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Partially Balanced incomplete block design associated with minimum perfect dominating sets of Clebesch graph

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ABSTRACT

A dominating set S of a graph G is perfect if each vertex of G is dominated by exactly one vertex in S. We study the minimum perfect dominating sets in the Petersen graph and in the Clebsch graph. In this paper we show that every minimum perfect dominating set in the Petersen graph and the Clebsch graph induces $K_{1,3}$ and C_4 respectively. Further we establish that these classes of minimum perfect dominating sets of Clebsch graph form Partially Balanced Incomplete Block Designs with the parameters (16, 40, 10, 4, 1, 4).

Keywords: Partially balanced incomplete block designs; Minimum perfect dominating set; Perfect domination number; strongly regular graphs.

1. Introduction

In combinatorial mathematics, a block design is a particular kind of hyper graph or set system which has applications to finite geometry, cryptography and algebraic geometry. In the class of incomplete block design, the *balanced incomplete block design* (here after called BIBD) given by Yates, in 1936, is the simplest one. A BIBD is one among the many variations that have been studied in block designs and it is a set of v elements arranged in b blocks of k elements each in such a way that each element occurs in exactly r blocks and every pair of unordered elements occurs in l blocks and it is denoted by the representation (v, b, r, k, l)-design. The connection between graph theory and designs were first observed by Berge [3]. Motivated by the works of Berge, Paola [14] has given a link between some classes of graphs and BIBD's. As the class of BIBD's do not fit for many experimental situations as these design requires large number of replications, to overcome this Bose and Nair [5] introduced a class of binary, equireplicate and proper designs called Partially Balanced Incomplete Block Designs (here after called PBIBD's) which was included as a special case of the BIBD's. Bose and Shimamoto [6] are first to introduce the concept of association schemes in PBIBD's. More about association schemes can be found in Bannai and Ito [2], Godsil and Royal [10] and Bailey [1] and a catalogue of different PBIB on two associate class designs can be found in Clatworthy [8]. Bose in his pioneering paper [4], used the graph theoretic method for the study of association schemes of PBIBD's and also shown that the concept of strongly regular graphs is isomorphic with the association schemes of PBIBD's (with two associate classes). Walikar et. al. [**?**] introduced the designs called (ν, β_0, μ)-designs whose blocks are maximum independent sets in regular graphs on ν vertices. Walikar *et. al.* [16] have also established the relation between minimum dominating sets of a graph with the blocks of PBIBD's. We know that it is possible to construct the strongly regular graph G with the parameters $(\nu, n_1, P^1_{11}, P^2_{11})$from a given PBIBD with two association schemes having parameters $(\nu, b, r, k, \lambda_1, \lambda_2)$(see Bose [4] and Rao [15]). In this paper, we prove that every minimum perfect dominating set (here after called MPDS) in Clebsch graph induces C_4. Further we establish that the set of all MPDS's forms PBIBD with parameters (16,40,10,4,1,4).

2. Background

In this section, we discuss some definitions and preliminary results (see Berge [3], Chartrand and Zhang [7], Harary [9], Godsil and Royl [10], Haynes et.al. [11]). Suppose $G = (V, E)$ is a graph with vertex set $V$and edge set E. Throughout the paper $G = (V, E)$stands for finite, connected and undirected simple graph. A vertex u is said to dominate a vertex ν if E contains an edge from u to ν or $u = \nu$. A set of vertices $S \subseteq V$ is called a dominating set of G if every vertex of G is dominated by at least one member of S. When each vertex of G is dominated by exactly one member of S, the set S is called a perfect dominating set (here after called PDS) of G. The perfect domination number $\gamma_p(G)$is the cardinality of smallest PDS of G. The Petersen graph G is a cubic strongly regular graph with the parameters (10,3,0,1) whose vertices are 2-element subset of a set $\{1, 2, 3, 4, 5\}$ and two vertices in G are adjacent if their intersection is empty (see Holtan and Sheehan [13]). The figure 1 shows the construction of Petersen graph G.

