International Journal of Applied Mathematics and Computation
Volume 3(2),pp 114–127, 2011
http://ijamc.psit.in

Positive Solutions for Higher Order Two-Point Boundary Value Problem

A. Kameswararao and S. Nageswararao
GVP College of Engineering for Women, Department of Mathematics, Madhurawada, Visakhapatnam, 530 048, A. P, India.
Email: kamesh_1724@yahoo.com

Department of Mathematics, Sri Prakash College of Engineering, Tuni, 533 401, A. P, India.
Email: sabbavarapu_nag@yahoo.co.in

Abstract:

In this paper, we consider a higher-order two-point boundary value problem. We study the existence of solutions of a non-eigenvalue problem and of at least one positive solution of an eigenvalue problem. Later we establish the criteria for the existence of at least two positive solutions of a non-eigenvalue problem.

Key words: Boundary value problem, eigenvalue interval, positive solutions, fixed point, cone.

1 Introduction

We are concerned with two-point boundary value problem (TPBVP)

$$\begin{cases} y^{(n)}(t) + f(t, y(t)) = 0, \quad t \in [a, b], \\ y^{(i)}(a) = 0, \quad 0 \le i \le n-2, \\ y^{(p)}(b) = 0, \quad (1 \le p \le n-1, \text{ but fixed}), \end{cases} \tag{1.1}$$

and the eigenvalue problem $y^{(n)}(t) + \lambda f(t, y(t)) = 0$ with the same boundary conditions where λ is a positive parameter, and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Corresponding author: **A. Kameswararao**

Today, the boundary value problems (BVPs) play a major role in many fields of engineering design and manufacturing. Major established industries such as the automobile, aerospace, chemical, pharmaceutical, petroleum, electronics and communications, as well as emerging technologies such as nanotechnology and biotechnology rely on the BVPs to simulate complex phenomena at different scales for design and manufactures of high-technology products. In these applied settings, positive solutions are meaningful. Due to their important role in both theory and applications, the BVPs have generated a great deal of interest over the recent years.

There is currently a great deal of interest in positive solutions for several types of boundary value problems. A large part of the literature on positive solutions to BVPs seems to be traced back to Krasnoselskii's work on nonlinear operator equations [16], especially the part dealing with the theory of cones in Banach space. In 1994, Erbe and Wang [12] applied Krasnoselskii's work to establish intervals of the parameter λ for which there is at least one positive solution. In 1995, Eloe and Henderson [6] obtained the solutions that are positive to a cone for the boundary value problem

$$\begin{aligned} u^{(n)}(t)+&a(t)f(u) = 0, \quad 0 < t < 1, \\ u^{(i)}(0) =&u^{(n-2)}(1) = 0, \quad 0 \le i \le n-2. \end{aligned}$$

In 2008, Shahed [21] established the existence of positive solutions to nonlinear nth order boundary value problems:

$$\begin{aligned} u^{(n)}(t)+&\lambda a(t)f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) =&u'''(0) = ... = u^{(n-1)}(0), \quad u'(1) = 0, \\ u(0) = u'(0) =&u''(0) = ... = u^{(n-2)}(0), \quad u'(1) = 0, \\ u(0) = u'(0) =&u''(0) = ... = u^{(n-1)}(0), \quad u''(1) = 0, \end{aligned}$$

where λ is a positive parameter.

In this paper, existence results of bounded solutions of a non-eigenvalue problem are first established as a result of the Schauder fixed-point theorem. Second, we establish criteria for the existence of at least one positive solution of the eigenvalue problem by using the Krasnosel'skii fixed-point theorem. Later, we investigate the existence of at least two positive solutions of TPBVP (1.1) by using the Avery-Henderson fixed-point theorem.

2 The Green's Function and Bounds

In this section, we construct the Green's function for the homogeneous boundary value problem corresponding to the boundary value problem (1.1). And then we prove some inequalities on bounds of the Green's function which are needed later.

