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Internal penny shaped crack subject to varying normal pressure in non-local elasticity

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Abstract:

In this paper, the problem of internal penny shaped crack subject to varying normal pressure in non-local elasticity is solved. The stress intensity factor is calculated and the corresponding results of the classical theory are obtained as particular cases.

Keywords Penny shaped crack, Normal pressure, Non-local elasticity.

1 Introduction

A systematic approach to non-local elasticity was presented by A.C. Eringen and Edelen [1], through both the balance laws and thermodynamic and non-local variational principles. A few problems that have been studied recently using the non-local theory of elasticity include stress concentration at the tip of crack [2], line crack subject to shear [3], line crack subject to anti plane shear [4], line crack subject to varying internal pressure [5], line crack subject to varying shear load [6] and line crack subject to antiplane varying shear [7].

In this paper an attempt is made to discuss the problem of flat crack in the form of circular disc called penny shaped crack in non-local elasticity. The resulting dual integral equations are completely solved. Removing the singularity that is present in the stress distribution then stress intensity factor is calculated. And the results tallies with the result obtained in classical elasticity [8].

2 Basic eqations of non-local elasticity

Basic equations of linear, homogeneous, isotropic, non-local elastic solids with vanishing body inertia forces, are

$$t_{k_l,k} = 0 \tag{2.1}$$

$$t_{k_l} = \int_V \left[\lambda^1 \left(\left|\hat{x}^1 - \hat{x}\right|\right) e_{rr}\left(\hat{x}^1\right) \delta_{\mathrm{kl}} + 2\mu^1 \left(\left|\hat{x}^1 - \hat{x}\right|\right) e_{kl}\left(\hat{x}^1\right)\right] dv\left(\hat{x}^1\right) \tag{2.2}$$

$$e_{k_l} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right) \tag{2.3}$$

Eringen [1] has obtained the forms of $\lambda^1 \left(\left|\hat{x}^1 - \hat{x}\right|\right)$ and $\mu^1 \left(\left|\hat{x}^1 - \hat{x}\right|\right)$ for which the dispersion curves of plane waves coincide within the entire Brillonin zone with those obtained in the Born-Von-Karman theory of lattice dynamics. Accordingly

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$\left(\lambda^1, \mu^1\right) = (\lambda, \mu)\ \alpha\ \left(\left|\overset{\wedge}{\mathrm{x}^1} - \hat{x}\right|\right)$ where

$$\alpha\ \left(\left|\overset{\wedge}{\mathrm{x}^1} - \hat{x}\right|\right) = \begin{cases} \alpha_0 \left(a - \left|\overset{\wedge}{\mathrm{x}^1} - \hat{x}\right|\right) & \text{if } \left|\overset{\wedge}{\mathrm{x}^1} - \hat{x}\right| \leq a \\ 0 & \text{if } \left|\overset{\wedge}{\mathrm{x}^1} - \hat{x}\right| > a \end{cases} \tag{2.4}$$

where **a** is the lattice parameter, λ and μ are the classical Lame' constants and α_0 is normalization constant to be determined from

$$\int_V \alpha\ \left(\left|\overset{\wedge}{\mathrm{x}^1} - \hat{x}\right|\right)\ \mathrm{dv}\ \left(\overset{\wedge}{\mathrm{x}^1}\right) = 1 \tag{2.5}$$

Where

$$\alpha\ \left(\left|\overset{\wedge}{\mathrm{x}^1} - \hat{x}\right|\right) = \alpha_0\ \exp\left[-\left(\frac{\beta}{\mathrm{a}}\right)^2 \left(x_k^1 - x_k\right)\left(x_k^1 - x_k\right)^2\right] \tag{2.6}$$

and β is a constant.

For two-dimensional case (2.6) has the specific form

$$\alpha\ \left(\left|\overset{\wedge}{\mathrm{x}^1} - \hat{x}\right|\right) = \frac{1}{\pi}\ \left(\frac{\beta}{\mathrm{a}}\right)^2 \exp\left\{-\left(\frac{\beta}{\mathrm{a}}\right)^2 \left[\left(x_1^1 - x_1\right)^2 + \left(x_2^1 - x_2\right)^2\right]\right\} \tag{2.7}$$