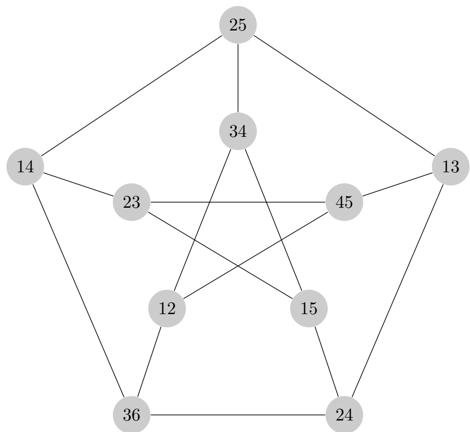

Figure 1. Construction of Peterson graph

The Clebsch graph G is a strongly regular Quintic graph on 16 vertices and 40 edges with parameters (16, 5, 0, 2). It is also known as the Greenwood-Gleason Graph. The Clebsch graph as vertices all subsets of $\{1, 2, 3, 4, 5\}$ of even cardinality; two vertices are adjacent if their symmetric difference has cardinality 4. The figure 2 shows the construction of Clebsch graph G.

Definition 2.1 ([11])**.** *A strongly regular graph G with parameters (n, k, λ, μ) is a graph on n vertices which is regular with valency k and has the following properties:*

- *any two adjacent vertices have exactly λ common neighbors;*
- *any two nonadjacent vertices have exactly μ common neighbors.*

So the Petersen graph and the Clebsch graph are strongly regular graph with the parameters (10, 3, 0, 1) and (16, 5, 0, 2) respectively. The four parameters n, k, λ and μ are not independent. Choose a

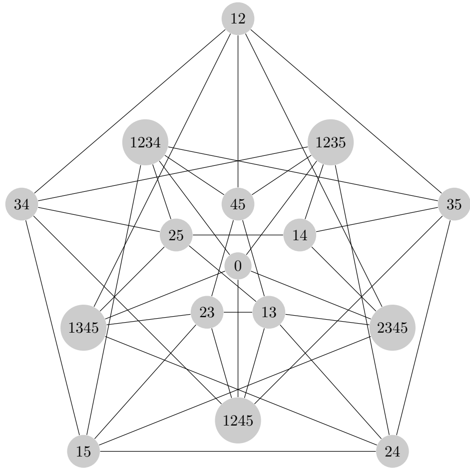

Figure 2. Construction of Clebsch graph

vertex ν; counting in two ways the ordered pairs (x, y) of adjacent vertices such that x is adjacent to ν but y is not, we obtain the following result.

Proposition 2.1 ([11]). *The parameters (n, k, λ, μ) of a strongly regular graph satisfies the equation, $k(k - \lambda - 1) = (n - k - 1)\mu$.*

Definition 2.2 ([15]). *Given ν symbols $1, 2, ..., \nu$, a relation satisfying the following conditions is called an m-class association scheme $(m \geq 2)$*

- *any two symbols are either 1^{st}, 2^{nd}, $\cdots$ or m^{th} associates; this relation being symmetric i.e., if the symbol $\alpha$is the i^{th} associate of β then β is the i^{th} associate of α.*
- *each symbol α has n_i i^{th} associates, the number n_i being independent of α.*
- *If α and β are two i^{th} associates, then the number of symbols that are j^{th} associates of α and k^{th} associates of β is P^i_{jk} and is independent of the pair of i^{th} associate α and β.*

Definition 2.3 ([15]). *Given ν treatment symbols $1, 2, \cdots, \nu$ and an association of m-classes with $m \geq 2$, we have a Partially balanced incomplete block design (PBIBD) if v treatment symbols can be arranged into b blocks each of them containing k symbols such that*

- *each of the symbol occurs in $r$blocks*
- *every symbol occurs at most once in a block.*
- *two symbols that are mutually i^{th}associates occur together in exactly $\lambda_i$blocks.*

The numbers ν, b, r, k, λ_i $(i = 1, 2, \cdots, m)$ are called the parameters of the first kind, whereas the numbers n_i and P^i_{jk}, $(i, j, k = 1, 2, \cdots, m)$ are called the parameters of the second kind. It can be easily seen that $vr = bk$and $\sum_{i=0}^{m} n_i\lambda_i = r(k - 1)$.

