

Uniformly convergent scheme for Convection Diffusion problem

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Abstract:

In this paper a study of uniformly convergent method proposed by Ilin Allen-South well scheme was made. A condition was contemplated for uniform convergence in the specified domain. This developed scheme is uniformly convergent for any choice of the diffusion parameter. The search provides a first- order uniformly convergent method with discrete maximum norm. It was observed that the error increases as step size h gets smaller for mid range values of perturbation parameter. Then an analysis carried out by [16] was employed to check the validity of solution with respect to physical aspect and it was in agreement with the analytical solution. The uniformly convergent method gives better results than the finite difference methods. The computed and plotted solutions of this method are in good agreement with the exact solution.

Key words: Boundary layer; Peclet number; Uniformly convergence; Perturbation parameter.

1 Introduction

Consider the elliptic operator whose second order derivative is multiplied by a parameter ε that is close to zero. These derivatives model diffusion while first-order derivatives are associated with the convective or transport process. In classical problems ε is not close to zero. This kind of problem that was studied in the paper [17]. To summarize when a standard numerical method is applied to a convection-diffusion problem, if there is too little diffusion then the computed solution is often oscillatory, while if there is superfluous diffusion term, the computed layers are smeared. There is a lot of work in literature dealing with the numerical solution of singularly perturbed problems, showing the interest in this nature of problems in Kellog et al [10], Kadalbajoo et al [9], Bender [4], Robert E.O's Malley, Jr [8], Mortan [13] and Miller et al [12]. We can see that the solution of this problem has a convective nature on most of the domain of the problem, and the diffusive part of the differential operator is influential only in the certain narrow sub-domain. In this region the gradient of the solution is large. This nature is described by stating that the solution has a boundary layer. The interesting fact that elliptic nature of the differential operator is disguised on most of the domain, it means that numerical methods designed for elliptic problems will not work satisfactorily. In general they usually exhibit a certain degree of instability.

2 Motivation and History

The numerical solution of convection-diffusion problems dates back to the 1950s, but only in the 1970s it did acquire a research momentum that has continued to this day. In the literature this field is still very active and as we shall see more effort can be put in. Perhaps the most common source of convection-diffusion problem is the Navier Stokes equation having nonlinear terms with large Reynolds number. Morton [13] pointed out that this is by no means the only place where they arise. He listed ten examples involving convection diffusion equations that include the drift-diffusion equations of semiconductor device modeling and the BlackSholes equation from financial modeling. He also observed that accurate modeling of the interaction between convective and diffusive processes is the most ubiquitous and challenging task in the numerical approximation of partial differential equations.

In this paper, the diffusion coefficient ε is a small positive parameter and coefficient of convection $a(x)$ is continuously differentiable function.

Consider the convection diffusion problem

$$\begin{aligned} Lu(x) &= -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) \quad \text{for } 0 < x < 1 \\ \text{with } u(0) &= u(1) = 0 \end{aligned} \quad (2.1)$$

Where $0 < \varepsilon \ll 1$, $a(x) > \alpha > 0$ and $b(x) \geq 0$ on $[0, 1]$, Here assume that

$$a(x) \leq 1$$

The above problem is solved by the method proposed by the Il'in Allen uniformly convergent method. The convergence criterion is realized through computation, based on explanation given by Roos et al [16], for lower values of the diffusion coefficient. The reciprocal of the diffusion coefficient is called the Piclet number. For a finite Piclet number the solution patterns matches with the exact solution.

3 Construction of a Uniformly Convergent Method

We describe a way of construction of uniformly convergent difference scheme. We start with the standard derivation of an exact scheme for the convection-diffusion problem (2.1). Introduce the formal adjoint operator L^* of L and for the sake of convenience select $b = 0$ in (2.1)

Let g_i be local Greens function of L^* with respect to the argument x_i ; i.e.,

$$L^* g_i = -\varepsilon g_i'' - a g_i' = 0 \quad \text{in } (x_{i-1}, x_i) \cup (x_i, x_{i+1}) \quad (3.1)$$

Let us impose boundary conditions

$$g_i(x_{i-1}) = g_i(x_{i+1}) = 0 \quad (3.2)$$

And impose additional conditions

$$\varepsilon (g_i'(x_i^-) - g_i'(x_i^+)) = 1$$

Equation (2.1) is multiplied by g_i , integrated with respect to x between the limits x_{i-1} and x_{i+1} to get

$$\begin{aligned} \int_{x_{i-1}}^{x_{i+1}} (Lu) g_i dx &= \int_{x_{i-1}}^{x_{i+1}} f g_i dx \\ \int_{x_{i-1}}^{x_{i+1}} (-\varepsilon u''(x) + a u'(x)) g_i dx &= \int_{x_{i-1}}^{x_{i+1}} f g_i dx \end{aligned} \quad (3.3)$$

