

# New approach for numerical solution of Fokker-Planck equations

**M. Zarebnia and S. Jalili**

*Department of Mathematics, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran*

## Abstract:

In this paper numerical solution of Fokker-Planck equations by means of the Chebyshev spectral collocation method is considered. Firstly, properties of the Chebyshev spectral collocation method required for our subsequent development are given and utilized to reduce the computation of different kinds of Fokker-Planck equations to some system of ordinary differential equations. Secondly, we use fourth-order Runge-Kutta formula for the numerical solution of the system of ordinary differential equations. The method is applied to a few test examples to illustrate the accuracy and the implementation of the method.

**Keywords:** Chebyshev polynomials; Spectral collocation method; Fokker-Planck equation, Kolmogorov equation.

## 1 Introduction

Fokker-Planck equation (FPE) arises in a number of different fields in natural science, including solid-state physics, quantum optics, chemical physics, theoretical biology and circuit theory. The Fokker-Planck equation was first used by Fokker and Planck [1] to describe the Brownian motion of particles. If a small particle of mass  $m$  is immersed in a fluid, the equation of motion for the distribution function  $W(x, t)$  is given by:

$$\frac{\partial W}{\partial t} = \gamma \frac{\partial v W}{\partial v} + \gamma \frac{KT}{m} \frac{\partial^2 W}{\partial^2 v}, \quad (1.1)$$

where  $v$  is the velocity for the Brownian motion of a small particle,  $t$  is the time,  $\gamma$  is the friction constant,  $K$  is Boltzmann's constant and  $T$  is the temperature of fluid [1]. Eq. (1) is one of the simplest type of Fokker-Planck equations. By solving (1) starting with distribution function  $W(x, t)$  for  $t = 0$  and subject to the appropriate boundary conditions, one can obtain the distribution function  $W(x, t)$  for  $t > 0$ .

The general Fokker-Planck equation for the variable  $x$  has the form [1]:

$$\frac{\partial u}{\partial t} = \left[ -\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] u, \quad (1.2)$$

with initial condition

$$u(x, 0) = f(x), \quad x \in D, \quad (1.3)$$

and boundary conditions

$$u(x, t) = g(t), \quad (x, t) \in \partial D \times [0, T], \quad (1.4)$$

where  $D = \{x : a < x < b\}$  and  $\partial D$  is its boundary and  $u(x, t)$  is unknown. In (2)  $B(x) > 0$  is called the diffusion coefficient and  $A(x)$  is the drift coefficient. The drift and diffusion coefficients may also depend on time, i.e.

$$\frac{\partial u}{\partial t} = \left[ -\frac{\partial}{\partial x} A(x, t) + \frac{\partial^2}{\partial x^2} B(x, t) \right] u. \quad (1.5)$$

Eq. (1) is seen to be a special case of the Fokker-Planck equation where the drift coefficient is linear and the diffusion coefficient is constant. Eq. (2) is an equation of motion for the distribution function  $u(x, t)$ . Mathematically, this equation is a linear second order partial differential equation of parabolic type. Roughly speaking, it is a diffusion equation with an additional first order derivative with respect to  $x$ . In the mathematical literatures, (2) is also called forward Kolmogorov equation. The similar partial differential equation is a backward Kolmogorov equation that is in the form [1]:

$$\frac{\partial u}{\partial t} = - \left[ A(x, t) \frac{\partial}{\partial x} + B(x, t) \frac{\partial^2}{\partial x^2} \right] u. \quad (1.6)$$

There is a more general form of FPE which is nonlinear FPE. Nonlinear FPE has important applications in various areas such as plasma physics, surface physics, population dynamic, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology and marketing [2]. The nonlinear FPE for one variable is in the following form.

$$\frac{\partial u}{\partial t} = \left[ -\frac{\partial}{\partial x} A(x, t, u) + \frac{\partial^2}{\partial x^2} B(x, t, u) \right] u. \quad (1.7)$$