Proposition 2.2 ([7]). *If G is a graph of order n, $\left[\frac{n}{\Delta+1}\right]$, where Δ denotes the maximum degree of any vertex of G.*

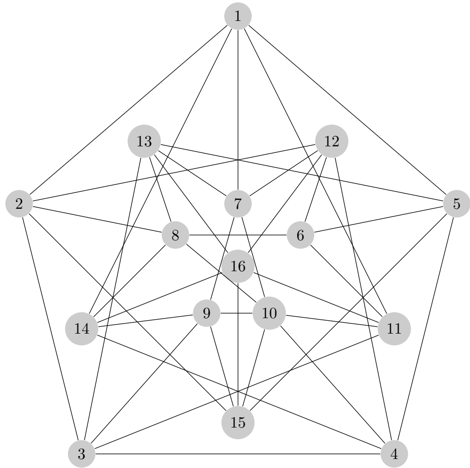

Figure 3. Construction of Clebsch graph with simplest node values.

Observation: For any graph G, $\gamma(G) \leq \gamma_p(G)$.

Lemma 2.3. *The perfect domination number of a Petersen graph is four.*

Proof. Let G be a Petersen graph. We know that $\gamma(G)$=3. From the observation 2.6, $\gamma_p(G) \geq 3$. To prove this, it is sufficient to show that $\gamma_p(G) \neq 2$ and $\gamma_p(G) \neq 3$. Clearly $\gamma_p(G) \neq 1$, since G is a cubic regular graph on 10 vertices. We now claim that $\gamma_p(G) \neq 2$. For if $\gamma_p(G) = 2$, let $S = \{u, v\}$ be MPDS. If u and ν are adjacent then they can cover maximum of six vertices including themselves and remaining four vertices of G are uncovered. On the other hand, if u and ν are non-adjacent then they have a common neighbor which is a contradiction to the fact that S is PDS. Thus, $\gamma_p(G) \neq 2$. Now, for if $\gamma_p(G) = 3$. Let $S = \{u, v, w\}$ be MPDS. Then, the vertices of S induce one of the following non-isomorphic sub graphs in G as shown in the figure 4.

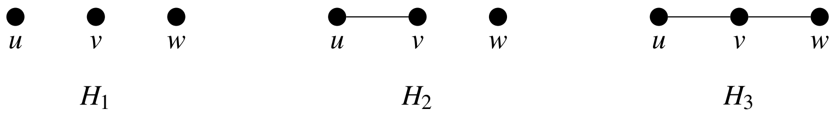

Figure 4.

As G is cubic strongly regular graph with parameters (10,3,0,1), any two nonadjacent vertices have exactly one common neighbor. If S induces any of the sub graphs H_1 or H_2 or H_3, then in all the three sub graphs the vertex u and w are nonadjacent and hence they must have common neighbor say z. Hence z is dominated by two of the vertices u and w of S, which is a contradiction to the fact that S is MPDS. Therefore, $\gamma_p(G) \neq 3$. On the other hand, by the regularity of G and since any two nonadjacent vertices have common neighbor, S must consist of four vertices to dominate G perfectly. Thus, $\gamma_p(G) = 4$, this proves the lemma. □

Lemma 2.4. *Every MPDS in Petersen graph induces a claw graph $K_{1,3}$.*

Proof. Let G be a Petersen graph. Then, by the above lemma 2.7 we have $\gamma_p(G) = 4$. Let $S = u_1, u_2, u_3, u_4$ be a MPDS of G. We claim that S is a claw graph $K_{1,3}$. For if, S is not a claw graph $K_{1,3}$

and then as G is triangle free cubic graph S induces one of the following non-isomorphic sub graphs in G as shown in the figure 5.

