International Journal of Applied Mathematics and Computation
Volume 3(2),pp 131–138, 2011
http://ijamc.psit.in

Viscous potential flow analysis of Kelvin-Helmholtz instability of cylindrical interface

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Abstract:

A linear analysis of Kelvin-Helmholtz instability of cylindrical interface is carried out using viscous potential flow theory. In the inviscid potential flow theory, the viscous term in Navier-Stokes equation vanishes as viscosity is zero. In viscous potential flow, the viscous term in Navier-Stokes equation vanishes as vorticity is zero but viscosity is not zero. Viscosity enters through normal stress balance in viscous potential flow theory and tangential stresses are not considered. Both asymmetric and axisymmetric disturbances are considered. A dispersion relation has been obtained and stability criterion is given in the terms of the critical value of relative velocity. A comparison between inviscid potential flow and viscous potential flow has been made. It has been observed that Reynolds number and inner fluid fraction both have destabilizing effect on the stability of the system.

Keywords: Fluid-fluid interfaces; hydrodynamic stability; viscous potential flow; interfacial flows; incompressible fluids.

1 Introduction

When two fluids of different physical properties are superposed one over other and are moving with a relative horizontal velocity, the instability occurs at the plane interface. It is called Kelvin-Helmholtz instability [1, 2]. Kelvin-Helmholtz instability occurs in various situations like mixing of clouds, meteor is entering on the earth's atmosphere etc.

Cylindrical geometry is very important while studying stability problems related to liquid jets and cooling of fuel rods by liquid coolants in the nuclear reactor. Nayak and Chakrborty [3] considered the Kelvin-Helmholtz instability of the cylindrical interface of inviscid fluids with heat and mass transfer and showed that plane geometry configuration is more stable than cylindrical one. The Kelvin-Helmholtz instability of a cylindrical flow with a shear layer has been considered by Wu and Wang [4].

Viscous potential theory has played an important role in studying various stability problems. Joseph and Liao [5] have shown that irrotational flow of a viscous fluid satisfies Navier-Stokes equations. Tangential stresses are not considered in viscous potential theory and viscosity enters through normal stress balance. In this theory no-slip condition at the boundary is not enforced so that two dimensional solutions satisfy three dimensional solutions. Various vortocity and circulation theorems of inviscid potential flow also hold well in viscous potential flow. Joseph et al. [6] studied viscous potential flow of Rayleigh-Taylor instability. Funada and Joseph [7] have done the viscous potential flow analysis of Kelvin-Helmholtz instability in a channel and found that the stability criterion for viscous potential flow is given by the critical value of the relative velocity.

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From the above study Funada and Joseph [7] concluded that the critical value of relative velocity is maximum when viscosity ratio equals the density ratio. Asthana and Agrawal [8] applied the viscous potential theory to analyze Kelvin-Helmholtz instability with heat and mass transfer and observed that heat and mass transfer has destabilizing effect on relative velocity when lower fluid viscosity is low while it has stabilizing effect when lower fluid viscosity is high.

The viscous potential flow analysis of capillary instability has been studied by Funada and Joseph [9]. They observed that viscous potential flow is better approximation of the exact solution than the inviscid model. Funada and Joseph [10] extended their study of capillary instability to the viscoelastic fluids of Maxwell types and observed that the growth rates are larger for viscoelastic fluids than for the equivalent Newtonian fluids. Stability of liquid jet into incompressible gases and liquids was computed by Funada et al. [11]. They consider both Kelvin-Helmholtz and capillary instabilities and observed that Kelvin-Helmholtz instability cannot occur in vacuum but capillary instability can occur in vacuum.

In this paper, viscous potential theory has been applied to analyze Kelvin-Helmholtz instability of cylindrical interface. Both fluids have been taken as viscous with different kinematic viscosities. Both asymmetric and axisymmetric disturbances are considered. A dispersion relation is derived and stability criteria are given in the terms of the critical value of relative velocity. The effect of Reynolds number on growth rate is observed. Finally a comparison has been made between the results of present study with the results obtained by inviscid potential flow.

