}{2}a_{44} \left(N_1 - \hat{N}_1\right)^2 + a_{46} \left(N_1 - \hat{N}_1\right) \left(N_2 - \hat{N}_2\right) - \frac{1}{3}a_{66} \left(N_2 - \hat{N}_2\right)^2 \\
& -\frac{1}{3}a_{44} \left(N_1 - \hat{N}_1\right)^2 + a_{24} \left(I_1 - \hat{I}_1\right) \left(N_1 - \hat{N}_1\right) - \frac{1}{4}a_{22} \left(I_1 - \hat{I}_1\right)^2.
\end{aligned}$$

Where, $a_{11} = \left(\beta_1 \hat{I}_2 + \nu_1 + \mu_1\right)$, $a_{22} = (\gamma_0 + \alpha_1 + \mu_1)$, $a_{33} = (\mu_1 + \delta_1)$,

$$a_{13} = \left(-\beta_1 \hat{I}_2\right), a_{14} = \left(\beta_1 \hat{I}_2\right), a_{15} = \beta_1 \left(N_1 - E_1 - I_1 - R_1\right),$$

$$a_{23} = (\gamma_0), a_{24} = (-\alpha_1), a_{25} = \beta_2 \left(N_2 - I_2\right),$$

$$a_{44} = (\mu_1), a_{56} = \left(\beta_2 \hat{I}_1\right), a_{55} = \left(\beta_2 \hat{I}_1 + \mu_2 + \alpha_2\right),$$

$$a_{66} = \left(\frac{s_0}{L(\hat{N}_1)}\right), a_{12} = \left(\nu_1 - \beta_1 \hat{I}_2\right), a_{46} = \left(s_1 - s_0 N_2 \eta(N_1)\right).$$

After maximizing the L.H.S and minimizing the R.H.S of the inequalities given in (6.1), then the term a_{15} , a_{25} and a_{46} can be written as $\beta_1 N_1$, $\beta_2 N_2$ and s_1. Then sufficient conditions for $\frac{dV}{dt}$ to be negative definite are,

$$(a_{12})^2 < \left(\frac{1}{4}\right) a_{11}\, a_{22} , (a_{23})^2 < \left(\frac{1}{2}\right) a_{22}\, a_{33} , (a_{13})^3 < \left(\frac{1}{2}\right) a_{11}\, a_{33} ,$$

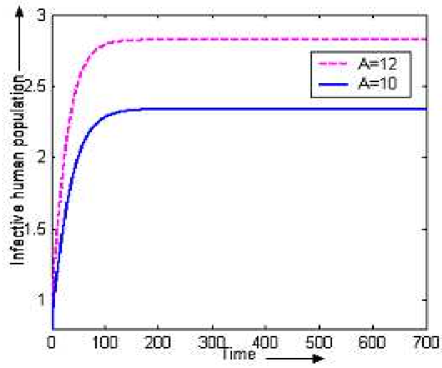

Figure 1: Variation of infective human population for different rates of immigration A

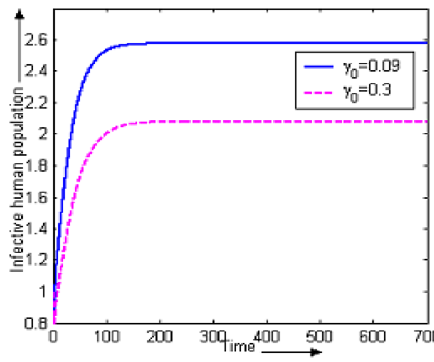

Figure 2: Variation of infective human population for different rates of recovery constant.

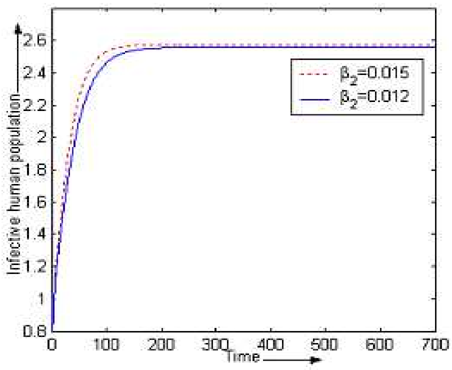

Figure 3: Variation of infective human population for different rates of the interaction coefficients of susceptible mosquito with infective human class.

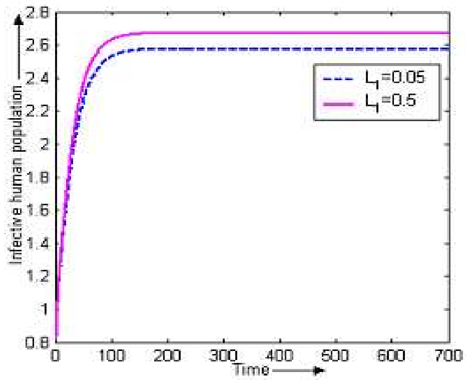

Figure 4: Variation of infective human population for different L_1.

$$(a_{14})^2 < \left(\frac{1}{3}\right) a_{11}\, a_{44}\ , (a_{15})^2 < \left(\frac{1}{3}\right) a_{11}\, a_{55}\ , (a_{25})^2 < \left(\frac{1}{3}\right) a_{22}\, a_{55}\ , \tag{6.2}$$

$$(a_{56})^2 < \left(\frac{2}{3}\right) a_{55}\, a_{66}\ , (a_{46})^2 < \left(\frac{2}{3}\right) a_{44}\, a_{66}\ , (a_{24})^2 < \left(\frac{1}{3}\right) a_{22}\, a_{44}\ .$$