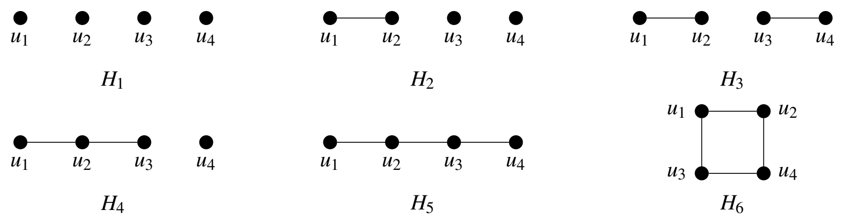

Figure 5.

To prove the lemma it is enough to show that following cases fails.
Case 1: $\langle S\rangle = H_1$ or $\langle S\rangle = H_2$ or $\langle S\rangle = H_3$ or $\langle S\rangle = H_4$ or $\langle S\rangle = H_5$

As G is cubic strongly regular graph with parameters (10, 3, 0, 1), any two nonadjacent vertices have exactly one common neighbor. By this property of G, we can see that if S induces any of the sub graphs H_1 or H_2 or H_3 or H_4 or H_5 then as the vertex u_1 is non-adjacent with the vertex u_4, they have a common neighbor say z. Hence the vertex $z$is dominated by the two vertices u_1 and u_4, which is a contradiction to the fact that S is MPDS.
Case 2: $\langle S\rangle = H_6$

As G is strongly regular graph with each of vertex of valency three, S can cover maximum of seven vertices in G and remaining three vertices of G and hence S is not a PDS. Thus, in all the above cases S is not PDS. Hence S must induce $K_{1,3}$. This proves the result. □

3. MPDS's in Clebsch graph

In this section, we study the MPDS in the Clebsch graph G. Let us partition the vertex set V of Clebsch graph G into two subsets $V_1 = N[z]$ and $V_2 = V - V1$, where z is any vertex in G. By the structure of Clebsch graph G, we can observe the following structural properties of G.
Observation 3.1: Clebsch graph is a triangle free strongly regular Quintic graph.
Observation 3.2: The sub graph on the non-neighbors of a point in the Clebsch graph is the Petersen graph.
Observation3.3: As G is strongly regular graph with parameters (16, 5, 0, 2), any two nonadjacent vertices are adjacent to two common vertices. Let u and ν be two nonadjacent vertices in G, then

- if $u, \mu \in V$, then they are adjacent to two common vertices namely z and other in V_2.
- if $u = z$ and $\mu \in V_2$, then both are adjacent to two nonadjacent vertices in V_1.
- if $u, \mu \in V_2$, then both are adjacent to one common vertex in V_2 and the other in V_1.

Lemma 3.1. *The perfect domination number of Clebsch graph is four.*

Proof. Let Gbe a Clebsch graph. Without loss of generality, let us partition the vertex set of G into two subsets $V_1 = N[z]$ and $V_2 = V - V_1$, as defined above. We now show that $\gamma_p(G) = 4$. To prove this we show that $\gamma_p(G) \neq 3$. For if $\gamma_p(G) = 3$, then let $S = \{u, v, w\}$ be any MPDS in G. As $G$is triangle free graph, S induces one of the following non-isomorphic sub graphs as shown in figure 6.