2 Problem Formulation

Our system of interest consists of two incompressible, viscous fluid layers separated by a cylindrical interface in an annular configuration, as demonstrated in Figure 1. Kim et al. [12] consider the problem of capillary instability with heat and mass transfer and in this problem fluids are not flowing. The undisturbed cylindrical interface is taken radius$R$. In the undisturbed state, inner fluid of density $\rho^{(1)}$ and viscosity $\mu^{(1)}$occupies the region $r_1 < r < R$and outer fluid of density $\rho^{(2)}$ and viscosity $\mu^{(2)}$occupies the region$R < r < r_2$. The inner and outer fluids have uniform velocities U_1 and U_2 respectively along the z-axis. The bounding surfaces $r = r_1$ and $r = r_2$ are considered to be rigid. After disturbance, the interface is given by

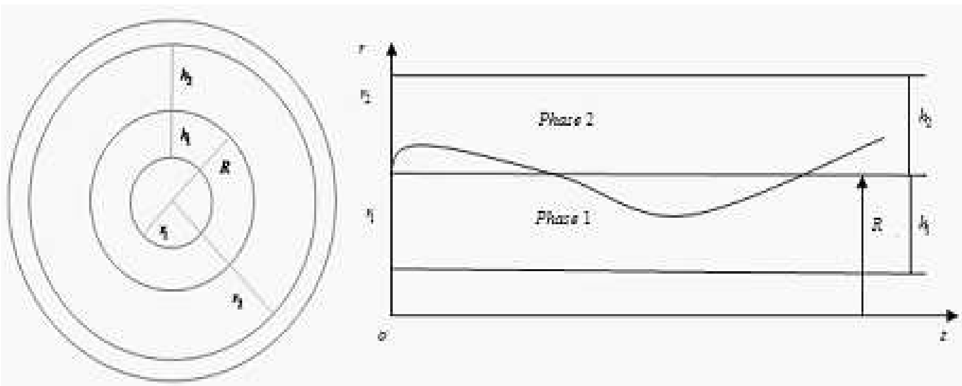

Figure 1: The equilibrium configuration of the fluid system.

$$F(r, \theta, z, t) = r - R - \eta(z, \theta, t) = 0 \tag{2.1}$$

In each fluid layer, velocity potential satisfies the Laplace equations:

$$\nabla^2 \phi^{(j)} = 0 \quad (j = 1, 2) \tag{2.2}$$

This can be rewritten as

$$\nabla^2\phi^{(j)} = \frac{\partial^2\phi^{(j)}}{\partial r^2} + \frac{1}{r}\frac{\partial\phi^{(j)}}{\partial r} + \frac{\partial^2\phi^{(j)}}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2\phi^{(j)}}{\partial\theta^2} = 0 \quad (j = 1, 2) \tag{2.3}$$

Conditions on the walls are given by

$$\frac{\partial\phi^{(1)}}{\partial r} = 0 \quad \text{at} \quad r = r_1, \tag{2.4}$$

$$\frac{\partial\phi^{(2)}}{\partial r} = 0 \quad \text{at} \quad r = r_2, \tag{2.5}$$

Kinematic conditions are given by:

$$\frac{\partial\eta}{\partial t} + U_j\frac{\partial\eta}{\partial z} = \frac{\partial\phi^{(j)}}{\partial r} \quad \text{at} \quad r = R \tag{2.6}$$

Interfacial condition for conservation of momentum is

$$\rho^{(1)}\left(\nabla\phi^{(1)}\cdot\nabla F\right)\left(\tfrac{\partial F}{\partial t} + \nabla\phi^{(1)}\cdot\nabla F\right) = \rho^{(2)}\left(\nabla\phi^{(2)}\cdot\nabla F\right)\left(\tfrac{\partial F}{\partial t} + \nabla\phi^{(2)}\cdot\nabla F\right)$$
$$+ \left(p_2 - p_1 - 2\mu^{(2)}\mathbf{n}\cdot\nabla\otimes\nabla\phi^{(2)}\cdot\mathbf{n} + 2\mu^{(1)}\mathbf{n}\cdot\nabla\otimes\nabla\phi^{(1)}\cdot\mathbf{n} + \sigma\nabla\cdot\mathbf{n}\right)|\nabla F|^2 \qquad \text{at } r = R + \eta \tag{2.7}$$