Employing (2.4) in (2.2) we write

$$t_{kl} = \int_V \alpha\left(\left|\overset{\wedge}{x^1} - \hat{x}\right|\right)\ \sigma_{\mathrm{kl}}\left(\overset{\wedge}{x^1}\right)\ \mathrm{dv}\left(\overset{\wedge}{x^1}\right) \tag{2.8}$$

where

$$\begin{aligned} \sigma_{kl} &= \lambda e_{rr}\left(\overset{\wedge}{x^1}\right)\delta_{kl} + 2\mu e_{kl}\left(\overset{\wedge}{x^1}\right) \\ &= \lambda\ \mathrm{u}_{\mathrm{r}_1 r}\left(\overset{\wedge}{x^1}\right)\delta_{kl} + \mu\left[u_{k,l}\left(\overset{\wedge}{x^1}\right) + u_{l,k}\left(\overset{\wedge}{x^1}\right)\right] \end{aligned} \tag{2.9}$$

It is the classical Hooke's law. Substituting (2.8) in (2.1) and using Green-Gauss theorem, we obtain

$$\int_V \alpha\ \left(\left|\overset{\wedge}{\mathrm{x}^1} - \hat{x}\right|\right)\sigma_{kl,k}\left(\overset{\wedge}{x^1}\right)\ \mathrm{dv}\left(\overset{\wedge}{x^1}\right) - \oint_{\partial v}\alpha\left(\left|\overset{\wedge}{\mathrm{x}^1} - \hat{x}\right|\right)\sigma_{kl}\left(\overset{\wedge}{x^1}\right)da_k\left(\overset{\wedge}{x^1}\right) = 0 \tag{2.10}$$

Here the surface integral may be dropped if then only surface of the body is at infinity.

3 Penny shaped crack under varying normal pressure

Let an infinite elastic solid be weakened by a penny shaped crack of radius r = 1lying in the XY-Plane. Let its center be the origin of coordinates. Let the solid be subjected to an internal pressure p(r). Using the cylindrical polar coordinates (r,θ, z), taking z-symmetry into account, the points on the plane z = 0, lying outside the circle r = 1, the normal component of displacement u_z is zero. Thus the boundary conditions for this problem are

$$\begin{aligned} &\sigma_{rz}(r,0) = 0 \quad \text{for } \mathrm{r} \geq 0 \\ &\mathrm{t_{zz}}(r,0) = -p(r) \quad \text{for } 0 \leq \mathrm{r} \leq 1 \\ &\mathrm{u_z}(r,0) = 0 \quad \text{for } \mathrm{r} > 1 \end{aligned} \tag{3.1}$$

Also all components of displacement and stress should tend to zero as $\left(r^2+z^2\right)^{1/2} \to \infty$ through positive values of z.

The general solution of (2.10) in cylindrical polar coordinates is given by

$$\begin{aligned} u_r &= -\tfrac{1}{2}\int_0^\infty \left(\tfrac{\mu}{\lambda+\mu} - \xi z\right)\ \mathrm{A}\left(\xi\right)\ \mathrm{e}^{-\xi z}\ \mathrm{J}_1\left(\xi r\right) d\xi \\ u_z &= \tfrac{1}{2}\int_0^\infty \left(\tfrac{\lambda+2\mu}{\lambda+\mu} + \xi z\right)\ \mathrm{A}\left(\xi\right)\ \mathrm{e}^{-\xi z}\ \mathrm{J}_0\left(\xi r\right) d\xi \qquad \text{and } \mathrm{u}_\theta = 0 \end{aligned} \tag{3.2}$$

where $\mathrm{A}(\xi)$ is to be determined from the boundary conditions (3.1). The components of strain in cylindrical polar coordinates are

$$\begin{aligned} &e_{rr} = \tfrac{\partial u_r}{\partial r},\ \mathrm{e}_{\theta\theta} = \tfrac{u_r}{r},\ \mathrm{e}_{\mathrm{zz}} = \tfrac{\partial u_z}{\partial z} \\ &e_{\theta z} = 0,\ \mathrm{e}_{\mathrm{zr}} = \tfrac{\partial u_r}{\partial z} + \tfrac{\partial u_z}{\partial r},\ \mathrm{e}_{\mathrm{r}\theta} = 0 \end{aligned} \tag{3.3}$$