Let $G_n(t,s)$ be the Green's function of the boundary value problem,

$$\begin{cases} -y^{(n)}(t) = 0, \\ y^{(i)}(a) = 0, \quad 0 \le i \le n-2, \\ y^{(p)}(b) = 0, \quad (1 \le p \le n-1, \text{ but fixed}). \end{cases} \tag{2.1}$$

Theorem 2.1. *The Green's function $G_n(t,s)$ for the boundary value problem (2.1) is given by*

$$G_n(t,s) = \begin{cases} \frac{(t-a)^{n-1}(b-s)^{n-p-1}}{(n-1)!(b-a)^{n-p-1}}, & a \le t \le s \le b, \\ \\ \frac{(t-a)^{n-1}(b-s)^{n-p-1}}{(n-1)!(b-a)^{n-p-1}} - \frac{(t-s)^{n-1}}{(n-1)!}, & a \le s \le t \le b. \end{cases}$$

Lemma 2.2. *For $(t,s) \in [a,b] \times [a,b]$, we have*

$$G_n(t,s) \le G_n(b,s). \tag{2.2}$$

Proof. For $a \le t \le s \le b$, we have

$$\begin{aligned} G_n(t,s) &= \frac{(t-a)^{n-1}(b-s)^{n-p-1}}{(n-1)!(b-a)^{n-p-1}} \\ &\le \frac{(b-a)^{n-1}(b-s)^{n-p-1}}{(n-1)!(b-a)^{n-p-1}} \\ &= G_n(b,s). \end{aligned}$$

Similarly, for $a \le s \le t \le b$, we have $G_n(t,s) \le G_n(b,s)$. Thus, we have

$$G_n(t,s) \le G_n(b,s), \quad \text{for all} \quad (t,s) \in [a,b] \times [a,b].$$

□

Lemma 2.3. *Let $I = \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]$. For $(t,s) \in I \times [a,b]$, we have*

$$G_n(t,s) \ge \gamma G_n(b,s). \tag{2.3}$$

Proof. The Green's function $G_n(t,s)$ for the boundary value problem (2.1) is clearly shows that

$$G_n(t,s) > 0 \text{ on } (a,b) \times (a,b).$$

For $a \le t \le s \le b$ and $t \in I$, we have

$$\begin{aligned} \frac{G_n(t,s)}{G_n(b,s)} &= \left(\frac{t-a}{b-a}\right)^{n-1} \\ &\ge \frac{1}{4^{n-1}}. \end{aligned}$$

Similarly, for $a \le s \le t \le b$ and $t \in I$ we have

$$\begin{aligned}
\frac{G_n(t,s)}{G_n(b,s)} &= \frac{(t-a)^{n-1}(b-s)^{n-p-1} - (t-s)^{n-1}(b-a)^{n-p-1}}{(b-a)^{n-1}(b-s)^{n-p-1} - (b-s)^{n-1}(b-a)^{n-p-1}} \\
&\ge \frac{(t-a)^{n-p-1}(b-s)^{n-p-1}[(t-a)^p - (t-s)^p]}{(b-a)^{n-1}(b-s)^{n-p-1} - (b-s)^{n-1}(b-a)^{n-p-1}} \\
&= \frac{1}{p}\left(\frac{t-a}{b-a}\right)^{n-2} \\
&\ge \frac{1}{p}\left(\frac{t-a}{b-a}\right)^{n-1} \\
&\ge \frac{1}{p.4^{n-1}}.
\end{aligned}$$

Therefore

$$G_n(t,s) \ge \gamma G_n(b,s), \quad \text{for} \quad (t,s) \in I \times [a,b],$$

where $\gamma = \min\left\{\frac{1}{4^{n-1}}, \frac{1}{p.4^{n-1}}\right\}$. □

3 Existence of Positive Solutions

In this section, first we obtain the existence of bounded solutions to the TPBVP (1.1). The proof of this result is based on an application of the Schauder fixed-point theorem.

Let $\mathcal{B}$ denote the Banach space $\mathcal{C}[a,b]$ with the norm

$$\| y \| = \max_{t \in [a,b]} | y(t) | .$$

Theorem 3.1. *Suppose the function $f(t,\xi)$ is continuous with respect to $\xi \in \mathbb{R}$. If $R > 0$ satisfies $Q\int_a^b G_n(b,s)ds \le R$, where $Q > 0$ satisfies*

$$Q \ge \max_{\|y\| \le R} | f(t, y(t) |,$$

for $t \in [a,b]$, then TPBVP (1.1) has a solution $y(t)$.

Proof. Let $\mathcal{P} = \{y \in \mathcal{B} : \| y \| \le R\}$. Note that $\mathcal{P}$ is closed, bounded and convex subset of $\mathcal{B}$ to which the Schauder fixed-point theorem is applicable.