A great deal of interest has been focused on Fokker-Planck equations, these are well addressed in [1-8]. The variational iteration method is implemented to solve linear and nonlinear Fokker-Planck equations, and some similar equations in [3]. Also, Alawneh et al. [4] by means of the variational iteration method, numerical solutions are computed for some stochastic models, without any linearization or weak assumptions. Two stochastic models, the Fokker-Planck equation for non-equilibrium statistical systems and the Black-Scholes model for pricing stock options, are solved numerically. The homotopy perturbation method (HPM) is applied for the space- and time-fractional Fokker-Planck equation in [5]. He in [6] is considered the regularities of the solutions to the Fokker-Planck-Boltzmann equation. Masud et al. [7] used an application of multi-scale finite element methods to the solution of the multi-dimensional Fokker-Planck equation. Finally, a semianalytic partition of unity finite element method (PUFEM) is presented to solve the transient Fokker-Planck equation (FPE) by Kumar et al. in [8]. In this paper a Chebyshev spectral collocation method is developed for the numerical solution of Equations (2) and (5)-(7) with initial and boundary conditions (3) and (4).

The layout of the paper is as follows. First, in Section 2 we review some of the main properties of chebyshev polynomials that are necessary for the formulation of the discrete system. In Section 3, we illustrate how the Chebyshev spectral collocation method may be used to replace equations (2)-(7) by explicit system of ordinary differential equations, which is solved by fourth-order RungeKutta method. In Section 4, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples.

## 2 Preliminaries

The goal of this section is to recall notations and definition of the Chebyshev polynomials, state some known results, and derive useful formulas that are important for this paper. These are

discussed thoroughly in [9].

The well known Chebyshev polynomial  $T_n(x)$  of the first kind is a polynomial in  $x$  of degree  $n$ , defined by the relation

$$T_n(x) = \cos n\theta \quad \text{when } x = \cos \theta. \quad (2.1)$$

If the range of the variable  $x$  is the interval  $[-1, 1]$ , then the range of the corresponding variable  $\theta$  can be taken as  $[0, \theta]$ . These ranges are traversed in opposite directions, since  $x = -1$  corresponds to  $\theta = \pi$  and  $x = 1$  corresponds to  $\theta = 0$ .

It is well known that  $\cos n\theta$  is a polynomial of degree  $n$  in  $\cos \theta$ , and indeed we are familiar with the elementary formulae

$$\begin{aligned} \cos 0\theta &= 1, & \cos 1\theta &= \cos \theta, & \cos 2\theta &= 2\cos^2 \theta - 1, \\ \cos 3\theta &= 4\cos^3 \theta - 3\cos \theta, & \cos 4\theta &= 8\cos^4 \theta - 8\cos^2 \theta + 1, & \dots \end{aligned}$$

We may immediately deduce from (7), that the first few Chebyshev polynomials are

$$\begin{aligned} T_0(x) &= 1, & T_1(x) &= x, & T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, & T_4(x) &= 8x^4 - 8x^2 + 1, & \dots \end{aligned}$$

In practice it is neither convenient nor efficient to work out each  $T_n(x)$  from first principles. Rather by combining the trigonometric identity

$$\cos n\theta + \cos(n-2)\theta = 2\cos \theta \cos(n-1)\theta,$$

with Definition (7), we obtain the fundamental recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \dots, \quad (2.2)$$

which together with the initial conditions

$$T_0(x) = 1, \quad T_1(x) = x, \quad (2.3)$$

recursively generates all the polynomials  $\{T_n(x)\}$  very efficiently.