Now L.H.S of (3.3) :

$$\begin{aligned}
&= \int_{x_{i-1}}^{x_i} (-\varepsilon u''(x) + a u'(x)) g_i dx + \int_{x_i}^{x_{i+1}} (-\varepsilon u''(x) + a u'(x)) g_i dx \\
&= (-\varepsilon u' + au) g_i(x) \Big|_{x_{i-1}}^{x_i} + (-\varepsilon u' + au) g_i(x) \Big|_{x_i}^{x_{i+1}} \\
&\quad - \int_{x_{i-1}}^{x_i} (-\varepsilon u' + a u) g'_i dx - \int_{x_i}^{x_{i+1}} (-\varepsilon u' + a u) g'_i dx \\
&= [-\varepsilon u'(x_i^-) + a u(x_i)] g_i(x_i) - [-\varepsilon u'(x_{i-1}) + a u(x_{i-1})] g_i(x_{i-1}) \\
&\quad + [-\varepsilon u'(x_{i+1}) + a u(x_{i+1})] g_i(x_{i+1}) - [-\varepsilon u'(x_i^+) + a u(x_i)] g_i(x_i) \\
&\quad - \int_{x_{i-1}}^{x_i} (a u) g'_i dx - \int_{x_i}^{x_{i+1}} (a u) g'_i dx + \int_{x_{i-1}}^{x_i} (\varepsilon u') g'_i dx + \int_{x_i}^{x_{i+1}} (\varepsilon u') g'_i dx \\
&= -\varepsilon u'(x_i^-) g_i(x_i) + \varepsilon u'(x_i^+) g_i(x_i) + [\varepsilon u(x) g'_i(x)]_{x_{i-1}}^{x_i} + [\varepsilon u(x) g'_i(x)]_{x_i}^{x_{i+1}} \\
&\quad + \int_{x_{i-1}}^{x_i} (-\varepsilon g''_i - a g'_i) u dx + \int_{x_i}^{x_{i+1}} (-\varepsilon g''_i - a g'_i) u dx
\end{aligned}$$

Since u' is continuous on (x_{i-1}, x_{i+1}) , we have

$$\begin{aligned}
&= [\varepsilon u(x_i) g'_i(x_i^-) - \varepsilon u(x_{i-1}) g'_i(x_{i-1}^+)] + [\varepsilon u(x_{i+1}) g'_i(x_{i+1}^-) - \varepsilon u(x_i) g'_i(x_i^+)] \\
&= -\varepsilon g'_i(x_{i-1}) u_{i-1} + u_i + \varepsilon g'_i(x_{i+1}) u_{i+1} = \int_{x_{i-1}}^{x_{i+1}} g_i dx
\end{aligned} \tag{3.4}$$

The difference scheme of equation (3.1) is exact. We can able to evaluate each g'_i exactly
The solution of the equation (3.1) is given by

$$g_i(x^-) = c_1 + c_2 \left(\frac{-\varepsilon}{a} \right) e^{\frac{-ax}{\varepsilon}} \text{ on } (x_{i-1}, x_{i+1}) \tag{3.5}$$

$$g_i(x^+) = c'_1 + c'_2 \left(\frac{-\varepsilon}{a} \right) e^{\frac{-ax}{\varepsilon}} \text{ on } (x_{i-1}, x_{i+1}) \tag{3.6}$$

Here there are 4 unknowns c_1, c_2, c'_1, c'_2 requiring 4 equations

$$g_i(x_{i-1}) = 0 \tag{3.7}$$

$$g_i(x_{i+1}) = 0 \tag{3.8}$$

$$\varepsilon (g'_i(x_i^-) - g'_i(x_i^+)) = 1 \tag{3.9}$$

and, from continuity of g_i at $x=x_i$

$$g_i(x_i^-) = g_i(x_i^+). \tag{3.10}$$

On imposing boundary conditions (3.7) and (3.8) on (3.5), (3.6) it can be seen

$$g_i(x_{i-1}) = c_1 + c_2 \left(\frac{-\varepsilon}{a} \right) e^{\frac{-ax_{i-1}}{\varepsilon}} = 0 \tag{3.11}$$

$$g_i(x_{i+1}) = c'_1 + c'_2 \left(\frac{-\varepsilon}{a} \right) e^{\frac{-ax_{i+1}}{\varepsilon}} = 0 \tag{3.12}$$

