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On the Preconditioning of the AOR Iterative Methods for M-matrices

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Abstract:

In this paper, we propose a preconditioned AOR iterative method for solving the systems of linear equations with M-matrix coefficient. Some numerical results are given to compare the proposed preconditioner with an available preconditioner.
Keywords: linear systems, preconditioner, convergence, M-matrix, AOR iterative method.

1 Introduction

Consider the system of linear equations

$$Ax = b, \tag{1.1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is nonsingular and $b \in \mathbb{R}^n$. Without loss of generality, let the matrix A of Eq. (1.1) can be written as

$$A = I - L - U,$$

where I is the identity matrix, $-L$ and $-U$ are strictly lower and upper triangular matrices obtained from A, respectively. The accelerated overrelaxation (AOR) iterative method [4] to solve Eq. (1.1) is given by

$$x^{(i+1)} = \mathcal{L}_{\gamma,\omega} x^{(i)} + \omega(I - \gamma L)^{-1} b, \tag{1.2}$$

with the iteration matrix

$$\mathcal{L}_{\gamma,\omega} = (I - \gamma L)^{-1}[(1-\omega)I + (\omega - \gamma)L + \omega U],$$

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and ω and γ are real parameters with $\omega \neq 0$. For certain values of the parameters ω and γ the AOR iterative method results in the successive overrelaxation (SOR), Gauss-Seidel and Jacobi methods [4]. To improve the convergence rate of the basic iterative methods, several preconditioned iterative methods have been proposed in the literature. In these methods the original system is transformed into the preconditioned form

$$PAx = Pb, \tag{1.3}$$

where $P = (p_{ij}) \in \mathbb{R}^{n\times n}$ is nonsingular and nonnegative with $p_{ii} = 1$, $i = 1, \cdots, n$. Then, a basic iterative method is applied to solve Eq. (1.3). If we use the AOR iterative method to solve (1.3), then we obtain

$$x^{(k+1)} = \tilde{\mathcal{L}}_{\gamma,\omega} x^{(k)} + \omega(\tilde{D} - \gamma\tilde{L})^{-1}\tilde{b}, \tag{1.4}$$

where

$$\tilde{\mathcal{L}}_{\gamma,\omega} = (\tilde{D} - \gamma\tilde{L})^{-1}[(1-\omega)\tilde{D} + (\omega-\gamma)\tilde{L} + \omega\tilde{U}],$$

and $\tilde{A} = PA = \tilde{D} - \tilde{L} - \tilde{U}$ with $\tilde{D}$, $\tilde{L}$ and $\tilde{U}$ being diagonal, strictly lower and strictly upper triangular matrices, respectively, and $\tilde{b} = Pb$.

For convenience, some notations, definitions and results that will be used in the sequel are given below. A matrix A is called nonnegative, semi-positive and positive if each entry of A is nonnegative, nonnegative but at least a positive entry and positive, respectively. We denote them by $A \geq 0$, $A > 0$ and $A \gg 0$. Similarly, for n-dimensional vectors, by identifying them with $n \times 1$ matrices, we can also define $x \geq 0$, $x > 0$ and $x \gg 0$. Additionally, we denote the spectral radius of A by $\rho(A)$.

Definition 1.1. ([1]) A matrix $A = (a_{ij})$ is said to be a *Z-matrix* if for any $i \neq j$, $a_{ij} \leq 0$; an *M-matrix* if it is a Z-matrix with $a_{ii} > 0$ for $i = 1, \ldots, n$, A is nonsingular and $A^{-1} \geq 0$.

Lemma 1.1. *([1]) Let A be a Z-matrix. A is an M-matrix if and only if there is a vector $x \gg 0$ such that $Ax \gg 0$.*

In [11], Wang and Song proposed the following interesting theorem concerning the preconditioned AOR iterative method.