Clenshaw and Curtis [10] introduced the following approximation of the function  $u(x, t)$ :

$$u(x, t) = \sum_{j=0}^N {}'' a_j T_j^*(x), \quad (2.4)$$

where  $T_j^*(x) = T_j((2x - (b+a))/(b-a))$  denotes the  $j$ th shifted Chebyshev polynomial of the first kind. Note the double prime indicating that the first and last terms of the sum are to be halved.

we can use the discrete orthogonality relation

$$\sum_{n=0}^N {}'' T_i^*(x_n) T_j^*(x_n) = \alpha_{i,j}, \quad (2.5)$$

where

$$\alpha_{i,j} = \begin{cases} 0, & i \neq j \ (\leq N), \\ \frac{1}{2}N, & 0 < i = j < N, \\ N, & i = j = 0, N, \end{cases} \quad (2.6)$$

and also, the collocation points  $x_n$  are given by

$$x_n = \frac{1}{2} \left( (a+b) - (b-a) \cos \left( \frac{\pi n}{N} \right) \right), \quad n = 0, 1, \dots, N. \quad (2.7)$$

We can invert the interpolating polynomial defined as (13) and find

$$a_j = \frac{2}{N} \sum_{n=0}^N {}''T_j^*(x_n) u(x_n, t). \quad (2.8)$$

The relation between the Chebyshev functions and the first derivative is given by [20]:

$$T_j^{*'}(x) = 2j\lambda \sum_{n=0, n+j \text{ odd}}^{j-1} c_n T_n^*(x), \quad (2.9)$$

where  $\lambda = \frac{2}{b-a}$  and

$$c_n = \begin{cases} 1, & 1 \leq n \leq N-1, \\ \frac{1}{2}, & n = 0, N. \end{cases} \quad (2.10)$$

### 3 The Chebyshev spectral collocation method

In this section, we use the spectral collocation method for Fokker-Planck equations of the form (2)-(7) with initial and boundary conditions (3) and (4) by using the Chebyshev polynomials.

#### 3.1 Forward Kolmogorov equation

Let us consider the Fokker-Planck equation (forward Kolmogorov equation)

$$\frac{\partial u}{\partial t} = \left[ -\frac{\partial}{\partial x} A(x, t) + \frac{\partial^2}{\partial x^2} B(x, t) \right] u(x, t), \quad (x, t) \in D \times [0, T], \quad (3.1)$$

with the initial condition

$$u(x, 0) = f(x), \quad x \in D, \quad (3.2)$$

and boundary conditions

$$u(x, t) = g(t), \quad (x, t) \in \partial D \times [0, T], \quad (3.3)$$

where  $D = \{x : a < x < b\}$  and  $\partial D$  is its boundary. For convenience, we rewrite the Eq. (18) as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} = & \left( -\frac{\partial A(x, t)}{\partial x} + \frac{\partial^2 B(x, t)}{\partial x^2} \right) u(x, t) + \left( -A(x, t) + 2\frac{\partial B(x, t)}{\partial x} \right) \frac{\partial}{\partial x} u(x, t) \\ & + B(x, t) \frac{\partial^2}{\partial x^2} u(x, t) \end{aligned} \quad (3.4)$$

We assume  $u(x, t)$  defined over the  $D \times [0, T]$  be the exact solution of the problem (18)-(20) that is approximated as follows:

$$u(x, t) = \sum_{j=0}^N {}''a_j T_j^*(x). \quad (3.5)$$

By considering the Equation (21) and the Chebyshev coefficients  $a_j$  that is defined by (15), we can obtain the first derivative of  $u(x, t)$  at the collocation points (14) as follows:

$$\begin{aligned}
 \frac{d}{dx}u(x_i, t) &= u_x(x_i, t) = \sum_{j=0}^N {}''a_j T_j^{*'}(x_i) \\
 &= \sum_{j=0}^N {}''\left(\frac{2}{N} \sum_{n=0}^N {}''T_j^*(x_n)u(x_n, t)\right) T_j^{*'}(x_i) \\
 &= \sum_{n=0}^N {}''\left(\frac{2}{N} \sum_{j=0}^N {}''T_j^{*'}(x_i) T_j^*(x_n)\right) u(x_n, t) \\
 &= \sum_{n=0}^N D_{i,n}^x u(x_n, t),
 \end{aligned} \tag{3.6}$$

where

$$D_{i,n}^x = \frac{2c_n}{N} \sum_{j=0}^N {}''T_j^{*'}(x_i) T_j^*(x_n), \quad i, n = 0, 1, \dots, N-1, N, \tag{3.7}$$

and also,  $T_j^{*'}(x_i)$  and  $c_n$  are defined by (16) and (17) respectively.

