

# Quadrature method for cylindrical wire antenna

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## Abstract:

The distributed current in the straight cylindrical antenna can be obtained by solving the Hallen equation with certain unknown constants. In this paper the Hallen equation is reduced to a Cauchy singular integral equation (CSIE). Quadrature method is then applied to the CSIE to obtain a linear system of equations. This approach enables to resolve the unknown constants with the condition that the current vanishes at the ends. This alternative approach is now well posed. A couple of examples are worked out and distributed current is computed.

**Key words:** Quadrature; cylindrical antenna; integral equation

## 1 Introduction

Consider the cylindrical antenna integral equation for a perfectly conducting tube of length  $2h$ , with a radius  $a$  and described by Hallen equation [1]

$$A(z) = \frac{-j\omega\epsilon}{k} \int_0^z E(z') \sin k(z-z') dz' + C_1 \cos kz + C_2 \sin kz \quad |z| \leq h \quad (1.1)$$

where

$$A(z) = \frac{1}{4\pi} \int_{-h}^{+h} K(z, z') I(z') dz' \quad (1.2)$$

We need to solve for the total axial current  $I(z')$  in (1.1). The kernel in (1.2) is

$$K(z, z') = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{e^{-jkP}}{P} d\phi' \quad (1.3)$$

where  $P = P(z - z', \phi') = \sqrt{[(z - z')^2 + 4a^2 \sin^2(\frac{\phi'}{2})]}$ . Here  $k$  is the wave number,  $\omega$  is the angular frequency and  $\epsilon$  characterises the medium. The antenna's length is aligned along the  $z$ -axis and the current flows along its length. Literature has extensively discussed the determination of the distributed current either from (1.1) or from the Pocklington equation (see [2, 3]). In (1.1),  $C_1 = A^1(0)$  and  $C_2 k = A^1(0)$  ( $^1$  denotes the derivative). These constants however depend on the unknown function. Rynne [2] has observed that the Hallen approach becomes well posed and is equivalent to the solution of the Pocklington equation provided the constants have certain specific

values and that the current vanishes at the ends. This suggestion is satisfied here while solving the CSIE using quadrature methods. This is the proposal in the paper.

In Section 2 we first briefly discuss the kernel and obtain equivalent CSIE from (1.1). To make  $I(z)$  vanish at the ends of the interval, an appropriate condition is obtained, which depends on current, the incident field and the kernel. This condition helps to determine the unknown constants appearing in the Hallen equation. Quadrature method is then applied to the CSIE in Section 3. The integral equation is finally replaced by a linear system of equations and the solution is the distributed current. In Section 4 of the paper we suggest separating the incident field  $E(z)$  into odd and even parts. Subsequently the unknown constants in the Hallen equation are either eliminated or determined by using the boundary condition. The computed current is illustrated for convergence against various dimensions of the linear system.

## 2 Cauchy Singular Integral Equation

The kernel in equation (1.2) can be replaced as  $K = K_r = \frac{e^{-jk_r r}}{r}$  here  $r = \sqrt{[(z - z')^2 + a^2]}$  for cylindrical antenna while  $a \ll h$  and  $a \ll \lambda$ . The equation (1.1) then becomes ill-posed [4]. The kernel in (1.3) has a logarithmic singularity as suggested by Schelkunoff [1] and derived by Pearson [5]. This singularity should be incorporated to obtain sensible solution. We extend the approach of Jones [6] to decompose  $K$  as:

$$K = K_1 + K_2 + K_3$$

where

$$K_1 = \frac{1}{\pi} \int_0^\pi (u^2 + a^2 \phi'^2)^{-1/2} d\phi' \quad (2.1)$$

$$K_2 = \frac{1}{\pi} \int_0^\pi [(u^2 + 4a^2 \sin^2(\frac{\phi'}{2}))^{-1/2} - (u^2 + a^2 \phi'^2)^{-1/2}] d\phi' \quad (2.2)$$

and

$$K_3 = \frac{-1}{2\pi} \int_{-\pi}^\pi \frac{1 - e^{jkP}}{P} d\phi' \quad (2.3)$$