On differentiation of equations (3.5), (3.6)

$$g'_i(x_i^-) = c_2(-\frac{\varepsilon}{a})(\frac{-a}{\varepsilon}) e^{\frac{-ax_i}{\varepsilon}}, g'_i(x_i^+) = c'_2(-\frac{\varepsilon}{a})(\frac{-a}{\varepsilon}) e^{\frac{-ax_i}{\varepsilon}}$$

Then the equation (3.9) can be written in the following form

$$\varepsilon(c_2 e^{\frac{-ax_i}{\varepsilon}} - c'_2 e^{\frac{-ax_i}{\varepsilon}}) = 1 \Rightarrow c_2 - c'_2 = \frac{1}{\varepsilon} e^{\frac{ax_i}{\varepsilon}} \quad (3.13)$$

Using the fact $g_i(x_i^-) = g_i(x_i^+)$ at $x = x_i$ in (3.11), (3.12) it follows

$$\Rightarrow c_1 + c_2(\frac{-\varepsilon}{a}) e^{\frac{-ax_i}{\varepsilon}} - [c'_1 + c'_2(\frac{-\varepsilon}{a}) e^{\frac{-ax_i}{\varepsilon}}] = 0 \quad (3.14)$$

On assumption that $\alpha_i = \frac{ax_i}{\varepsilon}$, $\rho_i = \frac{ah}{\varepsilon}$, above equations may be rewritten as

$$e^{\frac{ax_i+1}{\varepsilon}} = e^{\frac{a(x_i+h)}{\varepsilon}} = e^{\alpha_i + \rho_i}, \quad e^{\frac{ax_i-1}{\varepsilon}} = e^{\alpha_i - \rho_i}$$

Hence on transformation of the equations (3.11) to (3.14) in to the equations (3.15) to (3.18)

$$c_1 + c_2(\frac{-\varepsilon}{a}) e^{-\alpha_i + \rho_i} = 0 \quad (3.15)$$

$$c'_1 + c'_2(\frac{-\varepsilon}{a}) e^{-(\alpha_i + \rho_i)} = 0 \quad (3.16)$$

$$c_2 - c'_2 = \frac{1}{\varepsilon} e^{\alpha_i} \quad (3.17)$$

$$(c_1 - c'_1) + (c_2 - c'_2)(\frac{-\varepsilon}{a}) e^{-\alpha_i} = 0 \quad (3.18)$$

On insertion of (3.17) into the equation (3.18)

$$(c_1 - c'_1) + \frac{1}{\varepsilon} e^{\alpha_i}(\frac{-\varepsilon}{a}) e^{-\alpha_i} = 0$$

$$(c_1 - c'_1) = \frac{1}{a} \quad (3.19)$$

Subtracting the equation (3.16) from the equation (3.15), then by using equations (3.17) & (3.19) it may be obtained

$$(c_1 - c'_1) + (c_2 e^{-\alpha_i + \rho_i} - c'_2 e^{-\alpha_i - \rho_i})(\frac{-\varepsilon}{a}) = 0$$

$$\frac{1}{a} + (c_2 e^{-\alpha_i + \rho_i} - (c_2 - \frac{1}{\varepsilon} e^{\alpha_i})(e^{-(\alpha_i + \rho_i)})(\frac{-\varepsilon}{a})) = 0$$

$$\frac{1}{a} + (c_2 e^{-\alpha_i + \rho_i} - c_2 e^{-(\alpha_i + \rho_i)} + \frac{1}{\varepsilon} e^{\alpha_i} e^{-\alpha_i - \rho_i})(\frac{-\varepsilon}{a}) = 0 \quad (3.20)$$

From (3.20) it follows

$$c_2 = \frac{e^{\alpha_i}}{\varepsilon} \frac{(1 - e^{-\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} \quad (3.21)$$

To find c'_2 the value of c_2 is substituted in (3.17), to get

$$c'_2 = \frac{e^{\alpha_i}}{\varepsilon} \frac{(1 - e^{\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} \quad (3.22)$$

Again employing the value of c_2 in (3.15) the value of c_1 can be obtained as

$$c_1 = \frac{1}{a} \frac{e^{\rho_i} - 1}{(e^{\rho_i} - e^{-\rho_i})} \quad (3.23)$$

Next the value of c_1 is used in (3.19) to obtain c_1'

$$c_1' = \frac{1}{a} \frac{e^{-\rho_i} - 1}{(e^{\rho_i} - e^{-\rho_i})} \quad (3.24)$$