Having used the boundary conditions (20), we rewrite the Eq. (23) as follows:

$$u_x(x_i, t) = \sum_{n=0}^N D_{i,n}^x u(x_n, t) = D_{i,0}^x u(x_0, t) + \sum_{n=1}^{N-1} D_{i,n}^x u(x_n, t) + D_{i,N}^x u(x_N, t), \tag{3.8}$$

For the sake of simplicity, consider:

$$F_i(t) = D_{i,0}^x u(x_0, t) + D_{i,N}^x u(x_N, t),$$

thus we can write:

$$u_x(x_i, t) = F_i(t) + \sum_{n=1}^{N-1} D_{i,n}^x u(x_n, t), \tag{3.9}$$

Now for the second derivative of  $u(x, t)$  by similarly manner and using Equation (23), we obtain:

$$\begin{aligned}
 \frac{d^2}{dx^2}u(x_i, t) &= u_{xx}(x_i, t) = \sum_{n=0}^N D_{i,n}^x \left(\frac{d}{dx}u(x_n, t)\right) \\
 &= \sum_{n=0}^N D_{i,n}^x \left(\sum_{j=0}^N D_{n,j}^x u(x_j, t)\right) \\
 &= \sum_{j=0}^N \left(\sum_{n=0}^N D_{i,n}^x D_{n,j}^x\right) u(x_j, t).
 \end{aligned} \tag{3.10}$$

By assumption

$$D_{i,j}^{xx} = \sum_{n=0}^N D_{i,n}^x D_{n,j}^x, \quad i, j = 0, 1, \dots, N, \tag{3.11}$$

we have:

$$u_{xx}(x_i, t) = \sum_{j=0}^N D_{i,j}^{xx} u(x_j, t). \quad (3.12)$$

By using the boundary conditions (20), we obtain:

$$u_{xx}(x_i, t) = \sum_{j=0}^N D_{i,j}^{xx} u(x_j, t) = D_{i,0}^{xx} u(x_0, t) + \sum_{n=1}^{N-1} D_{i,n}^{xx} u(x_n, t) + D_{i,N}^{xx} u(x_N, t). \quad (3.13)$$

We consider the notation  $F_i^*(t)$  as follows:

$$F_i^*(t) = D_{i,0}^{xx} u(x_0, t) + D_{i,N}^{xx} u(x_N, t), \quad (3.14)$$

then we can write:

$$u_{xx}(x_i, t) = F_i^*(t) + \sum_{n=1}^{N-1} D_{i,n}^{xx} u(x_n, t). \quad (3.15)$$

Having replaced the  $\frac{\partial}{\partial x} u(x, t)$  and  $\frac{\partial^2}{\partial x^2} u(x, t)$  on the right-hand side of (21) with the Eqs. (26) and (32) respectively and also setting collocation points  $x = x_i$ ,  $i = 0, 1, \dots, N$  that are defined by (14) we get the collocation result as

$$\begin{aligned} u_t(x_i, t) = & \left( -A_x(x_i, t) + B_{xx}(x_i, t) \right) u(x_i, t) + \left( -A(x_i, t) + 2B_x(x_i, t) \right) u_x(x_i, t) \\ & + B(x_i, t) u_{xx}(x_i, t) \end{aligned} \quad (3.16)$$

