

Solution of fully fuzzy linear system with arbitrary coefficients

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Abstract:

Different methods are proposed for finding the non negative solution of fully fuzzy linear system (FFLS) i.e. fuzzy linear system with fuzzy coefficients involving fuzzy variables. To the best of our knowledge, there is no method in the literature for finding the non negative solution of FFLS without any restriction on the coefficient matrix. In this paper a new computational method is proposed to solve FFLS with arbitrary coefficients. To illustrate the proposed method, numerical examples are solved.

Key words: Fully fuzzy linear systems (FFLS); fuzzy matrix

1 Introduction

Several problems in various areas such as economics, engineering and physics come down to the solution of a linear system of equations. When the estimation of the system coefficients is imprecise and only some vague knowledge about the actual value of the parameters is available, it may be convenient to represent some or all of them with fuzzy numbers [17]. Fuzzy sets can provide solutions to vast range of problems including fuzzy topological spaces [7], hyperchaotic systems [18] etc.

For the definition of fuzzy numbers we follow Dubois and Prade [11]. Since there is no analytical formula for arithmetic operations on triangular fuzzy numbers, Dubois and Prade [11,12] have extended several useful operators on these type of fuzzy numbers.

A general model for solving a fuzzy linear system whose coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy vector was first proposed by Friedman et al. [13]. Another important kind of fuzzy linear system includes triangular fuzzy numbers in which all parameters are fuzzy and is named FFLS. Recently some other numerical iterative methods, including classic point iterative methods (such as Jacobi, Gauss Seidel, SOR etc.), steepest descent method and conjugate gradient method have been presented for solving fuzzy linear systems [2,3,4,5,6,14,16]. The direct methods based on LU decomposition have been proposed and analyzed by Abbasbandy et al. [1]. However till now, there does not exist any computational method for solving the FFLS

without any restriction on the coefficient matrix. Almost every method in the literature [8,9,10,15] presumes the non negativity of the coefficient matrix \tilde{A} of the FFLS $\tilde{A} \otimes \tilde{x} = \tilde{b}$.

In this paper a new computational method for finding the non negative solution of FFLS $\tilde{A} \otimes \tilde{x} = \tilde{b}$, where \tilde{A} is a fuzzy matrix with no restriction on its elements and \tilde{x} and \tilde{b} are fuzzy vectors with appropriate sizes, is proposed. The method is illustrated by solving numerical examples.

The rest of this paper is organized as follows: In section 2, shortcomings of the existing methods to solve FFLS are described. In section 3 some basic definitions are reviewed. In Section 4 a new method is proposed for solving FFLS and also to illustrate the proposed method, numerical examples are solved. In section 5 conclusions are discussed.

2 Shortcomings of existing methods

In this section the shortcomings in the existing methods [8,9,10,15,16] are pointed out:

1. The existing methods presume the non negativity of the coefficient matrix i.e. $\tilde{A} \geq 0$. This restriction creates difficulty in using the existing methods to solve FFLS occurring in real life situations for which the coefficient matrix may not be entirely non negative.
2. In all the existing methods, it is assumed that the system of equations is consistent and then the methods are developed i.e. consistency of the FFLS cannot be checked using the existing methods.
3. In the existing methods there is no provision for determining if the solution is unique or infinite and the nature of infinite solution.

To overcome the above shortcomings, in section 4, a new computational method is proposed for solving a FFLS.

3 Preliminaries

In this section, some necessary backgrounds and notions of fuzzy set theory are reviewed [15].

Definition 3.1. A fuzzy subset \tilde{A} of R is defined by its membership function

$$\mu_{\tilde{A}} : R \rightarrow [0, 1]$$

Which assigns a real number $\mu_{\tilde{A}}$ in the interval $[0, 1]$ to each element $x \in R$, where the value of $\mu_{\tilde{A}}$ at x shows the grade of membership of x in \tilde{A} .

Definition 3.2. A fuzzy number is a convex normalized fuzzy set of the real line R^1 whose membership function is piecewise continuous.

Definition 3.3. A fuzzy number \tilde{A} is called positive (negative), denoted by $\tilde{A} > 0$ ($\tilde{A} < 0$) if its membership function $\mu_{\tilde{A}}(x) = 0, \forall x \leq 0$ ($\forall x \geq 0$).

Using its mean value and left and right spreads such a fuzzy number is symbolically written as

$$\tilde{A} = (m, \alpha, \beta)$$

Clearly $\tilde{A} = (m, \alpha, \beta)$ is positive if and only if $m - \alpha \geq 0$.

Definition 3.4. A fuzzy set with the following membership function is named a triangular fuzzy number and in this paper we will use these fuzzy numbers.