In initial state we assume

$$\phi_0^{(j)} = U_j\mathrm{z} \qquad (j = 1,\ 2) \tag{2.8}$$

Using Bernoulli's equation in Equation (2.7) and subsequently linearizing it, we get

$$\left[\left[\rho\left(\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial z}\frac{\partial\phi_0}{\partial z}\right) + 2\mu\frac{\partial^2\phi}{\partial r^2}\right]\right] = -\sigma\left(\frac{\partial^2\eta}{\partial z^2} + \frac{\eta}{R^2} + \frac{1}{R^2}\frac{\partial^2\eta}{\partial\theta^2}\right) \tag{2.9}$$

$[\![x]\!] = x^{(2)} - x^{(1)}$in which superscripts refer to upper and lower fluid respectively.

3 Dispersion Relation

Let

$$\eta = A\exp\left[i\left(kz + m\theta - \omega t\right)\right] + c.c \tag{3.1}$$

where k, m and ωdenotes the wave number, azimuthal wave number and complex growth rate respectively.

On solving Eq.(2.3) with the help of boundary conditions, we get

$$\phi^{(1)} = \frac{1}{k}\left(\mathrm{i}kU_1 - \mathrm{i}\omega\right)AE^{(1)}\left(kr\right)\exp\left[i\left(kz + m\theta - \omega t\right)\right] + c.c \tag{3.2}$$

$$\phi^{(2)} = \frac{1}{k}\left(\mathrm{i}kU_2 - \mathrm{i}\omega\right)AE^{(2)}\left(kr\right)\exp\left[i\left(kz + m\theta - \omega t\right)\right] + c.c \tag{3.3}$$

where $c.c$ stands for complex conjugate of preceding term. $E^{(1)}\left(kr\right)$and $E^{(2)}\left(kr\right)$are given by $E^{(1)}\left(kr\right) = \frac{I_m(kr)K_m'(kr_1) - K_m(kr)I_m'(kr_1)}{I_m'(kR)K_m'(kr_1) - K_m'(kR)I_m'(kr_1)}$ and

$$E^{(2)}\left(kr\right) = \frac{I_m(kr)K_m'(kr_2) - K_m(kr)I_m'(kr_2)}{I_m'(kR)K_m'(kr_2) - K_m'(kR)I_m'(kr_2)}$$

Substituting the values of η, $\phi^{(1)}$and $\phi^{(2)}$ in Equation (2.9), we get the dispersion relation

$$D\left(\omega, k\right) = a_0\omega^2 + \left(a_1 + ib_1\right)\omega + a_2 + ib_2 = 0 \tag{3.4}$$

where

$$a_0 = \rho^{(1)} E^{(1)}(kR) - \rho^{(2)} E^{(2)}(kR)$$

$$a_1 = -2k\left(\rho^{(1)} U_1 E^{(1)}(kR) - \rho^{(2)} U_2 E^{(2)}(kR)\right)$$

$$b_1 = 2k^2\left(\mu^{(1)} F^{(1)}(kR) - \mu^{(2)} F^{(2)}(kR)\right)$$
$$a_2 = k^2\left(\rho^{(1)} U_1^2 E^{(1)}(kR) - \rho^{(2)} U_2^2 E^{(2)}(kR)\right) + \frac{\sigma k\left(1-m^2-k^2R^2\right)}{R^2}$$

$$b_2 = -2k^3\left(\mu^{(1)} U_1 F^{(1)}(kR) - \mu^{(2)} U_2 F^{(2)}(kR)\right)$$