$$u(x_i, 0) = f(x_i).$$

We denote

$$\begin{aligned} G_i(t, u(t)) = & \left( -A_x(x_i, t) + B_{xx}(x_i, t) \right) u(x_i, t) + \left( \left( -A(x_i, t) + 2B_x(x_i, t) \right) F_i(t) + B(x_i, t) F_i^*(t) \right) \\ & + \left( \left( -A(x_i, t) + 2B_x(x_i, t) \right) \sum_{n=1}^{N-1} D_{i,n}^x u(x_n, t) + B(x_i, t) \sum_{n=1}^{N-1} D_{i,n}^{xx} u(x_n, t) \right), \\ u(t) = & [u(x_1, t), u(x_2, t), \dots, u(x_{N-1}, t)]^T, \\ u_0 = & [u(x_1, 0), u(x_2, 0), \dots, u(x_{N-1}, 0)]^T, \\ du(t) = & [u_t(x_1, t), u_t(x_2, t), \dots, u_t(x_{N-1}, t)]^T, \end{aligned}$$

then the system of (33) can be given in the matrix form as:

$$\begin{aligned} du(t) &= G(t, u(t)), \\ u_0 &= P, \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} G(t, u(t)) &= [G_1(t, u(t)), G_2(t, u(t)), \dots, G_{N-1}(t, u(t))]^T, \\ P &= [f(x_1), f(x_2), \dots, f(x_{N-1})]^T. \end{aligned}$$

The above system is a system of ordinary differential equations. Solving this system by the fourth-order Runge-Kutta method, we can obtain an approximation to the solution of (18). The fourth-order Runge-Kutta method that is one of the well-known numerical methods for differential equations, can be presented as:

$$\begin{aligned}
 u1 &= h_t G(t_n, u(t_n)), \\
 u2 &= h_t G(t_n + h_t, u(t_n + \frac{u1}{2})), \\
 u3 &= h_t G(t_n + h_t, u(t_n + \frac{u2}{2})), \\
 u4 &= h_t G(t_n + h_t, u(t_n + u3)), \\
 u(t_{n+1}) &= u(t_n) + \frac{1}{6} (u1 + 2u2 + 2u3 + u4).
 \end{aligned} \tag{3.18}$$

### 3.2 Backward Kolmogorov equation

In this subsection, we consider the Fokker-Planck equation(backward Kolmogorov equation)

$$\frac{\partial u}{\partial t} = - \left[ A(x, t) \frac{\partial}{\partial x} + B(x, t) \frac{\partial^2}{\partial x^2} \right] u(x, t), \tag{3.19}$$

with the initial and boundary conditions (19) and (20). By considering approximate solution  $u(x, t)$  as in (22) and then setting  $x = x_i$  we get:

$$u_t(x_i, t) = - \left( A(x_i, t) u_x(x_i, t) + B(x_i, t) u_{xx}(x_i, t) \right). \tag{3.20}$$

Having replaced the  $u_x(x_i, t)$  and  $u_{xx}(x_i, t)$  on the right-hand sides of (37) with the Eqs. (26) and (32), we get:

$$u_t(x_i, t) = - \left[ A(x_i, t) \left( F_i(t) + \sum_{n=1}^{N-1} D_{i,n}^x u(x_n, t) \right) + B(x_i, t) \left( F_i^*(t) + \sum_{n=1}^{N-1} D_{i,n}^{xx} u(x_n, t) \right) \right] \tag{3.21}$$

$$u(x_i, 0) = f(x_i).$$

Now by assumption

$$G_i(t, u(t)) = - \left( A(x_i, t) F_i(t) + B(x_i, t) (F_i^*(t)) \right) - \left( A(x_i, t) \sum_{n=1}^{N-1} D_{i,n}^x u(x_n, t) + B(x_i, t) \sum_{n=1}^{N-1} D_{i,n}^{xx} u(x_n, t) \right),$$