Define $T : \mathcal{P} \to \mathcal{B}$ by

$$Ty(t) = \int_a^b G_n(t,s) f(s, y(s)) ds,$$

for $t \in [a,b]$. Obviously the solutions of the TPBVP (1.1) are the fixed points of operator T. It can be shown that $T : \mathcal{P} \to \mathcal{B}$ is continuous.

Claim that $T : \mathcal{P} \to \mathcal{P}$. Let $y \in \mathcal{P}$. By using Lemma 2.2, we get

$$\begin{aligned} \mid Ty(t) \mid &= \mid \int_a^b G_n(t,s) f(s, y(s)) ds \mid \\ &\leq \int_a^b \mid G_n(t,s) \mid\mid f(s,y(s)) \mid ds \\ &\leq Q \int_a^b G_n(b,s) ds \\ &\leq R, \end{aligned}$$

for every $t \in [a,b]$. This implies that $\parallel Ty \parallel \leq R$.

It can be shown that $T : \mathcal{P} \to \mathcal{P}$ is a compact operator by the Arzela-Ascoli theorem. Hence T has a fixed point in $\mathcal{P}$ by the Schauder fixed point theorem. □

Corollary 3.2. *If the function f is continuous and bounded on $[a,b] \times \mathbb{R}$, then the TPBVP (1.1) has a solution.*

Proof. Since the function $f(t,y)$ is bounded, it has a supremum for $t \in [a,b]$ and $y \in \mathbb{R}$. Let us choose $P > \sup\{\mid f(t,y) \mid : (t,y) \in [a,b] \times \mathbb{R}\}$. Pick R large enough such that $P < R$. Then there is a number $Q > 0$ such that

$$P > Q, \quad \text{where } Q \geq \max\{|f(t,y)| : \quad t \in [a,b], \quad \mid y \mid \leq R\}.$$

Hence

$$1 < \frac{R}{P} \leq \frac{R}{Q},$$

and thus the TPBVP (1.1) has a solution by Theorem 3.1. □

4 Existence of One Positive Solution

In this section, we consider the following TPBVP with parameter λ,

$$\begin{cases} y^{(n)}(t) + \lambda f(t, y(t)) = 0, & t \in [a,b], \\ y^{(i)}(a) = 0, & 0 \leq i \leq n-2, \\ y^{(p)}(b) = 0, & (1 \leq p \leq n-1, \text{ but fixed}). \end{cases} \tag{4.1}$$

We need the following fixed-point theorem to prove the existence of at least one positive solution to TPBVP (4.1).

Theorem 4.1. *[16] Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of $\mathcal{B}$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that either *(i)* $\| Tu \| \leq \| u \|, \ u \in \mathcal{P} \cap \partial\Omega_1, \qquad \| Tu \| \geq \| u \|, \ u \in \mathcal{P} \cap \partial\Omega_2;$ *or*

(ii) $\| Tu \| \geq \| u \|, \ u \in \mathcal{P} \cap \partial\Omega_1, \qquad \| Tu \| \leq \| u \|, \ u \in \mathcal{P} \cap \partial\Omega_2,$

holds. Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

We assume that $f \in \mathcal{C}([a,b] \times \mathbb{R}^+, \mathbb{R}^+)$, and the limits

$$f_0 = \lim_{y \to 0^+} \frac{f(t,y)}{y}, \qquad f_\infty = \lim_{y \to \infty} \frac{f(t,y)}{y}$$

exist uniformly in the extended reals. The case $f_0 = 0$ and $f_\infty = \infty$ is called the superlinear case, and the case $f_0 = \infty$ and $f_\infty = 0$ is called the sublinear case.

Theorem 4.2. *For each λ satisfying*

$$(a) \qquad \frac{1}{[\gamma^2 \int_{s \in I} G_n(b,s)ds] f_\infty} < \lambda < \frac{1}{\int_a^b G_n(b,s)ds] f_0}, \tag{4.2}$$

$$(b) \qquad \frac{1}{[\gamma^2 \int_{s \in I} G_n(b,s)ds] f_0} < \lambda < \frac{1}{\int_a^b G_n(b,s)ds] f_\infty}, \tag{4.3}$$

there exists at least one positive solution of the TPBVP (4.1). *Moreover, in the case f is superlinear(sublinear), then Eq.*(4.2)*(Eq.*(4.3)*) becomes* $0 < \lambda < \infty$.