□

7. Numerical simulation

In this section, we present numerical simulation to explain the applicability of the result discussed above. We choose the following parameters in model (2.3) are, $A = 10$, $\mu_1 = 0.8$, $\beta_1 = 0.009$, $\alpha_1 = 0.011$, $\beta_2 = 0.015$, $s_0 = 5$, $\nu_1 = 0.8$, $\alpha_2 = 0.001$, $\mu_2 = 0.001$, $\gamma_0 = 0.09$, $L_0 = 80$, $L_1 = 0.05$, $s_1 = 0.001$, $\delta_1 = 0.05$

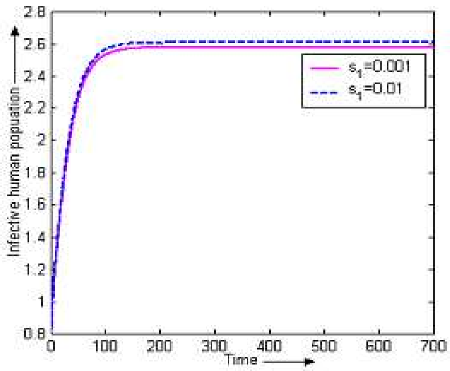

Figure 5: Variation of infective human population for different s_1

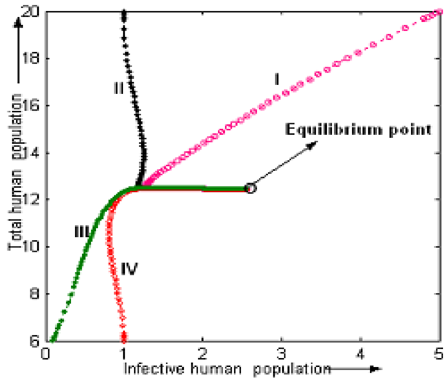

Figure 6: Variation of total human population with infective human population.

With these values of parameters, it can be checked that the interior equilibrium M_3 exists and is given by,

$\hat{E}_1 = 2.9013$, $\hat{I}_1 = 2.5760$, $\hat{R}_1 = 0.2728$, $\hat{N}_1 = 12.4646$, $\hat{I}_2 = 76.8161$, $\hat{N}_2 = 80.7920$.

The eigenvalues of the variational matrix corresponding to the interior equilibrium of the model are -5.01046462781, -0.8098302564, -0.1527458690, -0.0396957935, -1.755461224, -1.405230899.

Since all the eigenvalues are found to be negative, the interior equilibrium is locally asymptotically stable for the above set of parameters. Again with the set of parameters given above it can be verified that the conditions (5.1) and (6.1) in theorem (5.1) and (6.1) is satisfied. This shows that M_3 is locally and globally asymptotically stable respectively.

The results of numerical simulation are displayed graphically in Figs. (1-2) the effect of various parameters, i.e. A and γ_0 on the infective human population have been shown. It is noted that these figures that as these parameter value increase, the infective human population increases and decreases respectively.

In Figs. (3-4) shows the effect of various parameters, i.e. role of β_2 and L_1 on the infective human population. It is noted from that these figures that as a parameter increases, the infective human population increases. Also Fig. (**??**) show the effect of parameter s_1 on the infective human population. It shows that if value of the parameter s_1 increases, the infective human population increases. Simulation is performed for different initial positions in figures (6-7) to display global stability of the system. From these figures, it is clear that all the trajectories starting from different initial starts, reach the interior equilibrium M_3.

8. Conclusion

We analyzed a system of the ordinary differential equations model for the transmission of malaria, taking four variables for humans and two variables for mosquitoes. We showed that there exists a domain where the model is epidemiologically and mathematically well-posed. We proved the existence of an equilibrium point with no disease. We define a reproductive number, H_0, that is epidemiologically accurate in that it provides the expected number of new infections (in mosquitoes or humans) from one infectious individual (human or mosquito) over the duration of infectious period, given that all other members of the population are susceptible. We showed that if $H_0 < 1$then disease-free equilibrium point is locally asymptotically stable, and if $H_0 > 1$,then disease free equilibrium point is unstable. We also

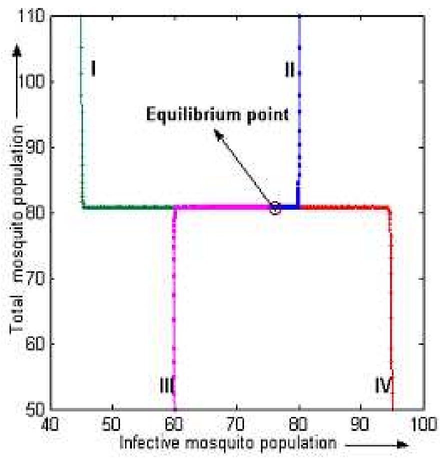

Figure 7: Variation of total mosquito population with infective mosquito population.

prove that an endemic equilibrium point exists for all $H_0 > 1$ and it is locally asymptotically stable. The criteria for global stability of an endemic equilibrium are also obtained. It is concluded from the computer simulation that if the constant immigration rate and the recovery rate coefficient of the human population increases, then the infective human population increases and decreases respectively. Also, when the interaction coefficients of susceptible mosquito with infective human class and the growth rate coefficients of mosquito population increases, then the infective human population increases.

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