The integral in (2.1) has a closed form expression:

$$K_1 = K_s + K_{1b} \quad (2.4)$$

where

$$K_s = \frac{-1}{a\pi} \ln |z - z'| \quad (2.5)$$

$$K_{1b} = \frac{1}{a\pi} \{ \ln a + \ln |\pi + [(\frac{u}{a})^2 + \pi^2]^{1/2}| \} \quad (2.6)$$

The kernel can thus be expressed as a sum of singular and non-singular part:

$$K = K_s + K_B \quad (2.7)$$

Here,

$$K_B = K_{1b} + K_2 + K_3 \quad (2.8)$$

The term  $K_s$  has the logarithmic singularity. In absence of any closed form expression for  $K_2$ , it is replaced by applying trapezoidal rule and the expression in [7] to evaluate  $K_3$ . Alternative approximation for  $K_B$  is given in [4]. Note that the first derivative of the bounded part  $K_B$  can be shown to be bounded and continuous, while its second derivative can be shown to have logarithmic singularity (see [3, 6]).

The equation (1.1) defined over  $[-h, h]$  is normalized to  $[-1, 1]$  in order to facilitate the application of the quadrature formulae. Differentiating the equation (1.1) with respect to  $z$ , and denoting  $\frac{\partial K(z, z')}{\partial z}$  as  $K^1(z, z')$ , we have the CSIE

$$(1/4\pi) \int_{-1}^{+1} K^1(z, z') I(z') dz' = (-j\varepsilon\omega) \int_0^z E(z') \cos k(z - z') dz' - C_1 k \sin kz + C_2 k \cos kz \quad (2.9)$$

The analytic theory of CSIE is discussed in detail in [8] as a Riemann-Hilbert problem. We shall use a more general notation instead of (2.9) for simplicity

$$(-1/\pi a) \int_{-1}^{+1} I(z')/(z - z') dz' + \int_{-1}^{+1} R(z, z') I(z') dz' = F(z) j z j = 1 \quad (2.10)$$

where by using (2.7) and (2.8) we have

$$R = K_B^{-1} \quad (2.11)$$

$$F(z) = (-4\pi j\varepsilon\omega) \int_0^z E(z') \cos k(z - z') dz' - C_1 k \sin kz + C_2 k \cos kz \quad (2.12)$$

The solution of (2.10) has integrable singularity at the ends of the interval.

$$I(z) = (1 - z^2)^{-1/2} c - (a/\pi) \int_{-1}^{+1} \chi(t) (1 - t^2)^{1/2} / (z - t) dt \quad (2.13)$$

where

$$c = (1/\pi) \int_{-1}^{+1} I(z') dz' \quad (2.14)$$

$$\chi(t) = F(t) - \int_{-1}^{+1} R(t, t') I(t') dt' \quad (2.15)$$

In physical problems, 'c' in (2.14) is usually known and hence closed form solution can be derived when  $R(z, z')$  is zero. The numerical solution of CSIE has a prolific literature (see [9]) ever since the work of Erdogan [10] appeared. The current that is confined to the wire is bounded and vanishes in the neighborhood of the ends. This physical condition implies that,

$$I(\pm 1) = 0 \quad (2.16)$$

To enable the current  $I(z')$  satisfy this relation we write [8]:

$$I(z') = (1 - z'^2)^{1/2} \Psi(z'), \quad (2.17)$$

The following condition needs to be satisfied by the solution of (2.13) (with (2.14) and (2.15)) in order to satisfy (2.16). This equation is given below which contains the constants  $C_1$  and  $C_2$  and specifies them.

$$\int_{-1}^{+1} (1 - z^2)^{-1/2} \chi(z) dz = 0 \quad (2.18)$$

The numerical solution of CSIE wherein the solution vanishes at the ends of the interval and in the absence of any unknown constants has been discussed in [11] - [14]. It needs to be mentioned that Tan [15] has also considered the Cauchy conversion (2.3) and has approximated the solution by Chebychev polynomials as in [10] while recently Bruno [16] has used them while solving (1.1).