Proof. Define $\mathcal{B}$ to be a Banach space of all continuous functions on $[a,b]$ equipped with the norm $\| . \|$ defined by

$$\| y \| = \max_{t \in [a,b]} | y(t) | .$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \Big\{ y \in \mathcal{B} : y(t) \geq 0, \qquad \min_{t \in I} y(t) \geq \gamma \| y \| \Big\},$$

where γ is as in Lemma 2.3. Define an operator T_λ by

$$T_\lambda y(t) = \lambda \int_a^b G_n(t,s) f(s, y(s)) ds,$$

for $t \in [a,b]$. The solutions of the TPBVP(4.1) are the fixed points of the operator T_λ.

Firstly, we show that $T_\lambda : \mathcal{P} \to \mathcal{P}$. Note that $y \in \mathcal{P}$ implies that $T_\lambda y(t) \geq 0$ on $[a,b]$ and

$$\begin{aligned} T_\lambda y(t) &= \lambda \int_a^b G_n(t,s) f(s, y(s)) ds \\ &\leq \lambda \int_a^b G_n(b,s) f(s, y(s)) ds. \end{aligned}$$

Note that by the nonnegative of f, we have

$$\| T_\lambda y \| \leq \lambda \int_a^b G_n(b,s) f(s, y(s)) ds$$

from which

$$\min_{t\in I} T_\lambda y(t) = \min_{t\in I} \lambda \int_a^b G_n(t,s) f(s, y(s)) ds$$
$$\geq \gamma\lambda \int_a^b G_n(b,s) f(s,y(s)) ds$$
$$\geq \gamma \parallel T_\lambda y \parallel, \quad y \in \mathcal{P}.$$

Hence $T_\lambda y \in \mathcal{P}$ and so $T_\lambda : \mathcal{P} \to \mathcal{P}$ which is what we want to prove. Therefore T_λ is completely continuous.

Assume that (a) holds. Since $\lambda < \frac{1}{[\int_a^b G_n(b,s)ds] f_0}$, there exists $\epsilon_1 > 0$ so that $0 < \lambda \leq \frac{1}{[\int_a^b G_n(b,s)ds](f_0+\epsilon_1)}$.

Using the definition of f_0, there is an $r_1 > 0$, sufficiently small, so that

$$f(t,y) < (f_0 + \epsilon_1)y, \quad \text{for} \quad 0 < y \leq r_1, \ t \in [a,b].$$

If $y \in \mathcal{P}$ with $\parallel y \parallel = r_1$, then

$$T_\lambda y(t) = \lambda \int_a^b G_n(t,s) f(s,y(s)) ds$$
$$\leq \lambda(f_0 + \epsilon_1) \int_a^b G_n(t,s) y(s) ds$$
$$\leq \lambda(f_0 + \epsilon_1) \parallel y \parallel \int_a^b G_n(b,s) ds$$
$$\leq \parallel y \parallel$$

for $t \in [a,b]$. So if we set $\Omega_1 = \{y \in \mathcal{B} : \parallel y \parallel \leq r_1\}$, then $\parallel T_\lambda y \parallel \leq \parallel y \parallel$, for $y \in \mathcal{P} \cap \partial\Omega_1$.

Now, we use assumption $\frac{1}{[\gamma^2 \int_{s\in I} G_n(b,s)ds] f_\infty} < \lambda$.

First, we consider the case when $f_\infty < \infty$. In this case pick an $\epsilon_2 > 0$ so that

$$\lambda\gamma^2 \int_{s\in I} G_n(b,s) ds (f_\infty - \epsilon_2) \geq 1.$$

Using the definition of f_∞, there exists $\overline{r}_2 > r_1$, sufficiently large, so that

$$f(t,y) > (f_\infty - \epsilon_2)y, \quad \text{for} \quad y \geq \overline{r}_2, \quad t \in [a,b].$$