and also

$$\begin{aligned}
 u(t) &= [u(x_1, t), u(x_2, t), \dots, u(x_{N-1}, t)]^T, \\
 u_0 &= [u(x_1, 0), u(x_2, 0), \dots, u(x_{N-1}, 0)]^T, \\
 du(t) &= [u_t(x_1, t), u_t(x_2, t), \dots, u_t(x_{N-1}, t)]^T,
 \end{aligned} \tag{3.22}$$

we may rewrite the system (38) in the form

$$\begin{aligned}
 du(t) &= G(t, u(t)), \\
 u_0 &= P,
 \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} G(t, u(t)) &= [G_1(t, u(t)), G_2(t, u(t)), \dots, G_{N-1}(t, u(t))]^T, \\ P &= [f(x_1), f(x_2), \dots, f(x_{N-1})]^T. \end{aligned} \quad (3.24)$$

Solving system of ordinary differential equations (40) by Runge-Kutta method (35), we can obtain an approximation to the solution of (36).

### 3.3 Nonlinear Fokker-Planck equation

Finally in this subsection, we illustrate how the spectral collocation method based on the chebyshev polynomial may be used to find approximate solution for the nonlinear Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \left[ -\frac{\partial}{\partial x} A(x, t, u) + \frac{\partial^2}{\partial x^2} B(x, t, u) \right] u(x, t), \quad (x, t) \in D \times [0, T], \quad (3.25)$$

with initial condition

$$u(x, 0) = f(x), \quad x \in D, \quad (3.26)$$

and boundary conditions

$$u(x, t) = g(t), \quad (x, t) \in \partial D \times [0, T], \quad (3.27)$$

where  $D = \{x : a < x < b\}$  and  $\partial D$  is its boundary.

We assume the solution

$$u(x, t) = \sum_{j=0}^N a_j T_j^*(x), \quad (3.28)$$

be the approximate solution of the problem (42). By considering righ-hand side of Eq.(42), We know that

$$-\frac{\partial}{\partial x} [A(x, t, u)u(x, t)], \quad \frac{\partial^2}{\partial x^2} [B(x, t, u)u(x, t)], \quad (3.29)$$

consist of  $u_x(x, t)$  and  $u_{xx}(x, t)$ . Similarly, by replecing  $u_x(x, t)$  and  $u_{xx}(x, t)$  as in (26) and (32) in the equation (42), applying the Chebyshev spectral collocation method, and setting the collocation points

$$x_i = \frac{1}{2} \left( (a+b) - (b-a) \cos \left( \frac{\pi n}{N} \right) \right), \quad i = 0, 1, \dots, N,$$

we get the collocation result in the general form as follows:

$$\begin{aligned} \frac{\partial u(x_i, t)}{\partial t} &= -\frac{\partial}{\partial x} [A(x, t, u(x, t))u(x, t)] \Big|_{x=x_i} \\ &\quad + \frac{\partial^2}{\partial x^2} [B(x, t, u)u(x, t)] \Big|_{x=x_i}, \quad (x, t) \in D \times [0, T], \\ u(x_i, 0) &= f(x_i). \end{aligned} \quad (3.30)$$



By using the notations (39) and (41), and also,

$$\begin{aligned} G_i(t, u(t)) &= -\frac{\partial}{\partial x} [A(x, t, u(x, t))u(x, t)] \Big|_{x=x_i} + \frac{\partial^2}{\partial x^2} [B(x, t, u)u(x, t)] \Big|_{x=x_i}, \\ i &= 1, \dots, N-1, \\ u(x_i, 0) &= f(x_i), \end{aligned}$$

we then rewrite the system (47) in the following form which is the system of ordinary differential equations.

$$\begin{aligned} du(t) &= G(t, u(t)), \\ u_0 &= P. \end{aligned} \tag{3.31}$$

Solving the system (48) by Runge-Kutta method (35), we can obtain an approximation to the solution of (42).