$$F^{(1)}(kR) = E^{(1)}(kR) - \frac{1}{kR} + \frac{m^2}{k^2R^2} E^{(1)}(kR)$$

$$F^{(2)}(kR) = E^{(2)}(kR) - \frac{1}{kR} + \frac{m^2}{k^2R^2} E^{(2)}(kR)$$

If both fluids are inviscid i.e. $\mu^{(1)} = \mu^{(2)} = 0$, the dispersion relation for inviscid potential flow (IPF) is given by;

$$\begin{array}{l}\left[\rho^{(1)} E^{(1)}(kR) - \rho^{(2)} E^{(2)}(kR)\right]\omega^2 + \left[-2k\left(\rho^{(1)} U_1 E^{(1)}(kR) - \rho^{(2)} U_2 E^{(2)}(kR)\right)\right]\omega \\ \left[k^2\left(\rho^{(1)} U_1^2 E^{(1)}(kR) - \rho^{(2)} U_2^2 E^{(2)}(kR)\right) + \frac{\sigma k\left(1-m^2-k^2R^2\right)}{R^2}\right] = 0\end{array} \tag{3.5}$$

4 Dimensionless Form of Dispersion Relation

$$\begin{array}{lll}\hat{k} = kh, & \hat{h}_2 = \frac{h_2}{h}, & \hat{U}_2 = \frac{U_2}{U_0} \\ \hat{h}_1 = \frac{h_1}{h} = \hat{h} = \beta, & \hat{U}_1 = \frac{U_1}{U_0}, & \hat{\rho} = \frac{\rho^{(2)}}{\rho^{(1)}}, \\ \hat{V} = \hat{U}_2 - \hat{U}_1, & \hat{\mu} = \frac{\mu^{(2)}}{\mu^{(1)}}, & \hat{\omega} = \frac{\omega h}{U_0}, \\ \hat{\sigma} = \frac{\sigma}{\rho^{(1)} h U_0^2}, & Re = \frac{hU_0}{\nu^{(1)}}, & \hat{z} = \frac{z}{h}, \\ \hat{R} = \frac{R}{h}, & \hat{r}_1 = \frac{r_1}{h}, & \hat{r}_2 = \frac{r_2}{h,}\end{array}$$

where Re denotes Reynolds number and $\hat{h} = \beta$is the inner fluid fraction.
The dimensionless form of Equation (3.4) is

$$D\left(\hat{\omega}, \hat{k}\right) = \hat{a}_0\hat{\omega}^2 + \left(\hat{a}_1 + i\hat{b}_1\right)\hat{\omega} + \hat{a}_2 + i\hat{b}_2 = 0 \tag{4.1}$$

$$\hat{a}_0 = E^{(1)}\left(\hat{k}\hat{R}\right) - \hat{\rho} E^{(2)}\left(\hat{k}\hat{R}\right)$$

$$\hat{a}_1 = -2\hat{k}\left(\hat{U}_1 E^{(1)}\left(\hat{k}\hat{R}\right) - \hat{\rho}\hat{U}_2 E^{(2)}\left(\hat{k}\hat{R}\right)\right)$$

$$\hat{b}_1 = \frac{2\hat{k}^2}{Re}\left(F^{(1)}\left(\hat{k}\hat{R}\right) - \hat{\mu} F^{(2)}\left(\hat{k}\hat{R}\right)\right)$$
$$\hat{a}_2 = \hat{k}^2\left(\hat{U}_1^2 E^{(1)}\left(\hat{k}\hat{R}\right) - \hat{\rho}\hat{U}_2^2 E^{(2)}\left(\hat{k}\hat{R}\right)\right) + \frac{\hat{\sigma}\hat{k}\left(1-m^2-\hat{k}^2\hat{R}^2\right)}{\hat{R}^2}$$

$$\hat{b}_2 = -\frac{2\hat{k}^3}{Re}\left(\hat{U}_1 F^{(1)}\left(\hat{k}\hat{R}\right) - \hat{\mu}\hat{U}_2 F^{(2)}\left(\hat{k}\hat{R}\right)\right)$$