We now show that there exists $r_2 \geq \overline{r}_2$ such that if $y \in \partial\mathcal{P}_{r_2}$, then
$\parallel T_\lambda y \parallel \geq \parallel y \parallel$. Let $r_2 = \max\{2r_1, \frac{1}{\gamma}\overline{r}_2\}$ and set $\Omega_2 = \{y \in \mathcal{B} : \parallel y \parallel \leq r_2\}$. If $y \in \mathcal{P} \cap \partial\Omega_2$, then

$$\min_{t\in I} y(t) \geq \gamma \parallel y \parallel = \gamma r_2 \geq \overline{r}_2,$$

and so

$$\begin{aligned}
T_\lambda y(t) &= \lambda \int_a^b G_n(t,s) f(s, y(s)) ds \\
&\geq \lambda(f_\infty - \epsilon_2) \int_a^b G_n(t,s) y(s) ds \\
&\geq \lambda(f_\infty - \epsilon_2) \int_{s\in I} G_n(t,s) y(s) ds \\
&\geq \lambda(f_\infty - \epsilon_2)\gamma^2 \parallel y \parallel \int_{s\in I} G_n(b,s) ds \\
&\geq \parallel y \parallel = r_2.
\end{aligned}$$

Consequently, $\parallel T_\lambda y \parallel \geq \parallel y \parallel$, for $t \in [a,b]$.

Finally, we consider the case $f_\infty = \infty$. In this case the hypothesis becomes $\lambda > 0$. Choose $N > 0$ sufficiently large so that

$$\lambda N \gamma \int_a^b G_n(b,s) ds \geq 1.$$

Hence there exists $\overline{r}_2 > r_1$ so that $f(t,y) > Ny$ for $y \geq \overline{r}_2$ and for all $t \in [a,b]$. Now define r_2 as before and assume $y \in \partial \mathcal{P}_{r_2}$. Then

$$\begin{aligned}
T_\lambda y(t) &= \lambda \int_a^b G_n(t,s) f(s, y(s)) ds \\
&\geq \lambda N \int_a^b G_n(t,s) y(s) ds \\
&\geq \lambda N \gamma \parallel y \parallel \int_a^b G_n(b,s) ds \\
&\geq \parallel y \parallel = r_2
\end{aligned}$$

for $t \in [a,b]$. Hence $\parallel T_\lambda y \parallel \geq \parallel y \parallel$ for $y \in \mathcal{P} \cap \partial\Omega_1$ and $\parallel T_\lambda y \parallel \leq \parallel y \parallel$ for $y \in \mathcal{P} \cap \partial\Omega_2$ hold. Then T_λ has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Now we show (b). Since $\frac{1}{\gamma^2[\int_{s\in I} G(b,s)ds]f_0} < \lambda$, there exits $\epsilon_3 > 0$ so that $\lambda\gamma^2 \int_{s\in I} G(b,s)ds(f_0 - \epsilon_3) \geq 1$.

From the definition of f_0, there exists an $r_3 > 0$ such that $f(t,y) \geq (f_0 - \epsilon_3)y$ for $0 < y \leq r_3$. If $y \in \mathcal{P}$ with $\parallel y \parallel = r_3$, then

$$\begin{aligned}
T_\lambda y(t) &= \lambda \int_a^b G_n(t,s) f(s, y(s)) ds \\
&\geq \lambda(f_0 - \epsilon_3) \int_a^b G_n(t,s) y(s) ds \\
&\geq \lambda\gamma^2 \parallel y \parallel (f_0 - \epsilon_3) \int_{s\in I} G_n(b,s) ds \\
&\geq \parallel y \parallel = r_3.
\end{aligned}$$

Hence $\parallel T_\lambda y \parallel \geq \parallel y \parallel$. So, if we set $\Omega_3 = \{y \in \mathcal{B} : \parallel y \parallel \leq r_3\}$, then $\parallel T_\lambda y \parallel \geq \parallel y \parallel$ for $y \in \mathcal{P} \cap \partial\Omega_3$.

Now, we use assumption $\frac{1}{\int_a^b G_n(b,s)ds f_\infty} > \lambda$. Pick an $\epsilon_4 > 0$ so that

$$\lambda \int_a^b G_n(b,s)ds(f_\infty + \epsilon_4) \leq 1.$$

Using the definition of f_∞, there exists an $\overline{r}_4 > 0$ such that $f(t,y) \leq (f_\infty + \epsilon_4)y$ for all $y \geq \overline{r}_4$. We consider the two cases.

Case I: Suppose $f(t,y)$ is bounded on $[a,b] \times (0,\infty)$. In this case, there is $L > 0$ such that $f(t,y) \leq L$ for $t \in [a,b]$, $y \in (0,\infty)$. Let $r_4 = \max\{2r_3, \lambda L \int_a^b G_n(b,s)ds\}$. Then for $y \in \mathcal{P}$ with $\| y \| = r_4$,

$$\begin{aligned} T_\lambda y(t) &= \lambda \int_a^b G_n(t,s) f(s,y(s))ds \\ &\leq \lambda L \int_a^b G_n(b,s)ds \\ &\leq \| y \| = r_4, \end{aligned}$$

so that $\| T_\lambda y \| \leq \| y \|$.