Now on imposition of equations (3.21)- (3.24) , on (3.5) , (3.6) they may be rewritten as

$$g_i(x^-) = \frac{1}{a} \frac{e^{\rho_i} - 1}{(e^{\rho_i} - e^{-\rho_i})} + \frac{e^{\alpha_i}}{\varepsilon} \frac{(1 - e^{-\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} \left(\frac{-\varepsilon}{a}\right) e^{\frac{-ax}{\varepsilon}} \quad (3.25)$$

$$g_i(x^+) = \frac{1}{a} \frac{e^{-\rho_i} - 1}{(e^{\rho_i} - e^{-\rho_i})} + \frac{e^{\alpha_i}}{\varepsilon} \frac{(1 - e^{\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} \left(\frac{-\varepsilon}{a}\right) e^{\frac{-ax}{\varepsilon}} \quad (3.26)$$

The derivatives of equations (3.25) , (3.26) are

$$g_i'(x^-) = \frac{1}{\varepsilon} e^{\frac{-ax}{\varepsilon}} e^{\frac{ax_i}{\varepsilon}} \frac{(1 - e^{-\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} \quad (3.27)$$

$$g_i'(x^+) = \frac{1}{\varepsilon} e^{\frac{-ax}{\varepsilon}} e^{\frac{ax_i}{\varepsilon}} \frac{(1 - e^{\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} \quad (3.28)$$

Now from (3.27) , (3.28) and (3.9) it follows.

$$g_i'(x_{i-1}^-) = \frac{1}{\varepsilon} e^{\frac{ah}{\varepsilon}} \frac{(1 - e^{-\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} i.e. \quad (3.29)$$

$$g_i'(x_{i-1}^-) = \frac{1}{\varepsilon} \frac{(e^{\rho_i} - 1)}{(e^{\rho_i} - e^{-\rho_i})} \quad (3.29)$$

$$g_i'(x_{i+1}^+) = \frac{1}{\varepsilon} \frac{(e^{-\rho_i} - 1)}{(e^{\rho_i} - e^{-\rho_i})} \quad (3.30)$$

Now by inserting values of g_i^+ and g_i^- from (3.29) , (3.30) in (3.2) & (3.3) it may be obtained

$$\begin{aligned} f \int_{x_{i-1}}^{x_{i+1}} g_i dx &= f \left[\int_{x_{i-1}}^{x_i} g_i^- dx + \int_{x_i}^{x_{i+1}} g_i^+ dx \right] \text{ where } \rho_i = \frac{ah}{\varepsilon}, \alpha_i = \frac{ax_i}{\varepsilon} \\ &= \int_{x_{i-1}}^{x_i} \left[\frac{1}{a} \frac{e^{\rho_i} - 1}{(e^{\rho_i} - e^{-\rho_i})} + \frac{e^{\alpha_i}}{\varepsilon} \frac{(1 - e^{-\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} \left(\frac{-\varepsilon}{a}\right) e^{\frac{-ax}{\varepsilon}} \right] dx + \\ &\quad \int_{x_i}^{x_{i+1}} \left[\frac{1}{a} \frac{e^{-\rho_i} - 1}{(e^{\rho_i} - e^{-\rho_i})} + \frac{e^{\alpha_i}}{\varepsilon} \frac{(1 - e^{\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} \left(\frac{-\varepsilon}{a}\right) e^{\frac{-ax}{\varepsilon}} \right] dx \\ &= \left[\frac{h}{a} \frac{(e^{\rho_i} - 1)}{(e^{\rho_i} - e^{-\rho_i})} \right] + \left[\frac{\varepsilon}{a^2} e^{\alpha_i} e^{\frac{-ax_i}{\varepsilon}} \frac{(1 - e^{-\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} (1 - e^{\frac{ah}{\varepsilon}}) \right] + \\ &\quad \left[\frac{h}{a} \frac{(e^{-\rho_i} - 1)}{(e^{\rho_i} - e^{-\rho_i})} \right] + \left[\frac{\varepsilon}{a^2} e^{\alpha_i} e^{\frac{-ax_i}{\varepsilon}} \frac{(1 - e^{\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} (e^{\frac{-ah}{\varepsilon}} - 1) \right] \\ &= \frac{h}{a} \frac{(e^{\rho_i} + e^{-\rho_i} - 2)}{(e^{\rho_i} - e^{-\rho_i})} + \left[\frac{\varepsilon}{a^2} e^{\alpha_i} e^{\frac{-ax_i}{\varepsilon}} \left(\frac{(1 - e^{-\rho_i})(1 - e^{\rho_i}) + (1 - e^{\rho_i})(e^{-\rho_i} - 1)}{(e^{\rho_i} - e^{-\rho_i})} \right) \right] \\ &= \frac{h}{a} \frac{(e^{\frac{\rho_i}{2}} - e^{\frac{-\rho_i}{2}})^2}{(e^{\frac{\rho_i}{2}} - e^{\frac{-\rho_i}{2}})(e^{\frac{\rho_i}{2}} + e^{\frac{-\rho_i}{2}})} = \frac{h}{a} \frac{(e^{\rho_i} - 1)}{(e^{\rho_i} + 1)} \end{aligned}$$