Equation (3.5) in dimensionless form cane written as;

$$\begin{array}{c}\left[E^{(1)}\left(\hat{k}\hat{R}\right) - \hat{\rho} E^{(2)}\left(\hat{k}\hat{R}\right)\right]\hat{\omega}^2 + \left(-2\hat{k}\left(\hat{U}_1 E^{(1)}\left(\hat{k}\hat{R}\right) - \hat{\rho}\hat{U}_2 E^{(2)}\left(\hat{k}\hat{R}\right)\right)\right)\hat{\omega} \\ + \hat{k}^2\left(\hat{U}_1^2 E^{(1)}\left(\hat{k}\hat{R}\right) - \hat{\rho}\hat{U}_2^2 E^{(2)}\left(\hat{k}\hat{R}\right)\right) + \frac{\hat{\sigma}\hat{k}\left(1-m^2-\hat{k}^2\hat{R}^2\right)}{\hat{R}^2} = 0\end{array} \tag{4.2}$$

Let $\hat{\omega} = \hat{\omega}_R + i\,\hat{\omega}_{\mathrm{I}}$, then Equation (4.1) is reduced to

$$\hat{a}_0\left(\hat{\omega}_{\mathrm{R}}^2 - \hat{\omega}_{\mathrm{I}}^2\right) + \left(\hat{a}_1\hat{\omega}_{\mathrm{R}} - \hat{b}_1\hat{\omega}_{\mathrm{I}}\right) + \hat{a}_2 = 0 \tag{4.3}$$

and

$$\hat{\omega}_{\mathrm{R}} = -\frac{\hat{a}_1\hat{\omega}_{\mathrm{I}} + \hat{b}_2}{2\hat{a}_0\hat{\omega}_{\mathrm{I}} + \hat{b}_1} \tag{4.4}$$

Eliminating the value of $\hat{\omega}_{\mathrm{R}}$from above equations we obtained a quartic equation in $\hat{\omega}_{\mathrm{I}}$as

$$A_4\hat{\omega}_{\mathrm{I}}^4 + A_3\hat{\omega}_{\mathrm{I}}^3 + A_2\hat{\omega}_{\mathrm{I}}^2 + A_1\hat{\omega}_{\mathrm{I}} + A_0 = 0 \tag{4.5}$$

where

$$A_4 = -4\hat{a}_0^3 \tag{4.6}$$

$$\mathrm{A}_3 = -8\hat{a}_0^2\hat{b}_1 \tag{4.7}$$

$$A_2 = 4\hat{a}_0^2\hat{a}_2 - 5\hat{a}_0\hat{b}_1^2 - \hat{a}_0\hat{a}_1^2 \tag{4.8}$$

$$A_1 = 4\hat{a}_0\hat{a}_2\hat{b}_1 - \hat{b}_1^3 - \hat{a}_1^2\hat{b}_1 \tag{4.9}$$

$$A_0 = \hat{a}_0\hat{b}_2^2 - \hat{a}_1\hat{b}_1\hat{b}_2 + \hat{a}_2\hat{b}_1^2 \tag{4.10}$$

Neutral curves are obtained by putting$\hat{\omega}_{\mathrm{I}}(k) = 0$. Equation (4.5) reduces to$A_0 = 0$, which in turn implies that

$$\hat{a}_0\hat{b}_2^2 - \hat{a}_1\hat{b}_1\hat{b}_2 + \hat{a}_2\hat{b}_1^2 = 0 \tag{4.11}$$

Substituting the values of $\hat{a}_0, \hat{a}_1, \hat{b}_1, \hat{a}_2, \hat{b}_2$in the above equation we get

$$\hat{V}^2 = \frac{\hat{\sigma}}{\hat{k}\hat{R}^2}\left(1 - m^2 - \hat{k}^2\hat{R}^2\right)\frac{\left(\hat{\mu}F^2\left(\hat{k}\hat{R}\right) - F^{(1)}\left(\hat{k}\hat{R}\right)\right)^2}{\left(\hat{\rho}E^{(2)}\left(\hat{k}\hat{R}\right)F^{(1)^2}\left(\hat{k}\hat{R}\right) - \hat{\mu}^2E^{(1)}\left(\hat{k}\hat{R}\right)F^{(2)^2}\left(\hat{k}\hat{R}\right)\right)} \tag{4.12}$$

Here relative velocity $\hat{V}$is given by$\hat{V} \equiv \hat{U}_2 - \hat{U}_1$.