While differentiating (1.1) and deriving the CSIE (2.9), the derivatives of the kernel in (2.7) and (2.8) are obtained as follows.

$$K^1_s = (-1/\pi a)(1/u) \quad (2.19)$$

$$K^1_{1b} = [\{\pi + ((u/a)^2 + \pi^2)^{-1/2}\}(u/a^2)\{(u/a)^2 + \pi^2\}^{-1/2}](1/\pi a) \quad (2.20)$$

$K_2$  is approximated using trapezoidal rule on the variable  $\varphi'$  over  $[-1,1]$  and then its derivative is obtained. Also, the derivative of  $K_3$  (composed of real and imaginary parts) is first obtained and approximated to a known accuracy by applying McLaurin expansion (see [7]).

As mentioned earlier other alternative approximate form for  $K_B^1$  as derived from [4] can also be used.

### 3 Quadrature and numerical solution

The Cauchy integral in (2.10) is approximated as follows [11] :

$$\begin{aligned} \int_{-1}^{+1} (1 - z'^2)^{1/2} \Psi(z') / (z' - z) dz' &= \int_{-1}^{+1} (1 - z'^2)^{1/2} (\Psi(z') - \Psi(z)) / (z' - z) dz' - z \Psi(z) \\ &\approx (\pi / (n + 1)) \sum_{k=1}^n (1 - t_k^2) \Psi(t_k) / (t_k - z) - \Psi(z) \{-z + T_{n+1}(z) / U_n(z)\} - z \Psi(z) \end{aligned} \quad (3.1)$$

here

$$U_n(t_k) = 0; t_k = \cos(k\pi / (n + 1)); k = 1, 2, \dots, n \quad (3.2)$$

When we select  $z$  such that,

$$z = x_r, T_{n+1}(x_r) = 0; x_r = \cos((2r - 1)\pi / 2(n + 1)); r = 1, 2, \dots, (n + 1) \quad (3.3)$$

We notice the terms trailing the summation in (3.1) vanishes. The Gauss-Chebyshev quadrature is applied to the regular integral in (2.10) for  $z = x_r$  to obtain

$$\int_{-1}^{+1} (1 - z'^2)^{1/2} R(x_r, z') \Psi(z') dz' = \pi / (n + 1) \sum_{k=1}^n (1 - t_k^2) R(x_r, t_k) \Psi(t_k) \quad (3.4)$$

Thus effectively we have reduced (2.10) as

$$(1/a(n + 1)) \sum_{k=1}^n (1 - t_k^2) \Psi(t_k) / (x_r - t_k) + \pi(n + 1) \sum_{k=1}^n (1 - t_k^2) R(x_r, t_k) \Psi(t_k) = F(x_r) \quad (3.5)$$

where  $r = 1, 2, \dots, (n + 1)$ . The above linear system has  $(n + 1)$  equations in  $(n + 2)$  unknowns, namely  $\Psi(t_k)$ ,  $k = 1, 2, \dots, n$ ,  $C_1$  and  $C_2$ . The current  $I(z')$ , at  $z = t_k$  can be obtained using (2.17). Quadrature methods have the advantage of being simpler and eliminate the need of the evaluation of integrals though the collocation points are restricted while arriving at the linear system of equations.

Next we use

$$\int_{-1}^{+1} (1 - z'^2)^{-1/2} F(z') dz' = \pi / (n + 1) \sum_{r=1}^{n+1} F(x_r) \quad (3.6)$$

along with the approximation (3.4) in (2.18) to get

$$\pi / (n + 1) \sum_{r=1}^{n+1} F(x_r) - (\pi / (n + 1)) \sum_{k=1}^n (1 - t_k^2) R(x_r, t_k) \Psi(t_k) = 0 \quad (3.7)$$