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \frac{m-x}{\alpha}, & m - \alpha \leq x \leq m, \alpha > 0, \\ 1 - \frac{x-m}{\beta}, & m \leq x \leq m + \beta, \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.5. Two fuzzy numbers $\tilde{M} = (m, \alpha, \beta)$ and $\tilde{N} = (n, \gamma, \delta)$ are said to be equal if and only if $m = n$, $\alpha = \gamma$ and $\beta = \delta$.

Definition 3.6. For two Fuzzy numbers $\tilde{M} = (m, \alpha, \beta)$ and $\tilde{N} = (n, \gamma, \delta)$ the formula for extended addition becomes:

$$(m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta).$$

The formula for extended opposite becomes:

$$-\tilde{M} = - (m, \alpha, \beta) = (-m, \beta, \alpha).$$

Definition 3.7. A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix, if each element of \tilde{A} is a fuzzy number. \tilde{A} will be positive (negative) and denoted by $\tilde{A} > 0$ ($\tilde{A} < 0$) if each element of \tilde{A} be positive (negative). \tilde{A} will be non positive (non negative) and denoted by $\tilde{A} \leq 0$ ($\tilde{A} \geq 0$) if each element of \tilde{A} be non positive (non negative). We may represent $n \times m$ fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})_{n \times m}$ that $(\tilde{a}_{ij}) = (m_{ij}, \alpha_{ij}, \beta_{ij})$

Definition 3.8. Let $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{B} = (\tilde{b}_{ij})$ be two $m \times n$ and $n \times p$ fuzzy matrices. We define $\tilde{A} \otimes \tilde{B} = \tilde{C} = (\tilde{c}_{ij})$ which is the $m \times p$ matrix where

$$\tilde{c}_{ij} = \sum_{k=1, \dots, n}^{\oplus} \tilde{a}_{ik} \otimes \tilde{b}_{kj}$$

4. Proposed method

In this section a new computational method is proposed to find non negative solutions of FFLS $\tilde{A} \otimes \tilde{x} = \tilde{b}$.

Consider the $n \times n$ FFLS:

$$\begin{aligned} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) &= \tilde{b}_1 \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) &= \tilde{b}_2 \\ &\vdots \\ (\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) &= \tilde{b}_n \end{aligned}$$

The matrix form of the above equation is

$$\tilde{A} \otimes \tilde{x} = \tilde{b}$$

where the coefficient matrix $\tilde{A} = (\tilde{a}_{ij})$, $1 \leq i, j \leq n$ is a $n \times n$ fuzzy matrix. Let $\tilde{a}_{ij} = (m_{ij}, \alpha_{ij}, \beta_{ij})$, $\tilde{x} = (x_i, y_i, z_i) \geq 0$, $\tilde{b} = (b_i, g_i, h_i)$ be triangular fuzzy numbers, then the FFLS can be rewritten as

$$\sum_{j=1}^n (m_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_j, y_j, z_j) = (b_i, g_i, h_i) \quad \forall i = 1, 2, \dots, n$$

The extended multiplication operation can be defined as follows:

For $i, j = 1, 2, \dots, n$

$$\begin{aligned} (m_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_j, y_j, z_j) &= (f_{ij}, p_{ij}, q_{ij}) \\ f_{ij} &= m_{ij}x_j \end{aligned}$$

$$\begin{aligned}
p_{ij} &= f_{ij} - r_{ij} \\
q_{ij} &= s_{ij} - f_{ij} \\
r_{ij} &= \min((m_{ij} - \alpha_{ij})(x_j - y_j), (m_{ij} - \alpha_{ij})(x_j + z_j)) \\
s_{ij} &= \max((m_{ij} + \beta_{ij})(x_j - y_j), (m_{ij} + \beta_{ij})(x_j + z_j))
\end{aligned}$$

Hence the given system of equations can be rewritten as:

$$\sum_{j=1}^n (f_{ij}, p_{ij}, q_{ij}) = (b_i, g_i, h_i) \quad \forall i = 1, 2, \dots, n$$

or

$$\begin{aligned}
\sum_{j=1}^n f_{ij} &= b_i \quad \forall i = 1, 2, \dots, n \\
\sum_{j=1}^n p_{ij} &= g_i \Rightarrow \sum_{j=1}^n r_{ij} = \sum_{j=1}^n f_{ij} - g_i = b_i - g_i \quad \forall i = 1, 2, \dots, n \\
\sum_{j=1}^n q_{ij} &= h_i \Rightarrow \sum_{j=1}^n s_{ij} = \sum_{j=1}^n f_{ij} + h_i = b_i + h_i \quad \forall i = 1, 2, \dots, n
\end{aligned}$$

This splitting converts the $n \times n$ FFLS into $3n \times 3n$ linear system of equations that can be easily solved using standard methods of solving equations like Cramer's rule, LU decomposition etc.