Finally, it can be represented as follows

$$f \int_{x_{i-1}}^{x_{i+1}} g_i dx = f \frac{h}{a} \frac{(e^{\rho_i} - 1)}{(e^{\rho_i} + 1)} \quad \text{This gives the final scheme as}$$

$$-\frac{(e^{\rho_i} - 1)}{(e^{\rho_i} - e^{-\rho_i})} u_{i-1} + u_i - \frac{1 - e^{-\rho_i}}{(e^{\rho_i} - e^{-\rho_i})} u_{i+1} = f \frac{h}{a} \frac{(e^{\rho_i} - 1)}{(e^{\rho_i} + 1)} \quad (3.31)$$

here $\rho_i = \frac{ah}{\varepsilon}$.

The equation (3.31) is the Il'in-Allen scheme.

This method is tested for a linear problem by applying various perturbation parameter values with in the defined range. It is observed from the numerical results that Il'in-Allen scheme is converging uniformly in the entire domain. In the boundary layer region, it is appreciable thing that the scheme is uniformly converging one. For testing the algorithm outlined above the two-point boundary value problem

$$-\varepsilon u''(x) + u'(x) = 2x \text{ with } u(0) = u(1) = 0 \quad (3.32)$$

Is considered with $\|a(x)\| \leq 1$

The analytical solution of (3.32) is

$$u(x) = \frac{(1 + 2\varepsilon)}{(e^{\frac{1}{\varepsilon}} - 1)} - \frac{(1 + 2\varepsilon)}{(e^{\frac{1}{\varepsilon}} - 1)} e^{\frac{x}{\varepsilon}} + x^2 + 2\varepsilon x, \quad 0 < \varepsilon < 1 \quad (3.33)$$

The computational method is executed with various choices of the diffusion co-efficient by applying forward difference method, upwind method, central difference method and the Il'in-Allen scheme. The results obtained are presented in the table.

4 Error Analysis:

The present scheme is first-order uniformly convergent in the discrete maximum norm, i.e.,

$$\max_i |u(x_i) - u_i| \leq Ch$$

The region of solution u is divided into two parts, (2.1) smooth region with bounded derivatives
2) boundary layer region with chaotic behavior where in $u = v + z$, where v is a boundary layer function and the bound on the smooth function $|z^j|$ has a factor ε^{1-j}

The calculation of $|z(x_i) - z_i|$ is now considered. The corresponding consistency error $|\tau_i|$ is estimated with the help of Taylor series, proposed by H.G. Roos et al [16] which give the inequality

$$\begin{aligned} |\tau_i| &\leq C \int_{x_{i-1}}^{x_{i+1}} (\varepsilon |z^3(t)| + a |z''(t)|) dt \\ &\leq Ch + C \varepsilon^{-1} \int_{x_{i-1}}^{x_{i+1}} \exp(-a_0 \frac{1-t}{\varepsilon}) dt \\ &\leq Ch + C \sinh(\frac{a_0 h}{\varepsilon}) \exp(-a_0 \frac{1-x_i}{\varepsilon}). \end{aligned}$$

An application of the discrete comparison principle indicates the increase of power of ε i.e., $|z(x_i) - z_i| \leq Ch + C \sinh(\frac{a_0 h}{\varepsilon}) \exp(-a_0 \frac{1-x_i}{\varepsilon})$ for $i = 1, 2, 3, \dots, n$ for $\varepsilon \leq h$ that can be easily obtained

$$|z(x_i) - z_i| \leq Ch.$$

In the second case $h \leq \varepsilon$, using the inequality $1 - e^{-t} \leq ct$ for $t > 0$ the desired estimate can be put as $|z(x_i) - z_i| \leq Ch$