Interestingly while summing all the  $(n+1)$  equations in (3.5) and using the formula

$$\sum_{r=1}^{n+1} 1 / (t_k - x_r) = 0 \quad (3.8)$$

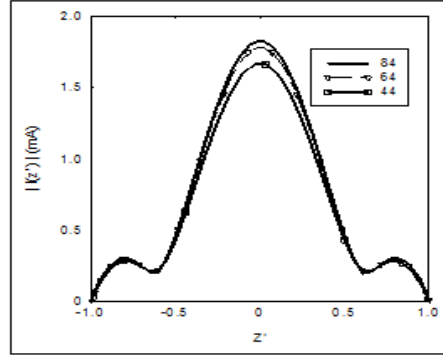


Figure 1: Quadrature Method :  $2h = 1.2 \lambda$  and  $a = 0.001\lambda$

The first summation term in (3.5) vanishes and we arrive at (3.7). Thus, we have an important observation that the solution of (3.5) for  $\Psi(t_k)$ ,  $k = 1, 2, \dots, n$ ,  $C_1$  and  $C_2$  obtained by discretising (2.10) also satisfies the equation (3.7) which is nothing but the discretisation of the boundary condition (2.18). Thus unknown constants in (3.5) satisfy boundary condition in (2.18) and attain specific values as suggested by Rynne in being well posed. Next, as any constant term (from the even component of the right side terms) appearing in (1.1) vanishes on differentiation to (2.9), we propose the following to establish the equivalence of CSIE to (1.1). An additional condition is obtained after multiplying either side of (1.1) by  $(1 - z'^2)^{-1/2}$  and then integrating over  $[-1, 1]$  in order to remove the singularity by using the formula

$$(1/\pi) \int_{-1}^{+1} \ln jz - z'j(1 - z'^2)^{-1/2} dz = -\ln 2, \quad (3.9)$$

We have

$$(1/4\pi) \int_{-1}^{+1} \{-\ln 2/a + \int_{-1}^{+1} K_B(1 - z'^2)^{-1/2} dz\} I(z') dz' = \int_{-1}^{+1} \{g(z) + C_1 \cos kz(1 - z'^2)^{-1/2} dz \quad (3.10)$$

where  $g(z)$  denotes the first right side term in (1.1). This equivalence approach has been suggested in [17] where the logarithmic integral equation has a solution that possesses integrable singularity. Notice that the term containing  $C_2$  vanishes. Then applying the quadrature (3.6) for the integrals in (3.10) we get

$$\sum_{k=1}^n (1/4(n+1)) - \ln 2/a + \pi/(n+1) \sum_{r=1}^{n+1} K_B(x_r, t_k)(1 - t_k^2) \Psi(t_k) = \pi(n+1) \left\{ \sum_{r=1}^{n+1} g(x_r) + C_1 \cos kx_r \right\} \quad (3.11)$$

The linear system in (3.5) along with that in (3.11) has  $(n+2)$  equations to solve for as many unknowns. Alternatively the value of  $C_1$  from (3.11) can be substituted in (3.5) to solve for the remaining  $(n+1)$  unknowns in  $(n+1)$  equations. This shall become clear in the next section.

## 4 Case Study

The external source field in (1.1) could in general be separated as a sum of even and odd parts. The incident field is set to be a plane wave and that a constant say  $C_0$  (an even function). In (2.10), it can be shown that  $R(v) = -R(u)$ , where  $v = -(u)$ . Whenever the input function in (2.10) is odd, the solution  $\Psi(z)$  is then an even function. In (1.1) the constant  $C_2$  is absent and hence can be written as