Remark 4.1. The $n \times n$ FFLS $\tilde{A} \otimes \tilde{x} = \tilde{b}$ will have a feasible non negative solution $\tilde{x} = (x_i, y_i, z_i) \quad \forall i = 1, 2, \dots, n$ iff $(x_i, y_i, z_i) \geq 0$ i.e. $x_i - y_i \geq 0, y_i \geq 0, z_i \geq 0 \quad \forall i = 1, 2, \dots, n$.

Example 4.1. Let us consider the following FFLS and solve it by the proposed method

$$\begin{aligned}
(4, 6, 1) \otimes (x_1, y_1, z_1) \oplus (4, 2, 4) \otimes (x_2, y_2, z_2) &= (24, 26, 21) \\
(3, 2, 1) \otimes (x_1, y_1, z_1) \oplus (2, 3, 1) \otimes (x_2, y_2, z_2) &= (14, 18, 13) \\
(x_1, y_1, z_1) &\geq 0, (x_2, y_2, z_2) \geq 0
\end{aligned}$$

Solution: The given 2×2 FFLS can be converted into a 6×6 linear system of equation as follows:

$$\begin{aligned}
4x_1 + 4x_2 &= 24 \\
3x_1 + 2x_2 &= 14 \\
-2(x_1 + z_1) + 2(x_2 - y_2) &= 24 - 26 \\
(x_1 - y_1) - (x_2 + z_2) &= 14 - 18 \\
5(x_1 + z_1) + 8(x_2 + z_2) &= 24 + 21 \\
4(x_1 + z_1) + 3(x_2 + z_2) &= 14 + 13
\end{aligned}$$

The matrix form of this system of equations is:

$$\begin{bmatrix}
4 & 0 & 0 & 4 & 0 & 0 \\
3 & 0 & 0 & 2 & 0 & 0 \\
-2 & 0 & -2 & 2 & -2 & 0 \\
1 & -1 & 0 & -1 & 0 & -1 \\
5 & 0 & 5 & 8 & 0 & 8 \\
4 & 0 & 4 & 3 & 0 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
y_1 \\
z_1 \\
x_2 \\
y_2 \\
z_2
\end{bmatrix}
=
\begin{bmatrix}
24 \\
14 \\
-2 \\
-4 \\
55 \\
27
\end{bmatrix}$$

On solving the above linear system of equations, the following solution is obtained

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

Putting the values of $x_1, y_1, z_1, x_2, y_2, z_2$ in $\tilde{x}_1 = (x_1, y_1, z_1)$ and $\tilde{x}_2 = (x_1, y_1, z_1)$ we get,
 $\tilde{x}_1 = (2, 1, 1)$ and $\tilde{x}_2 = (4, 2, 1)$

Both the fuzzy numbers are positive and the solution is unique and feasible.

Example 4.2. Let us consider the following FFLS and solve it by the proposed method

$$(-1, 1, 0) \otimes (x_1, y_1, z_1) \oplus (-2, 0, 1) \otimes (x_2, y_2, z_2) = (-6, 4, 2)$$

$$(-2, 2, 0) \otimes (x_1, y_1, z_1) \oplus (-4, 0, 2) \otimes (x_2, y_2, z_2) = (-12, 8, 4)$$

$$(x_1, y_1, z_1) \geq 0, (x_2, y_2, z_2) \geq 0$$

In matrix form

$$\begin{bmatrix} (-1, 1, 0) & (-2, 0, 1) \\ (-2, 2, 0) & (-4, 0, 2) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \end{bmatrix} = \begin{bmatrix} (-6, 4, 2) \\ (-12, 8, 4) \end{bmatrix}$$

The given 2×2 FFLS can be converted into a 6×6 linear system of equation as follows:

$$-1x_1 - 2x_2 = -6$$

$$-2x_1 - 4x_2 = -12$$

$$-2(x_1 + z_1) - 2(x_2 + z_2) = -6 - 4$$

$$-4(x_1 + z_1) - 4(x_2 + z_2) = -12 - 8$$

$$-1(x_1 - y_1) - 1(x_2 - y_2) = -6 + 2$$

$$-2(x_1 - y_1) - 2(x_2 - y_2) = -12 + 4$$

The matrix form of this system of equations is:

$$\begin{bmatrix} -1 & 0 & 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & -4 & 0 & 0 \\ -2 & 0 & -2 & -2 & 0 & -2 \\ -4 & 0 & -4 & -4 & 0 & -4 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ -2 & 2 & 0 & -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -12 \\ -10 \\ -20 \\ -4 \\ -8 \end{bmatrix}$$

The linear system of equation has infinite solutions. One of the possible solution is:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{x}_1 = (2, 0, 1) \text{ and } \tilde{x}_2 = (2, 0, 0).$$

Both the fuzzy numbers are positive. Hence the solution is feasible.

4 Conclusion

In this paper, a new computational method for finding the non negative solutions of FFLS with arbitrary coefficient matrix, is presented. The proposed method is easy to understand and apply in real life situations. The method is illustrated with the help of numerical examples.

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