5 Results and Discussion

We have considered the situation when water is lying in the inner region and air is lying in the outer region. Following parametric values have been taken.

$$\hat{\rho} = 0.0012, \hat{\mu} = 0.018, \hat{\sigma} = 72.3,$$

The radii of inner and outer cylinders are 1 cm and 2 cm respectively. For numerical computation length scale h is taken as 1 cm and the reference velocity $U_0$is taken as 1 cm/s. Neutral curve for relative velocity divide the plane into a stable region (below the curve) and an unstable region (above the curve). We have taken the asymmetric case ($m = 1$) for numerical computation purpose form figures 2- 6.

In Figure 2, neutral curves have been drawn for relative velocity $\hat{V}$ with$\hat{\rho} = 0.0012$, $\hat{\mu} = 0.018$, $Re = 1000$for different values of inner fluid fractionβ. It is observed that as inner fluid fraction increases, stable region (below region) decreases. So it concludes that the inner fluid fraction has destabilizing effect on the stability of the system. Figure 3 shows the neutral curves for relative velocity for the different values of viscosity ratio and density ratio. We observe that the stable region is maximum when viscosity ratio is equal to the density ratio. Similar result was obtained by Funada and Joseph [7] for plane geometry.

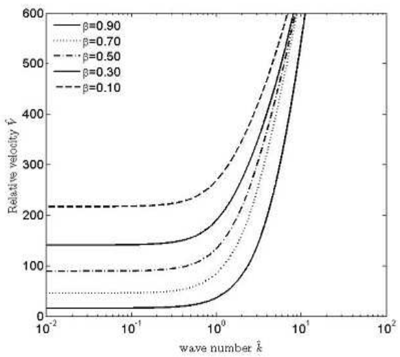

Figure 2: Neutral curve for relative velocity with $\hat{\rho}$ = 0.0012, $\hat{\mu}$ = 0.018, Re = 1000 for different value of inner fluid fractionβ.

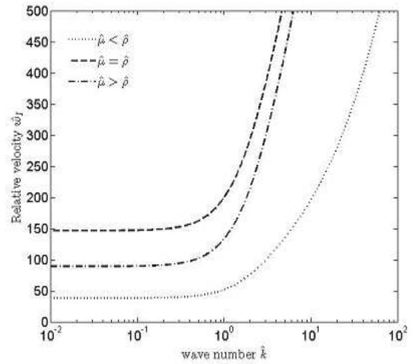

Figure 3: Neutral curve for relative velocity with Re = 1000, β = 0.5 for different value of density ratio $\hat{\rho}$ and viscosity ratio$\hat{\mu}$.

In Figure 4, growth rate curves for various values of Reynolds number have been plotted and observe that growth rate curve increases with Reynolds number. Hence Reynolds number has destabilizing effect. On increasing Reynolds number viscosity of inner fluid will decrease and less resistance to fluid flow will take place. So the flow will become unstable. Also if density of the lower fluid increases Reynolds number increases, so lower fluid density has destabilizing effect on the stability of the system. The characteristic velocity and length scale both are also playing destabilizing role as Reynolds number increases with increasingU_0or h.

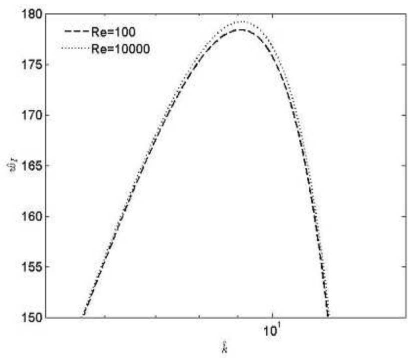

Figure 4: Imaginary part of growth rate $\hat{\omega}_I$ $vs.$ $\hat{k}$ with$\hat{\rho}$ = 0.0012, $\hat{\mu}$ = 0.018, $\hat{h}$ = 0.5and$\hat{V}$ = 900 for different values of Re.