Case II: Suppose $f(t,y)$ is unbounded on $[a,b] \times (0,\infty)$. In this case,

$$g(r) = \max\{f(t,y) : t \in [a,b],\ 0 \leq y \leq r\}$$

satisfies

$$\lim_{r \to \infty} g(r) = \infty.$$

We can therefore choose

$$r_4 = \max\{2r_3, \overline{r}_4\}$$

such that

$$g(r_4) \geq g(r)$$

for $0 \leq r \leq r_4$ and hence for $y \in \mathcal{P}$ and $\| y \| = r_4$, we have

$$\begin{aligned} T_\lambda y(t) &= \lambda \int_a^b G_n(t,s) f(s,y(s))ds \\ &\leq \lambda \int_a^b G_n(t,s) g(r_4)ds \\ &\leq \lambda (f_\infty + \epsilon_4) r_4 \int_a^b G_n(b,s)ds \\ &\leq r_4 = \| y \|, \end{aligned}$$

and again we hence have $\| T_\lambda y \| \leq \| y \|$ for $y \in \mathcal{P} \cap \partial\Omega_4$, where $\Omega_4 = \{y \in \mathcal{B} : \| y \| \leq r_4\}$ in both cases. It follows from part (ii) of Theorem 4.1 that T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_4 \backslash \Omega_3)$, such that $r_3 \leq \| y \| \leq r_4$. The proof of part (b) of this theorem is complete. Therefore, the TPBVP (4.1) has at least one positive solution. $\square$

5 Existence of Two Positive Solutions

In this section, using Theorem 5.1 (Avery-Henderson fixed-point theorem) we prove the existence of at least two positive solutions of the TPBVP(1.1).

Theorem 5.1. *[4]. Let $\mathcal{P}$ be a cone in a real Banach space. If φ and ψ are increasing, non-negative continuous functionals on $\mathcal{P}$, let θ be a non-negative continuous functional on $\mathcal{P}$ with $\theta(0) = 0$ such that, for some positive constants r and γ,*

$$\psi(u) \le \theta(u) \le \varphi(u) \text{ and } \quad \| u \| \le \gamma\psi(u)$$

for all $u \in \overline{\mathcal{P}(\psi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\theta(\lambda u) \le \lambda\theta(u), \quad for \quad all \quad 0 \le \lambda \le 1 \quad and \quad u \in \partial\mathcal{P}(\theta, q).$$

If $T : \overline{\mathcal{P}(\psi, r)} \to \mathcal{P}$ is a completely continuous operator satisfying

(i) $\psi(Tu) > r$ *for all* $u \in \partial\mathcal{P}(\psi, r)$,

(ii) $\theta(Tu) < q$ *for all* $u \in \partial\mathcal{P}(\theta, q)$,

(iii) $\mathcal{P}(\varphi, p) \neq \{\}$ *and* $\varphi(Tu) > p$ *for all* $u \in \partial\mathcal{P}(\varphi, p)$,

then T has at least two fixed points u_1 and u_2 such that

$$p < \varphi(u_1) \quad with \quad \theta(u_1) < q \quad and \quad q < \theta(u_2) \quad with \quad \psi(u_2) < r.$$

Let the Banach space $\mathcal{B} = \mathcal{C}[a, b]$ with the norm $\| \, . \, \|$ defined by $\| \, y \, \| = \max_{t\in[a,b]} \mid y(t) \mid$. Again define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \Big\{ y \in \mathcal{B} : y(t) \ge 0, \quad \min_{t\in I} y(t) \ge \gamma \parallel y \parallel \Big\},$$

where γ is as in Lemma 2.3, and the operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Ty(t) = \int_a^b G_n(t, s) f(s, y(s)) ds.$$

Let the non-negative, increasing, continuous functionals ψ, θ, and φ be defined on the cone $\mathcal{P}$ by

$$\psi(y) = \min_{t\in I} y(t), \qquad \theta(y) = \max_{t\in I} y(t), \qquad \varphi(y) = \max_{t\in[a,b]} y(t) \tag{5.1}$$

and let $\mathcal{P}(\psi, r) = \{y \in \mathcal{P} : \psi(y) < r\}$.