Case 1 Let $\langle S\rangle = G$

As $G$is a strongly regular graph with parameters (16, 5, 0, 2), any pair of nonadjacent vertices have

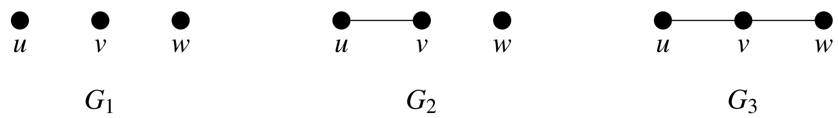

Figure 6.

two common neighbors. By this property and the regularity of G, we have $\bigcup_{z\in S} N[z] = 10$. Hence there are at least six vertices in G uncovered by S. Hence S is not a PDS.

Case 2 Let $\langle S\rangle = G_2$
The two vertices u and μ in G_2 can cover ten vertices of G and the remaining six vertices induce 3 $K_2's$ in G. By choosing one vertex w in 3 $K_2's$, we can cover two more vertices in G. Therefore there are four vertices of G which are uncovered by S. Hence S is not a PDS.

Case 3 Let $\langle S\rangle = G_3$
The two nonadjacent vertices $u$and $w$have common neighbor in G and they can cover eleven vertices of G. As the vertex μ is adjacent to both u and w and by the regularity of G, we have $\bigcup_{z\in S} N[z] = 13$. Thus, there are three vertices of G, which are uncovered by $S$and hence S is not a PDS. Thus in all the above cases S is not a PDS, hence $\gamma_p(G) = 4$ holds. In fact, the three vertices of MPDS in a Petersen graph induced by V_2 and one vertex in V_1 forms a MPDS in G, which gives $\gamma_p(G) = 4$. This proves the result.

□

Theorem 3.2. *If S is a MPDS in the Clebsch graph G, then S induces C_4.*

Proof. By the above lemma 3.6, we have $\gamma_p(G) = 4$. Let $S = \{u_1, u_2, u_3, u_4\}$ be a MPDS in G. We prove that the set S induces C_4 in G. As G is triangle free graph, the following are the possible non-isomorphic graphs induced by S as shown in figure 7.

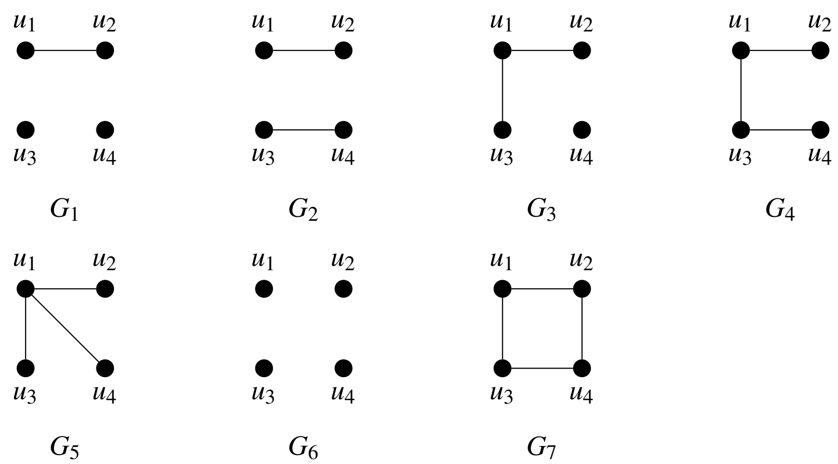

Figure 7.

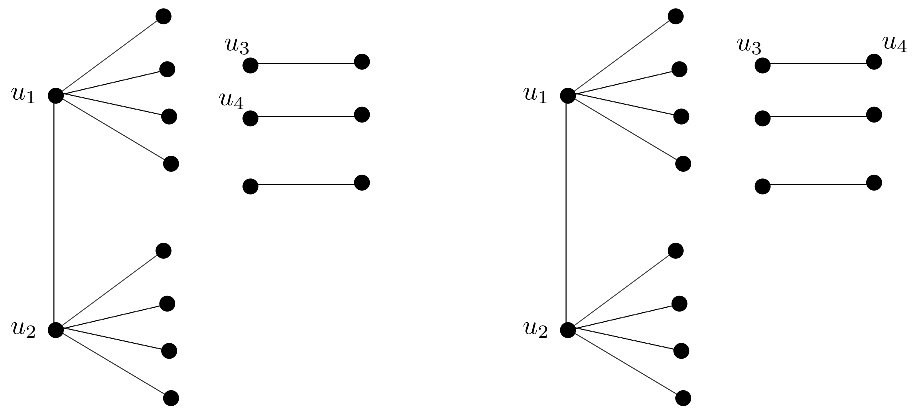

Fig.8

Figure 8.