## 4 Numerical examples

In order to illustrate the performance of the chebyshev spectral collocation method in solving the problems (2)-(7) and efficiency of the presented method, the following examples are considered. We assume  $u_i$  and  $u_i^*$  be exact and approximate solutions and use the maximum of absolute and relative errors, defined as

$$\|E\|_\infty = \max_{0 < i < N} |u_i - u_i^*|, \tag{4.1}$$

and

$$\|E\| = \sqrt{\frac{\sum_{i=1}^{N-1} (u_i^* - u_i)^2}{\sum_{i=1}^{N-1} (u_i)^2}}. \tag{4.2}$$

The numerical results are tabulated in Tables 1-4.

**Example 1.** Consider the Fokker-Planck equation [3]

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left[ -\frac{\partial}{\partial x} A(x, t) + \frac{\partial^2}{\partial x^2} B(x, t) \right] u(x, t), \\ A(x, t) &= e^t \left( \coth(x) \cosh(x) + \sinh(x) \right) - \coth(x) \\ B(x, t) &= e^t \cosh(x), \end{aligned} \tag{4.3}$$

subject to initial condition

$$u(x, 0) = \sinh x, \quad 0 \leq x \leq 1, \tag{4.4}$$

and boundary conditions

$$u(0, t) = 0, \quad u(1, t) = e^t \sinh 1, \quad t > 0. \tag{4.5}$$

which has the exact solution given by

$$u(x, t) = e^t \sinh x. \tag{4.6}$$

We solve (51) for different values of  $t$ ,  $h_t = 10^{-4}$ . The maximum of absolute and relative errors are tabulated in Table 1 for  $N = 4, 6, 8, 10$ .

Table 1: Results for Example 1.

$N$	$\ E\ _\infty$			$\ E\ $		
	$t = 0.05$	$t = 0.1$	$t = 0.2$	$t = 0.05$	$t = 0.1$	$t = 0.2$
4	1.61818E-5	1.81697E-5	2.02421E-5	2.87354E-6	4.19616E-6	5.46189E-6
6	2.05446E-8	1.63740E-8	2.35967E-8	1.74520E-9	1.63486E-9	1.24567E-9
8	1.21707E-11	1.26541E-11	1.34075E-11	3.58668E-13	1.29937E-13	5.01469E-13
10	8.99858E-12	1.05249E-11	1.44167E-11	1.46569E-12	1.62906E-12	2.01518E-12

Table 2: Results for Example 2.

$N$	$\ E\ _\infty$			$\ E\ $		
	$t = 0.05$	$t = 0.1$	$t = 0.2$	$t = 0.05$	$t = 0.1$	$t = 0.2$
5	4.44089E-15	1.77636E-14	3.197444E-14	8.79340E-16	3.91766E-15	6.57221E-15
8	8.17124E-13	5.0518E-12	2.737190E-11	2.21091E-14	1.33760E-13	7.34940E-13
10	8.57856E-11	6.33937E-9	4.829490E-7	1.59534E-12	1.23751E-10	9.96470E-9

**Example 2.** Consider the backward Kolmogorov equation [2]

$$\frac{\partial u}{\partial t} = - \left[ A(x, t) \frac{\partial}{\partial x} + B(x, t) \frac{\partial^2}{\partial^2 x} \right] u(x, t), \quad (4.7)$$

$$\begin{aligned} u(x, 0) &= (x + 1), \\ u(-10, t) &= -9e^t, \quad u(10, t) = 11e^t, \end{aligned}$$

where  $A(x, t) = -x - 1$  and  $B(x, t) = x^2 e^t$ . The exact solution to Eq. (55) is given by

$$u(x, t) = e^t(x + 1). \quad (4.8)$$

We solve the example 2 for different values of  $t$ ,  $h_t = 10^{-4}$ . The maximum of absolute and relative errors are tabulated in Table 2 for  $N = 5, 8, 10$ .