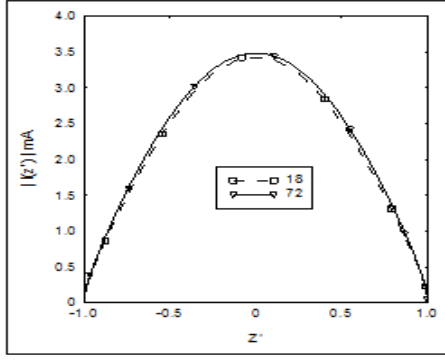


Figure 2: Quadrature Method :  $2h = \lambda/2$  and  $a = \lambda/5000$

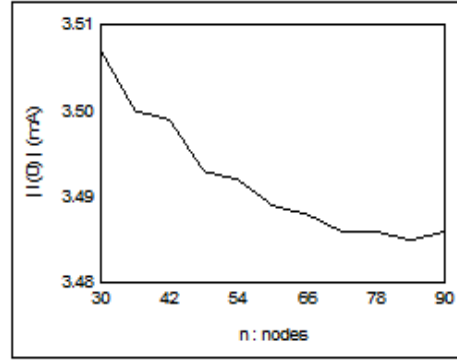


Figure 3: Convergence  $I(0)$

$$A(z) = (C_0 \frac{j\omega\epsilon}{k^2} (1 - \cos kz) + C_1 \cos kz) \quad (4.1)$$

The differentiation leads to a CSIE which is

$$\frac{1}{4\pi} \int_{-1}^{+1} K^1(z, z') I(z') dz' = C_0 \frac{j\omega\epsilon}{k^2} (k \sin kz) - C_1 k \sin kz \quad (4.2)$$

The additional equation in (3.11) is

$$\int_{-1}^{+1} I(z') dz' \{ -\ln 2/a + \int_{-1}^{+1} K_B (1 - z^2)^{-1/2} dz \} = (4\pi C_0 j\omega\epsilon/k^2)(\pi - d) + C_1 d \quad (4.3)$$

where  $d = \int_{-1}^{+1} (\cos kz)(1 - z^2)^{-1/2} dz$ . Equation (4.3) enables to eliminate  $C_1$  in (4.2). The equation (4.2) is reduced to a linear system as in (3.11). It contains  $(n+1)$  equations with only  $n$  unknowns;  $\Psi(t_k), k = 1, 2, \dots, n$ . It is interesting to note that if  $n$  is an even number, then when  $z = 0$  (that is  $x_r = 0$ ) the particular linear equation in (3.11) is trivial as  $\Psi(z)$  is an even function. Ignoring that equation, the remaining  $n$  equations enable to solve uniquely for  $\Psi(t_k)$ , else if  $n$  is an odd number then we end up having an over determined system of equations and adopt the suggestion in [13] to obtain an optimal solution.

We give the outline of the convergence of this solution. After eliminating  $C_1$ , the system in (3.5) is obtained. We find in (4.2) that  $R = K_B^1$  is continuous and that the input function is also continuous. Then the theorem in Elliot [18] (p142) assures that the numerical solution of (3.5) converges to the exact solution at the discrete points  $t_k$ .

We considered an example where the antenna length,  $2h$  is  $1.2\lambda$ . We choose  $\lambda = \frac{2}{1.2}$  to make  $h = 1$ . The radius is  $0.001\lambda$  and the incident plane wave is constant. Behaviour of the absolute value of the current (in milli Amps) is depicted in Fig 1 for various values of nodes,  $n$ . For values of  $n$  higher than 94, there was no change in the second decimal.

Next, when we change  $\lambda = 4$  and set radius equal to  $(\frac{5\lambda}{1000})$ . This is to solve (1.1) as in [19, 20] and is a case of a cylinder whose length is half wave length. The result when  $n$  is 18 or 72 is given in Fig 2 (a) and is normalized to  $\lambda = 1$  and this agrees with that in [19, 20]. The value of  $I(0)$ , obtained by interpolation, is seen in Fig 2 (b) converging for  $n$  greater than 70 to 3.486. This value is 3.485 when  $n$  is 120 or 150.

Before concluding we notice that whenever the source field is an odd function, the additional condition (4.1) is trivial and  $C_1 = 0$ . Equations in (3.5) directly determine the current and constant  $C_2$ . However, the convergence analysis mentioned earlier is not applicable because of the presence of the constant  $C_2$  and is beyond the scope of this paper (see [14]).

## 5 Conclusion

The Cauchy singular integral equation (CSIE) with the pair of unknown constants and an additional equation are deduced from the Hallen equation having a kernel containing logarithmic singularity. The quadrature method based method to solve the CSIE is formulated. This approach allows in resolving the unknown constants besides satisfying the boundary condition while computing the distributed current. The proposed alternative approach is thus well posed [2]. Results are presented for a couple of applications.

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