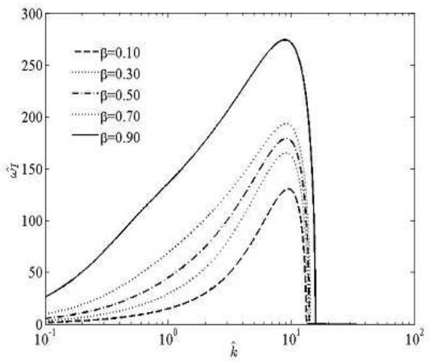

Figure 5: Imaginary part of growth rate $\hat{\omega}_I$ $vs.$ $\hat{k}$ with$\hat{\rho}$ = 0.0012, $\hat{\mu}$ = 0.018, Re = 850and$\hat{V}$ = 900 for different values of β.

Figure 5 shows the behavior of growth rate curves for different values of inner fluid fraction. It has been observed that the growth rate curves increases as inner fluid fraction β increases. It also concludes that the inner fluid fraction β has destabilizing effect on the stability of the system.

We have compared our results with the results obtained by inviscid potential flow in the Figure 6. It has been observed that viscosity has stabilizing effect on the stability of the system. We have compared growth rate of axisymmetric and asymmetric perturbation in Figure 7. It is concluded that axisymmetric perturbation is more stable. Funada et al. [11] have also concluded that asymmetric perturbation will dominate at low Reynolds number.

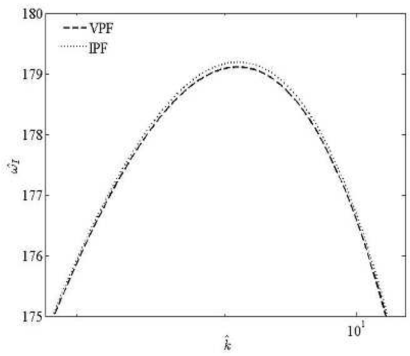

Figure 6: Comparison of growth rates for inviscid potential flow and viscous potential flow when$\hat{\rho} = 0.0012,\ \hat{\mu} = 0.018,\ Re = 1000,\ \hat{V} = 900$.

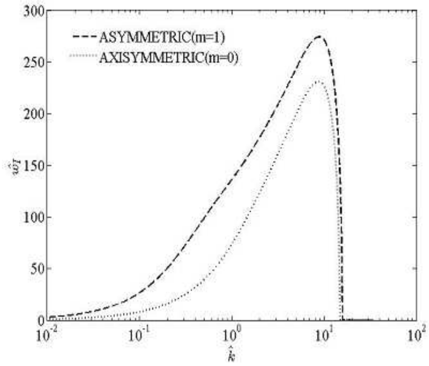

Figure 7: Comparison of growth rates for Asymmetric disturbance and Axisymmetric disturbance when$\hat{\rho} = 0.0012,\ \hat{\mu} = 0.018,\ Re = 1000,\ \hat{V} = 900$.

6 Conclusion

Viscous potential flow analysis of Kelvin-Helmholtz instability of cylindrical interface has been made. The dispersion relation for both asymmetric and axisymmetric disturbances is a quadratic equation in growth rate. The stability condition is given in the terms of critical value of relative velocity. It is observed the critical value of relative velocity is maximum when density ratio is equal to the viscosity ratio. Reynolds number and inner fluid fraction both have destabilizing effect on the stability of the system. A comparison between inviscid potential flow (IPF) and viscous potential flow (VPF) has been made. It is concluded that VPF is more stable than IPF.

Acknowledgement

Author is thankful to Council of Scientific and Industrial Research (CSIR), New Delhi for their financial support during this work.

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