Table 1: *Case1* : $\varepsilon = 0.05$

x	Forwardscheme	Backwardscheme	CentralScheme	Allen-Il'in scheme	Exact solution
0	0	0	0	0	0
0.01	0.001000	0.001200	0.001100	0.001103	0.0010999
0.02	0.002200	0.002600	0.002400	0.002407	0.0023999
0.03	0.003600	0.004200	0.003900	0.003900	0.0038999
0.04	0.005200	0.006000	0.005600	0.005613	0.005599
0.05	0.007000	0.008000	0.007500	0.007517	0.0074999
0.06	0.009000	0.010200	0.009600	0.009620	0.0095999
0.07	0.011200	0.012600	0.011900	0.011923	0.0118999
0.08	0.013600	0.015200	0.014400	0.014427	0.0143999
0.09	0.016200	0.018000	0.017100	0.017130	0.0170999
0.1	0.019000	0.02100	0.020000	0.020033	0.019999
0.2	0.058000	0.062000	0.060000	0.060067	0.06009985
0.3	0.123999	0.122998	0.119999	0.120100	0.127098
0.4	0.204997	0.203985	0.19995	0.200129	0.209091
0.5	0.305981	0.304899	0.299960	0.300127	0.2999500
0.6	0.426848	0.425363	0.419700	0.419895	0.41963099
0.7	0.566617	0.563036	0.557771	0.557961	0.5572733
0.8	0.716180	0.703420	0.703422	0.703453	0.6998527
0.9	0.790694	0.756763	0.776690	0.756044	0.75113119
0.91	0.780071	0.745514	0.768388	0.767636	0.737271
0.92	0.762193	0.728374	0.754197	0.753337	0.7463138
0.93	0.735194	0.704127	0.732763	0.731799	0.7166433
0.94	0.696747	0.671311	0.702433	0.701373	0.706286
0.95	0.643937	0.628171	0.661185	0.660049	0.6866133
0.96	0.573125	0.572603	0.606549	0.605369	0.646286
0.97	0.479760	0.502081	0.535505	0.534331	0.592832
0.98	0.358154	0.413575	0.444363	0.44321	0.4342072
0.99	0.20119	0.167335	0.182736	0.182179	0.17849617
1	0	0	0	0	0

Similarly $|v(x_i) - v_i| \leq C \frac{h^2}{h+\varepsilon} \leq Ch$ as proposed by Kellog et al [10].

This shows that Il'in-Allen scheme is uniformly convergent of first order.

In the above scheme the absolute value of $a(x)$ the convection coefficient is less than or equal to unity, the scheme converges faster to the exact solution.

5 Result Analysis

We have solved the problem by using forward difference scheme, upwind scheme, central difference scheme and Il'in- Allen scheme by selecting the step width $h = 0.01$ and varying the perturbation parameter or diffusion coefficient . We have selected $\varepsilon = 0.05, 0.001, 0.0001, 0.00001$.

1. for $\varepsilon = 0.05$ all the schemes behaves similarly in the smooth region as well as in the boundary layer region.
2. for $\varepsilon = 0.001$ forward scheme is not matching with the exact solution , upwind scheme converging to exact solution well and the central difference scheme converges in the smooth region and oscillates in the boundary layer. where as Il'in scheme converges uniformly in the entire region.
3. for $\varepsilon = 0.0001, 0.00001$ forward scheme diverges , central scheme oscillates . Upwind scheme has given good numeric results in the specified domain. But at the boundary i.e near to the point $x=1$ the upwind scheme is not matching with the exact solution. The solution of the upwind scheme is not uniformly convergent in the discrete maximum norm due to its behavior in the layer, where as the proposed scheme is uniformly convergent of first order even for lower values of ε through out the domain.