Two vertices u_1 and u_2 of G_1 can cover ten vertices in G and the remaining six vertices induce $3K_2$'s in G. As u_3 and u_4 are nonadjacent, we can cover four other vertices of G and hence at least two vertices in G are not covered by S. Therefore S is not a PDS (Refer figure 8).

Case 2: $\langle S \rangle = G_3$ or $\langle S \rangle = G_4$ or $\langle S \rangle = G_5 = K_{1,3}$.

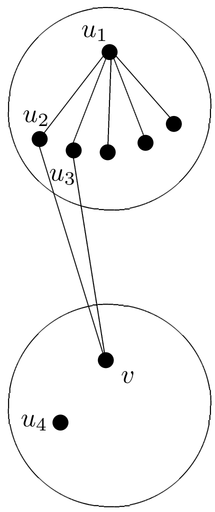

Fig.9

If S induces any of the graphs G_3 or G_4 or G_5. In all the cases without loss of generality, let us assume that $u_1, u_2, u_3 \in V$ and $u_4 \in V_2$. By the structure of Clebsch graph, u_2 and u_3 must be adjacent to a common vertex in V_2 and this vertex must be other than u_4 as none of the vertices u_1, u_2 and u_3 are adjacent to u_4. This contradicts the fact that S is a PDS. Hence, S is not a PDS (Refer figure 9).

Case 3 : $\langle S \rangle = G_6 = \overline{K}_4$

In this case we prove that if S induces K_4 then S is not a PDS. To prove this we consider the following possibilities for the set S (See figure 10).

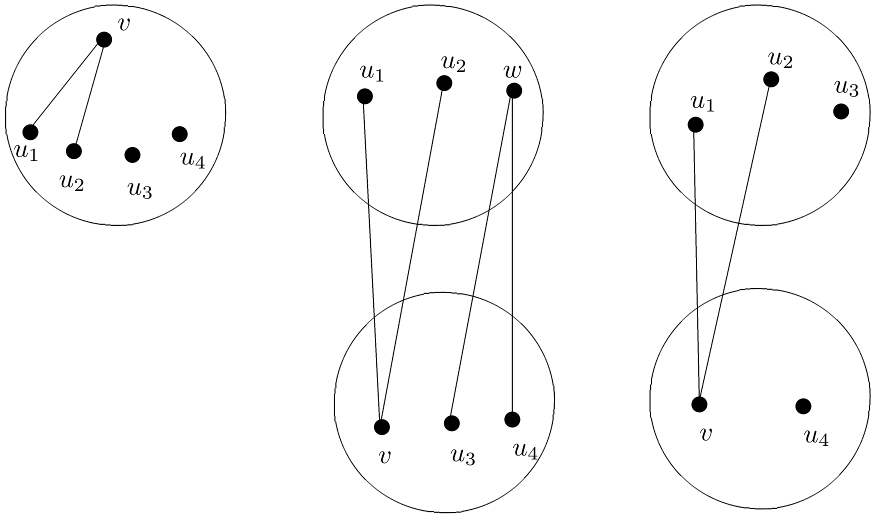

Fig.10

(i) $S \subset V_1$ or $S \subset V_2$.

(ii) Partition S into two subsets V_1 and V_2 such that $u_1, u_2 \in V_1$ and $u_3, u_4 \in V_2$.