Define

$$C = \gamma \int_{s\in I} G_n(b, s) ds, \qquad D = \int_a^b G_n(b, s) ds.$$

In the next theorem, we will assume

(H) $f \in \mathcal{C}([a, b] \times [0, \infty), [0, \infty))$.

Theorem 5.2. *Assume (H) holds. Suppose there exist positive numbers $0 < p < q < r$ such that the function f satisfies the following conditions:*

(D1) $f(t, y) > \frac{p}{C}$ *for* $t \in I$ *and* $y \in [\gamma p, p]$,

(D2) $f(t, y) < \frac{q}{D}$ *for* $t \in [a, b]$ *and* $y \in [0, \frac{q}{\gamma}]$,

(D3) $f(t, y) > \frac{r}{C}$ *for* $t \in I$ *and* $y \in [r, \frac{r}{\gamma}]$,

where γ is defined in Lemma 2.3. Then the TPBVP (1.1) *has at least two positive solutions y_1 and y_2 such that*

$$p < \max_{t\in[a,b]} y_1(t) \quad with \quad \max_{t\in I} y_1(t) < q,$$
$$q < \max_{t\in I} y_2(t) \quad with \quad \min_{t\in I} y_2(t) < r.$$

Proof. From Lemma 2.2 and Lemma 2.3, $T\mathcal{P} \subset \mathcal{P}$. Moreover, T is completely continuous. From (5.1), for each $y \in \mathcal{P}$ we have

$$\psi(y) \leq \theta(y) \leq \varphi(y), \tag{5.2}$$

$$\| y \| \leq \frac{1}{\gamma} \min_{t\in I} y(t) = \frac{1}{\gamma}\psi(y) \leq \frac{1}{\gamma}\theta(y) \leq \frac{1}{\gamma}\varphi(y). \tag{5.3}$$

For any $y \in \mathcal{P}$, (5.2) and (5.3) imply

$$\psi(y) \leq \theta(y) \leq \varphi(y), \qquad \| y \| \leq \frac{1}{\gamma}\psi(y).$$

For all $y \in \mathcal{P}$, $\lambda \in [0, 1]$ we have

$$\theta(\lambda y) = \max_{t\in I}(\lambda y)(t) = \lambda \max_{t\in I} y(t) = \lambda\theta(y).$$

It is clear that $\theta(0) = 0$.

We now show that the remaining conditions of Theorem 5.1 are satisfied.

Firstly, we shall verify that condition (iii) of Theorem 5.1 is satisfied. Since $0 \in \mathcal{P}$ and $p > 0$, $\mathcal{P}(\varphi, p) \neq \{\}$. Since $y \in \partial\mathcal{P}(\varphi, p)$, $\gamma p \leq y(t) \leq \| y \| = p$ for $t \in I$. Therefore,

$$\begin{aligned}
\varphi(Ty) &= \max_{t\in[a,b]} Ty(t) \\
&\geq Ty(t) \\
&= \int_a^b G_n(t, s) f(s, y(s)) ds \\
&\geq \gamma \int_{s\in I} G_n(b, s) f(s, y(s)) ds \\
&\geq \frac{p}{C}\gamma \int_{s\in I} G_n(b, s) ds \\
&\geq p
\end{aligned}$$

using hypothesis $(D1)$.

Now we shall show that condition (ii) of Theorem 5.1 is satisfied. Since $y \in \partial\mathcal{P}(\theta, q)$, from (5.3) we have that $0 \le y(t) \le \| y \| \le \frac{q}{\gamma}$ for $t \in [a, b]$. Thus

$$\begin{aligned}
\theta(Ty) &= \max_{t\in I} Ty(t) \\
&= \max_{t\in I} \int_a^b G_n(t,s) f(s,y(s))ds \\
&\le \int_a^b G_n(b,s) f(s,y(s))ds \\
&\le \frac{q}{D} \int_a^b G_n(b,s)ds \\
&\le q
\end{aligned}$$

using hypothesis $(D2)$.