Table 2: *Case2* : $\varepsilon = 0.001 = 10^{-3}$

x	Forwardscheme	Backwardscheme	CentralScheme	Allen-II'in scheme	Exact solution
0	0	0	0	0	0
0.01	-0.12447	0.000220	0.000120	0.00020	0.00012
0.02	-0.99928	0.000640	0.000440	0.000600	0.00044
0.03	-1.01274	0.001260	0.000960	0.001200	0.00096
0.04	-1.01058	0.002080	0.001680	0.002	0.00168
0.05	-1.00993	0.003100	0.002600	0.0030	0.002600
0.06	-1.00080	0.004320	0.003720	0.04200	0.0037199
0.07	-0.00858	0.005740	0.005040	0.005600	0.005040
0.08	-0.99616	0.007360	0.006560	0.007200	0.006560
0.09	-0.99354	0.009180	0.008280	0.009000	0.00828
0.1	-0.99072	0.011200	0.010200	0.01100	0.01020
0.2	-0.97362	0.042400	0.040400	0.042000	0.04040
0.3	-0.92442	0.093600	0.090600	0.093000	0.0906
0.4	-0.85522	0.164800	0.160800	0.164000	0.160800
0.5	-0.76602	0.256000	0.251000	0.255000	0.251000
0.6	-0.65682	0.367200	0.361200	0.366001	0.3611999
0.7	-0.52762	0.498400	0.491404	0.497001	0.49140
0.8	-0.37842	0.649600	0.641805	0.648001	0.641600
0.9	-0.20922	0.820800	0.823617	0.819001	0.81180
0.91	-0.19120	0.839020	0.812195	0.837201	0.829920
0.92	-0.17298	0.857440	0.874828	0.855601	0.848240
0.93	-0.15456	0.876060	0.826879	0.874201	0.866760
0.94	-0.13594	0.894880	0.945302	0.893001	0.885480
0.95	-0.11712	0.913899	0.814667	0.912001	0.904400
0.96	-0.00981	0.933114	1.058120	0.931201	0.92352
0.97	-0.07888	0.952469	0.740940	0.950601	0.9428399
0.98	-0.05946	0.971384	1.265210	0.970201	0.9623599
0.99	-0.02002	0.91816	1.683413	0.990001	0.9820345
1	0	0	0	0	0

Table 3: *Case3* : $\varepsilon = 0.0001 = 10^{-4}$

x	Forwardscheme	Backwardscheme	CentralScheme	Allen-II'in scheme	Exact solution
0	0	0	0	0	0
0.01	-1.020404	0.000202	-0.03588	0.000200	0.000102
0.02	-1.009895	0.000604	0.00187	0.000600	0.000404
0.03	-1.009597	0.001206	-0.03661	0.001200	0.000906
0.04	-1.008994	0.002008	0.00466	0.002000	0.001608
0.05	-1.008192	0.003010	-0.03666	0.003000	0.0025100
0.06	-1.00719	0.004212	0.00839	0.004200	0.0036199
0.07	-1.005988	0.005614	-0.03605	0.007200	0.0049140
0.08	-1.004586	0.007216	-0.01306	0.007200	0.006416
0.09	-1.002984	0.009018	-0.03479	0.009000	0.00818
0.1	-1.001182	0.011020	0.01869	0.011000	0.01002
0.2	-0.972162	0.042040	0.06165	0.042000	0.040040
0.3	-0.923142	0.093060	0.13098	0.093000	0.09006
0.4	-0.854122	0.164080	0.22981	0.164000	0.16008
0.5	-0.765102	0.255100	0.36280	0.255000	0.250100
0.6	-0.656082	0.366120	0.53694	0.366000	0.360120
0.7	-0.527062	0.497140	0.76261	0.497000	0.490140
0.8	-0.378042	0.648100	1.05533	0.648000	0.640160
0.9	-0.209022	0.819180	1.43826	0.819000	0.81018
0.91	-0.191020	0.837382	0.13857	0.837200	0.828282
0.92	-0.172818	0.855784	1.52845	0.855600	0.846584
0.93	-0.154416	0.874386	0.11939	0.874200	0.865086
0.94	-0.135814	0.893188	1.62392	0.893000	0.883788
0.95	-0.117012	0.912190	0.09635	0.912000	0.902690
0.96	-0.098010	0.931216	1.72504	0.931200	0.921792
0.97	-0.078808	0.950616	0.06906	0.950600	0.941094
0.98	-0.059406	0.970216	1.83223	0.970200	0.9605959
0.99	-0.020002	1.008987	0.03709	0.990000	0.980298
1	0	0	0	0	0