(iii) Partition Sinto two subsets $V_1$and V_2such that $u_1, u_2, u_3 \in V_1$and $u_4 \in V_2$.

In the first sub case, as $S \subset V_1$ or $S \subset V_2$ and two vertices in S are non-adjacent, hence they have two common neighbors in V_1 or V_2 accordingly as $S \subset V_1$ or $S \subset V_2$, which is a contradiction to the fact that S is PDS. Hence S is not a PDS. On the other hand, in the second and third sub case, there is at least one vertex in V_1 which is non-adjacent to a vertex in V_2. By the structure of Clebsch graph, these non-adjacent vertices will have common neighbors in V_1 and V_2 which contradicts that S is a PDS. Thus, S is not a PDS.

Case 3: $\langle S \rangle = G_7 = C_4$

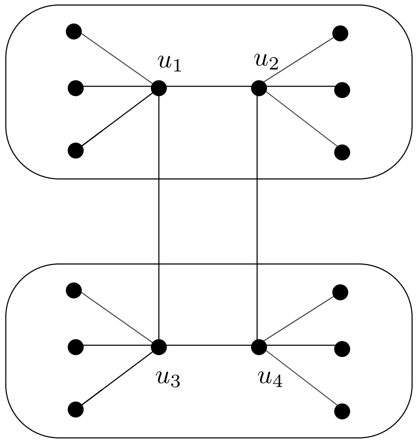

Fig.11

As G is strongly regular graph with each of the vertex of degree five. Each of the vertices of S will cover three distinct vertices in G as $N[u_i] \bigcup N[u_j] = \phi$, for every pair of u_i and u_j adjacent vertices of S in G. Thus, S covers every vertex of G. i.e., $\bigcup_{i=1}^{4} N[u_i] = V$. Hence S is a MPDS of G (Refer figure 11). This completes the proof.

□

The following is the list of MPDS which induces C_4in Clebsch graph G.
$\{1,2,3,11\}$, $\{1,2,15,5\}$, $\{1,2,12,7\}$, $\{1,2,8,14\}$, $\{1,5,4,14\}$, $\{1,5,6,11\}$, $\{1,5,13,7\}$, $\{1,7,9,14\}$, $\{1,7,10,11\}$, $\{1,11,16,14\}$, $\{2,3,4,12\}$, $\{2,3,13,8\}$, $\{2,3,9,15\}$, $\{2,8,6,12\}$, $\{2,8,10,15\}$, $\{2,12,16,15\}$, $\{3,4,5,13\}$, $\{3,4,14,9\}$, $\{3,4,10,11\}$, $\{3,9,6,11\}$, $\{3,9,7,13\}$, $\{3,11,16,13\}$, $\{4,5,6,12\}$, $\{4,5,15,10\}$, $\{4,10,7,12\}$, $\{4,10,8,14\}$, $\{4,12,16,14\}$, $\{5,6,8,13\}$, $\{5,6,9,15\}$, $\{5,13,16,15\}$, $\{6,9,7,12\}$, $\{6,8,14,9\}$, $\{6,8,10,11\}$, $\{6,11,16,12\}$, $\{7,10,8,13\}$, $\{7,9,15,10\}$, $\{7,12,16,13\}$, $\{8,13,16,14\}$, $\{9,14,16,15\}$, $\{10,11,16,15\}$.

4. PBIBD's associated with MPDS's of Clebsch graph

Let us define the 2 class association scheme of Clebsch graph by using the defintion 2.3 as follows: Let S be the MPDS's of Clebsch graph which induces C_4. Then, S is the set of blocks of PBIBD with parameters of first kind as (16,40,10,4,1,4) and parameters of second kind as

$$P_1 = \begin{pmatrix} P_{11}^1 & P_{12}^1 \\ P_{21}^1 & P_{22}^1 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & 6 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} P_{11}^2 & P_{12}^2 \\ P_{21}^2 & P_{22}^2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$$

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