Substituting (3.2) into (3.3), we have

$$\begin{aligned} e_{rr} &= -\tfrac{1}{2}\int_0^\infty \left(\tfrac{\mu}{\lambda+\mu} - \xi z\right)\ \mathrm{A}\left(\xi\right)\ \mathrm{e}^{-\xi z}\ \left[\mathrm{J}_0\left(\xi r\right) - \tfrac{1}{r}J_1\left(\xi r\right)\right] d\xi \\ e_{\theta\theta} &= -\tfrac{1}{2r}\int_0^\infty \left(\tfrac{\mu}{\lambda+\mu} - \xi z\right)\ \mathrm{A}\left(\xi\right)\ \mathrm{e}^{-\xi z}\ \mathrm{J}_1\left(\xi r\right) d\xi \\ e_{zz} &= -\tfrac{1}{2}\int_0^\infty \left(\tfrac{\mu}{\lambda+\mu} + \xi z\right)\ \mathrm{A}\left(\xi\right)\ \mathrm{e}^{-\xi z}\ \mathrm{J}_0\left(\xi r\right) d\xi \\ e_{\theta z} &= 0 \\ e_{r\theta} &= 0 \\ e_{zr} &= -\tfrac{1}{2}z\int_0^\infty A\left(\xi\right)\ \xi^2\ e^{-\xi z}\ \ J_1\left(\xi r\right)\ \ d\xi \end{aligned} \tag{3.4}$$

Using (3.2), from (2.9) the local stresses takes the following form:

$$\begin{aligned} \sigma_{rz} &= -\mu z H_1\ \left[\xi\ \mathrm{A}\left(\xi\right)\ e^{-\xi z},\ \xi \to \mathrm{r}\right] \\ \sigma_{\theta z} &= 0 \\ \sigma_{zz} &= -\mu\ \mathrm{H}_0\left[A\left(\xi\right)\left(1+\xi z\right) e^{-\xi z}, \xi \to \mathrm{r}\ \right] \end{aligned} \tag{3.5}$$

where $H_\upsilon\left[G\left(r,z\right), r \to \xi\right]$ is the Hankel transform.

$\int_0^\infty r\ \mathrm{G}\left(\mathrm{r,z}\right)\ \mathrm{J}_\upsilon\left(r\xi\right)$ dr of order υ with respect to the variable r of an axisymmetric function G(r,z).

The z-components of the non-local stress field can be expressed as

$$\begin{aligned} t_{rz} &= \int_V \alpha\left(\left|\hat{x}^1 - \hat{x}\right|\left[\sigma_{rz}\left(\hat{x}^1\right)\cos\left(\theta^1-\theta\right) + \sigma_{\theta z}\left(\hat{x}^1\right)\sin\left(\theta-\theta^1\right)\right]\right) dv\left(\hat{x}^1\right) \\ t_{\theta z} &= \int_V \alpha\left(\left|\hat{x}^1 - \hat{x}\right|\left[\sigma_{rz}\left(\hat{x}^1\right)\sin\left(\theta^1-\theta\right) + \sigma_{\theta z}\left(\hat{x}^1\right)\cos\left(\theta^1-\theta\right)\right]\right) dv\left(\hat{x}^1\right) \\ t_{zz} &= \int_V \alpha\left(\left|\hat{x}^1 - \hat{x}\right|\right)\ \sigma_{\mathrm{zz}}\left(\hat{x}^1\right) dv\left(\hat{x}^1\right) \end{aligned} \tag{3.6}$$

where we introduced the cylindrical coordinates at $\hat{x}$and$\hat{x}^1$ by

$$\begin{aligned} &x_1 = r\cos\theta,\ \mathrm{x}_2 = r\sin\theta,\ \mathrm{x}_3 = z \\ &x_1^1 = r^1\cos\theta^1,\ \mathrm{x}_2^1 = r^1\sin\theta^1,\ \mathrm{x}_3^1 = z^1 \quad \text{and} \\ &\left|\hat{\mathrm{x}}^1 - \hat{x}\right| = \left[\left(r^1\right)^2 + r^2 - 2rr^1\cos\left(\theta^1-\theta\right) + \left(z^1-z\right)^2\right]^{1/2} \\ &dv\left(\hat{\mathrm{x}}^1\right) = r^1\ \mathrm{dr}^1\ \mathrm{d}\theta^1\ \mathrm{dz}^1 \end{aligned} \tag{3.7}$$

Using $(3.5)_2$, (3.6) takes the following form:

$$
\begin{aligned}
t_{rz} &= \int_V \alpha\left(\left|\hat{x}{}^1-\hat{x}\right|\right) \ \sigma_{rz}\left(\hat{x}{}^1\right)\ \cos\left(\theta^1-\theta\right)\ \mathrm{dv}\left(\hat{x}{}^1\right) \\
t_{\theta z} &= \int_V \alpha\left(\left|\hat{x}{}^1-\hat{x}\right|\right) \ \sigma_{\theta z}\left(\hat{x}{}^1\right)\ \sin\left(\theta^1-\theta\right)\ \mathrm{dv}\left(\hat{x}{}^1\right) \\
t_{zz} &= \int_V \alpha\left(\left|\hat{x}{}^1-\hat{x}\right|\right) \ \sigma_{zz}\left(\hat{x}{}^1\right)\ \mathrm{dv}\left(\hat{x}{}^1\right)
\end{aligned}
\tag{3.8}
$$