**Example 3.** For the sake of comparison, we consider the following forward Kolmogorov equation discussed by Odibat et al [11]. The authors used the variational iteration (VIM) and Adomian decomposition methods (ADD) to obtain their numerical solution.

$$\frac{\partial u}{\partial t} = \left[ -\frac{\partial}{\partial x} A(x, t) + \frac{\partial^2}{\partial^2 x} B(x, t) \right] u(x, t), \quad (4.9)$$

where  $A(x, t) = \frac{x}{6}$  and  $B(x, t) = \frac{x^2}{12}$ . The exact solution to Eq. (57) is given by

$$u(x, t) = x^2 e^{\frac{t}{2}}. \quad (4.10)$$

The initial and boundary conditions are taken from the exact solutions.

We compare the results with the ADM and VIM [11] applied to same equation. For this purpose, we consider the same parameter values for the Fokker-Planck equation (52) as considered in [11], namely;  $t = 0.2, 0.4, 0.6$ . Table 3 exhibits the compared results ( $h_t = 10^{-3}$ ).

**Example 4.** Consider the flowing nonlinear Fokker-Plank equation[5]

$$\frac{\partial u}{\partial t} = \left[ -\frac{\partial}{\partial x} A(x, t, u) + \frac{\partial^2}{\partial^2 x} B(x, t, u) \right] u(x, t), \quad (4.11)$$

Table 3: Table 3: Results for Example 3.

$t$	$x$	Present method	ADM	VIM	Exact
0.2	0.25	0.069073	0.069062	0.069062	0.069073
	0.5	0.276290	0.276259	0.276250	0.276293
	0.75	0.621659	0.621563	0.621563	0.621659
	1	1.105170	1.105000	1.105000	1.105171
0.4	0.25	0.076338	0.076250	0.076250	0.076338
	0.5	0.305351	0.305000	0.305000	0.305351
	0.75	0.687039	0.686250	0.686250	0.687039
	1	1.221400	1.220000	1.220000	1.221403
0.6	0.25	0.084366	0.084062	0.084063	0.084366
	0.5	0.337465	0.336250	0.336250	0.337465
	0.75	0.759296	0.631709	0.756562	0.759296
	1	1.349860	1.345000	1.345000	1.349859

Table 4: Table 4: Results for Example 4.

$N$	$h_t$	$\ E\ _\infty$			$\ E\ $		
		$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 0.1$	$t = 0.15$	$t = 0.2$
5	0.0001	4.24105E-14	4.98490E-14	5.73985E-14	2.45812E-14	2.78877E-14	2.96201E-14
10	0.0001	1.46505E-11	1.71874E-11	2.01721E-11	3.47143E-12	3.87402E-12	4.32464E-12
10	0.0005	2.37042E-8	3.11860E-8	4.25258E-8	5.57912E-9	6.97858E-9	9.04676E-9
15	0.0001	1.04837E-9	1.40039E-9	1.95562E-9	1.55443E-10	1.97393E-10	2.62032E-10

$$u(x, 0) = x^2,$$

$$u(0, t) = 0, \quad u(1, t) = e^t,$$

where  $A(x, t, u) = \frac{4u(x, t)}{x} - \frac{x}{3}$  and  $B(x, t, u) = u(x, t)$ . The exact solution to Eq. (59) is given by

$$u(x, t) = x^2 e^t. \quad (4.12)$$

We solved example 4 for different values of  $N$  and  $h_t$ . The maximum of absolute and relative errors on the Chebyshev collocation points are tabulated in Table 4 for  $t = 0.1, 0.15, 0.2$ .

## 5 Conclusion

The Chebyshev spectral collocation method is used to solve the Fokker-Planck equations with initial and boundary conditions. From the numerical results and Tables 1-4, we can say that errors are very small and they are very better than the results of another methods cited in this article.

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