Finally using hypothesis $(D3)$, we shall show that condition (i) of Theorem 5.1 is satisfied. Since $y \in \partial\mathcal{P}(\psi, r)$, from (5.3) we have that $\min_{t\in I} y(t) = r$ and $r \le \| y \| \le \frac{r}{\gamma}$. Then

$$\begin{aligned}
\psi(Ty) &= \min_{t\in I} Ty(t) \\
&= \min_{t\in I} \int_a^b G_n(t,s) f(s,y(s))ds \\
&= \int_a^b \min_{t\in I} G_n(t,s) f(s,y(s))ds \\
&\ge \gamma \int_{s\in I} G_n(b,s) f(s,y(s))ds \\
&\ge \gamma \frac{r}{C} \int_{s\in I} G_n(b,s)ds \\
&\ge r.
\end{aligned}$$

This completes the proof. □

References

[1] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[2] R. P. Agarwal, M. Bohner and P. Wong, Eigenvalues and eigenfunctions of discrete conjugate boundary value problems, *Comp. Math. Appl.*, **38**(1999), 159-183.

[3] D. R. Anderson and J. M. Davis, Multiple solutions and eigenvalues for third-order right focal boundary value problems, *J. Math. Anal. Appl.*, **267**(2002), 135-157.

[4] R. I. Avery and J. Henderson, Two positive fixed points of nonlinear operators on ordered Banach spaces, *Comm. Appl. Nonlinear Anal.*, **8**(2001), 27-36.

[5] J. M. Davis, J. Henderson, K. R. Prasad and W. Yin, Eigenvalue intervals for nonlinear right focal problems, *Appl. Anal.*, **74**(2000), 215-231.

[6] P. W. Eloe and J. Henderson, Positive solutions for higher order ordinary differential equations, *Elec. J. Diff. Eqns.*, **1995**(1995), No. 3, 1-8.

[7] P. W. Eloe and J. Henderson, Positive solutions for (n-1, 1) conjugate boundary value problems, *Nonlinear Anal.*, **28**(1997), 1669-1680.

[8] P. W. Eloe and J. Henderson, Positive solutions and nonlinear multipoint conjugate eigenvalue problems, *Elec. J. Diff. Eqns.,* **1997**(1997), No. 3, 1-11.

[9] P. W. Eloe and J. Henderson, Positive solutions and nonlinear (k, n-k) conjugate eigenvalue problems, *Diff. Eqns. Dyn. Sys.*, **6**(1998), 309-317.

[10] L. H. Erbe, S. Hu and H. Wang, Multiple positive solutions of some boundary value problems, *J. Math. Anal. Appl.*, **184**(1994), 640-648.

[11] L. H. Erbe and M. Tang, Existence and multiplicity of positive solutions to nonlinear boundary value problems, *Diff. Eqns. Dyn. Sys.*, **4**(1996), 313-320.

[12] L. H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.*, **120**(1994), 743-748.

[13] P. Hartman, *Ordinary Differntial Equations*, John Wiley and Sons, 1964.

[14] J. Henderson and H. Wang, Positive solutions for nonlinear eigenvalue problems, *J. Math. Anal. Appl.*, **208**(1997), 252-259.

[15] D. Jiang and H. Liu, Existence of positive solutions to (k, n-k) conjugate boundary value problems, *Kyushu. J. Math.*, **53**(1998), 115-125.

[16] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff Ltd, Groningen, The Netherlands (1964).

[17] E. R. Kaufmann, Multiple positive solutions for higher order boundary value problems, *Rocky. Mtn. J. Math.*, **28**(1998), No. 3, 1017-1028.

[18] H. J. Kuiper, On positive solutions of nonlinear elliptic eigenvalue problems, *Rend. Circ. Mat. Palermo.*, **20**(1971), 113-138.

[19] W. C. Lian, F. H. Wong and C. C. Yeh, On the existence of positive solutions of nonlinear second order differential equations, *Proc. Amer. Math. Soc.*, **124**(1996), 1117-1126.

[20] K. R. Prasad, A. Kameswararao and P. Murali, Eigenvalue intervals for two-point general third order differential equation, *Bull. Inst. Math. Academia Sinica.*, **5**(2010), No. 1, 55-68.

[21] M. F. Shahed, Positive solutions of boundary value problems for nth order ordinary differential equations, *Elec. J. Qual. Theory. Diff. Eqns.*, **2008**(2008), No. 1, 1-9.

[22] P. J. Y. Wong and R. P. Agarwal, Characterization of eigenvalues for difference equations subject to lidstone conditions, *Japan. J. Industrial. Applied. Math.*, **19**(2002), No.1, 1-28.