The evaluation of the integrals in (3.8) depends on the form of the function $\alpha\left(\left|\hat{x}{}^1-\hat{x}\right|\right)$. In this study, we take

$\alpha\left(\left|\hat{x}{}^1-\hat{x}\right|\right) = \alpha_0 \exp\left[-\left(\frac{\beta}{\mathrm{a}}\right)^2 \ \left|\hat{x}{}^1-\hat{x}\right|^2\right]$ where $\alpha_0 = \pi^{-3/2}\left(\frac{\beta}{a}\right)^3$ and using standard integrals [9]we can express (3.8) as

$$
\begin{aligned}
t_{rz} &= -\mu e^{-mz^2}\int_0^\infty A(\xi)\,\xi^2\left[\frac{1}{2(\pi m)^{1/2}} - \frac{\gamma}{4m}e^{\left(\frac{\gamma^2}{4m}\right)\left(1-\phi\left(\frac{\gamma}{2\sqrt{m}}\right)\right)}\right]e^{\frac{\xi^2}{4m}}J_1(\xi r)\,d\xi \\
t_{\theta z} &= 0 \\
t_{zz} &= -\mu\frac{e^{-mz^2}}{4}\int_0^\infty A(\xi)\,\xi\left[e^{\left(\frac{\gamma^2}{4m}\right)\left(1-\phi\left(\frac{\gamma}{2\sqrt{m}}\right)\right)\left(1-\frac{\xi r}{2m}\right)} + (\pi m)^{-1/2}\,\xi\right] e_0^{-\frac{\xi^2}{4m}J}(\xi r)\ d\xi
\end{aligned}
\tag{3.9}
$$

where $\gamma = \xi - 2mz$ and $\phi(\mathrm{z})$ is the error function defined by

$$
\phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp\left(-t^2\right)\ \mathrm{dt}
$$

The normal component of non-local stress in the plane z=0 can be expressed as

$$
\begin{aligned}
t_{zz}(r,0) = -\left(\tfrac{\mu}{4}\right)\int_0^\infty \xi\ \mathrm{A}(\xi)\ \mathrm{K}(\xi,\mathrm{m})\ \mathrm{J}_0(\xi r)\ \mathrm{d}\xi \quad \text{where} \\
\mathrm{K}(\xi,\mathrm{m}) = \left[1-\phi\left(\tfrac{\xi}{2\sqrt{m}}\right)\right]\left(1-\tfrac{\xi^2}{2m}\right) + (\pi m)^{-1/2}\quad \xi\exp\left(-\tfrac{\xi^2}{4m}\right)
\end{aligned}
\tag{3.10}
$$

Thus, the boundary condition (3.1) and (3.1) reduces to the dual integral equations like

$$
\begin{aligned}
&\int_0^\infty \xi\ \mathrm{A}(\xi)\ \ [1+\mathrm{k}(\in\xi)\ ]\ \mathrm{J}_0(\xi r)\ \mathrm{d}\xi\ = 4\tfrac{\mathrm{p(r)}}{\mu}\ \text{ for } 0\le \mathrm{r}\le 1 \\
&\int_0^\infty \mathrm{A}(\xi)\ \ \mathrm{J}_0(\xi r)\ \mathrm{d}\xi\ = 0\ \text{ for } \mathrm{r}>1
\end{aligned}
\tag{3.11}
$$

where

$\in = \frac{1}{2\sqrt{m}}$

$k(\in\xi) = K(\xi,m) - 1 = -2(\in\xi)^2\ \ [1-\phi(\in\xi)] - \phi(\in\xi) + 2\pi^{-1/2}(\in\xi)\ \exp\left(-\in^2\xi^2\right)$

For $\in = 0$, $\mathrm{k}(\in\xi) = 0$ and the dual integral equations (3.11) reduces to the dual integral equations in classical elasticity [8].