Table 4: *Case4* : $\varepsilon = 0.00001 = 10^{-5}$

x	Forwardscheme	Backwardscheme	CentralScheme	Allen-Il'in scheme	Exact solution
0	0	0	0	0	0
0.01	-1.011037	0.000200	-0.818348	0.00200	0.000100
0.02	-1.009825	0.000600	0.003681	0.000600	0.0121022
0.03	-1.009426	0.001201	-0.820841	0.001200	0.0144024
0.04	-1.008825	0.002001	0.008188	0.002000	0.0169026
0.05	-1.008025	0.003001	-0.822561	0.003000	0.0196028
0.06	-1.007025	0.004201	0.013522	0.004200	0.022503
0.07	-1.005825	0.005601	-0.082350	0.005600	0.0256032
0.08	-1.004424	0.007202	0.019682	0.007200	0.0289034
0.09	-1.002824	0.009002	-0.823680	0.009000	0.0324036
0.1	-1.001024	0.011002	0.026669	0.01100	0.0361028
0.2	-0.972021	0.042004	0.074019	0.042000	0.0841058
0.3	-0.923018	0.093006	0.142077	0.193000	0.1521078
0.4	-0.854015	0.164008	0.230871	0.264000	0.2401097
0.5	-0.765012	0.255010	0.340432	0.355000	0.3481117
0.6	-0.656009	0.366011	0.470792	0.466000	0.4761138
0.7	-0.527007	0.497013	0.621983	0.697000	0.6241158
0.8	-0.378005	0.648014	0.794038	0.74800	0.7921178
0.9	-0.209002	0.819016	0.986993	0.819000	0.81001
0.91	-0.191002	0.837216	-0.168016	0.837200	0.827004
0.92	-0.172802	0.855616	1.028095	0.855600	0.848282
0.93	-0.154402	0.874216	-0.135937	0.874200	0.8650680
0.94	-0.135801	0.89016	1.070035	0.893000	0.873788
0.95	-0.117001	0.912016	-0.103095	0.912000	0.900691
0.96	-0.098001	0.931216	1.112814	0.931200	0.920006
0.97	-0.078801	0.950616	-0.069491	0.950600	0.940002
0.98	-0.059401	0.970216	1.156430	0.970200	0.9600231
0.99	-0.039800	1.008987	-0.035126	0.99000	0.980098
1	0	0	0	0	0

4. For finite value of the Peclet number Il'in-Allen scheme behaves well with the exact solution in the region $[0,1]$.
5. The standard finite difference scheme of upwind and central scheme on equally spaced mesh does not converge uniformly. Because, the point wise error is not necessarily reduced by successive uniform improvement of the mesh in contrast to solving unperturbed problems. The standard central difference scheme is of order $O(h^2)$.It is numerically unstable in the boundary layer region and gives oscillatory solutions unless the mesh width is small comparatively with the diffusion coefficient but it is practically not possible as diffusion coefficient is very small.
6. For any value of x in $[0,1]$, $a(x)=1$ Il'in- Allen scheme converges uniformly. This has been thoroughly verified through computation.

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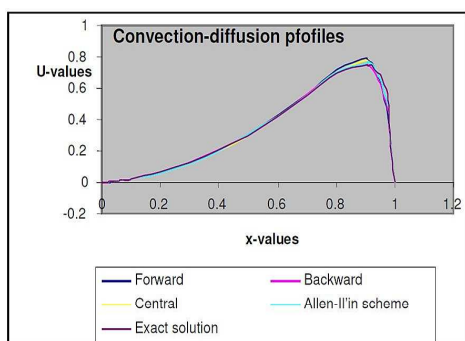


Figure 1

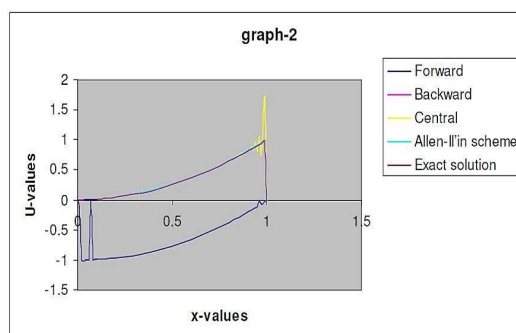


Figure 2

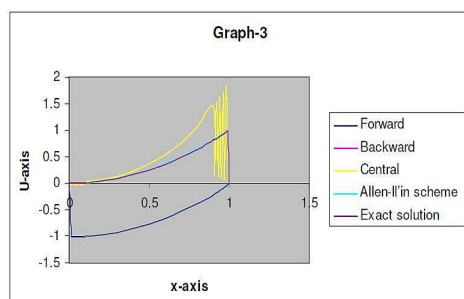


Figure 3

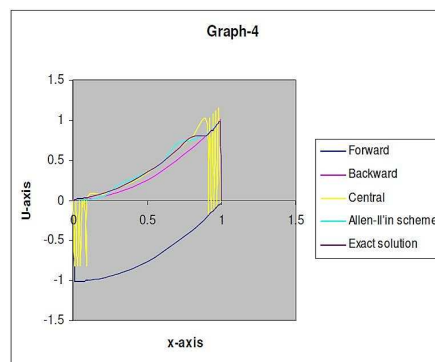


Figure 4

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