The solution of the dual integral equations (3.11) can be obtained by reducing the problem to that of solving the Fredholm integral equation of the second kind [10].

$$
h(r) + \int_0^1 h(u)\ \mathrm{L}(\mathrm{r},\mathrm{u})\,du = \frac{4}{\mu\sqrt{\pi}}\int_0^r \frac{u\ \mathrm{p}(\mathrm{u})}{\sqrt{(r^2-u^2)}}du
\tag{3.12}
$$

for the function h(r), where

$L(r,u) = (2\pi)^{-1/2}\{k_c(|r-u|) - k_c(|r+u|)\}$ where $k_c(\xi)$is the Fourier cosine transform of $k(\in t)$. Then $A(\xi)$ is given by

$$A(\xi) = \frac{2}{\sqrt{\pi}} \int_0^1 h(t)\ \sin(\xi \mathrm{t})\ \mathrm{dt} \tag{3.13}$$

For $\in$=0, k($\in$t) =0 and (3.12) gives

$$h_0(r) = \frac{4}{\mu\sqrt{\pi}} \int_0^r \frac{u\ \mathrm{p(u)}}{\sqrt{(r^2-u^2)}} du \tag{3.14}$$

Hence from (3.13) and (3.14), $A_0(\xi)$ is given by

$$A_0(\xi) = \frac{4}{\mu} \int_0^1 g(t)\ \sin\ (\xi \mathrm{t})\ \mathrm{dt} \quad \text{where } \mathrm{g(t)} = \frac{2}{\pi} \int_0^t \frac{u\ \mathrm{p(u)}}{\sqrt{(t^2-u^2)}} du$$

Since $\in$ is very small number, $k(\in \xi)$ is very small compared to unity and hence it is sufficient to take the solution of (3.11) to calculate the stress field as

$$A(\xi) = \frac{4}{\mu} \int_0^1 g(t)\ \sin\ (\xi \mathrm{t})\ \mathrm{dt} \tag{3.15}$$

Neglecting the higher order terms in the expression of $k(\in \xi)$ and retaining upto second order and by using (3.15), we can express (3.10) as

$$t_{zz}(r,0) = -\int_0^1 g(t) \left[\int_0^\infty \xi\ \mathrm{J}_0(\xi r)\ \left(1 - 2\in^2 \xi^2\right)\ \sin(\xi \mathrm{t})\ \mathrm{d}\xi\right] \mathrm{dt} \tag{3.16}$$

Since $\int_0^\infty J_0(\xi r)\ \cos(\xi \mathrm{t})\ \mathrm{d}\xi = \frac{H(r-t)}{\sqrt{(r^2-t^2)}}$, (3.16) can be expressed as

$$t_{zz}(r,0) = \int_0^1 g(t) \left[\frac{\mathrm{d}}{\mathrm{dt}} \frac{H(r-t)}{\sqrt{(r^2-t^2)}} + 2\in^2 \frac{d^3}{dt^3} \frac{H(r-t)}{\sqrt{(r^2-t^2)}}\right] \mathrm{dt} \tag{3.17}$$

For r > t and H(r-t) = 1, integrating by parts we get

$$t_{zz}(r,0) = \frac{g(1)}{\sqrt{(r^2-1)}} - \int_0^1 \frac{g^1(t)}{\sqrt{(r^2-t^2)}} dt + 2\in^2 \left[\frac{r^2 g(1)}{(r^2-1)^{5/2}} - \int_0^1 \frac{g^1(t)}{\sqrt{(r^2-t^2)}} \left(\frac{t^2}{(r^2-t^2)} + 1\right) dt\right] \tag{3.18}$$

Now if we choose the internal pressure p(r) in such a way that g(t) is differentiable in the neighbourhood of t = 1, (**??**) takes the following form:

$$t_{zz}(r,0) = \frac{g(1)}{\sqrt{(r^2-1)}} - 0(1) + 2\in^2 \left[\frac{r^2 g(1)}{(r^2-1)^{5/2}} - 0(1)\right] \quad \text{as r} \to 1^+ \tag{3.19}$$

If the crack is of radius c, then (3.19) becomes

$$t_{zz}(r,0) = \frac{g(c)}{\sqrt{(r^2-c)}} - 0(1) + 2\in^2 \left[\frac{r^2 g(c)}{(r^2-c)^{5/2}} - 0(1)\right] \tag{3.20}$$

Then by removing the singularity of order (5/2), the non-local stress intensity factor I^1 is given by

$$I^1 = \frac{4 \in^2}{\pi c} \int_0^c \frac{r \ \mathrm{p}\left(\mathrm{r}\right)}{\sqrt{(c^2 - r^2)}} \ \mathrm{dr}$$

For $\in = 0$, (3.19) takes the following form:

$$t_{zz}\left(r, 0\right) = \frac{g\left(1\right)}{\sqrt{(r^2 - 1)}} - \int_0^1 \frac{g^1\left(t\right)}{\sqrt{(r^2 - t^2)}} dt \tag{3.21}$$

Removing the singularity of order (1/2), the local stress intensity factor I is given by
I = g(**??**) which coincides with the classical result [8].

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