Theorem 1.2. *([11]) Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix. Assume that $0 \leq \gamma \leq \omega \leq 1$, $\omega \neq 0$, $P = (p_{ij}) \geq 0$ is a nonsingular preconditioner with $p_{ij} = -\alpha_{ij}a_{ij}$, $0 \leq \alpha_{ij} \leq 1$, for $1 \leq i \neq j \leq n$ and $p_{ii} = 1$ for $1 \leq i \leq n$. Then, we have*

$$\rho(\tilde{\mathcal{L}}_{\gamma,\omega}) \leq \rho(\mathcal{L}_{\gamma,\omega}) < 1.$$

The preconditioner P for the special cases of the parameters α_{ij} results in many known preconditioners. As mentioned in [11], if $\alpha_{ij} = 0$, $j \neq 1$, then P is reduced to the preconditioner proposed first by Milaszewicz [8] and generalized in [5]. If $\alpha_{ij} = 0$, $j \neq i+1$, and $j \neq i$, then P is reduced to the preconditioner presented in [3] and parameterized in [6]. If $\alpha_{ij} = 0$ for $i > j$, then P is the preconditioner proposed in [7], etc. Many of these preconditioners are lower or upper triangular matrices. In [10], Usui et al. proposed the preconditioner $P = I + L$. In this paper, we improve this preconditioner.

The rest of the paper is organized as follows. In section 2, we our main results. Section 3 is devoted to some numerical results. Concluding remarks are given in section 4.

2 Main results

In this section, we assume that the matrix A of Eq. (1.1) is an M-matrix and the preconditioner P is of the form $P = I + L$. By Theorem 1.2, P is a suitable preconditioner for Eq. (1.1). We first give the next lemma.

Theorem 2.1. *If A is an M-matrix and $P = I + L$, then PA is also an M-matrix.*

Proof. First, we see that

$$(PA)_{ij} = a_{ij} - \sum_{k=1}^{i-1} a_{ik}a_{kj} \leq 0.$$

This shows that the matrix PA is a Z-matrix. Since A is an M-matrix, by Lemma 1.1, there exists $x \gg 0$ such that $Ax \gg 0$. Evidently, $PAx \gg 0$ and Lemma 1.1 shows that PA is an M-matrix. □

We now investigate the properties of the matrix $\tilde{A} = PA$. We have

$$\tilde{A} = (I + L)A = I - U - L^2 - LU.$$

Let $LU = \bar{D} + \bar{L} + \bar{U}$, where $\bar{D}$, $\bar{L}$ and $\bar{U}$ are diagonal, strictly lower and strictly upper triangular matrices, respectively. Obviously, $\bar{D}, \bar{L}, \bar{U} \geq 0$. The matrix $\tilde{A}$ can be written as

$$\tilde{A} = (I - \bar{D}) - (L^2 + \bar{L}) - (U + \bar{U}).$$

By Theorem 2.1, $\tilde{A}$ is an M-matrix. Therefore,

$$0 < (I - \bar{D})_{ii} \leq 1, \quad i = 1, \ldots, n.$$

This shows that $I - \bar{D}$ is nonsingular. Now, we have

$$(I - \bar{D})^{-1}PA = I - (I - \bar{D})^{-1}(L^2 + \bar{L}) - (I - \bar{D})^{-1}(U + \bar{U}).$$

It is easy to see that the matrix $(I - \bar{D})^{-1}PA$ is an M-matrix. Hence, by Theorem 1.2,

$$I + (I - \bar{D})^{-1}(U + \bar{U}),$$

is a suitable preconditioner for the system

$$(I - \bar{D})^{-1}PAx = (I - \bar{D})^{-1}Pb. \tag{2.1}$$

On the other hand, we have

$$I + (I - \bar{D})^{-1}(U + \bar{U}) \geq I + (I - \bar{D})^{-1}U \geq I + U =: Q.$$

This shows that, one may use the matrix $Q = I + U$ as a left preconditioner for the system (2.1). In this case, we should compute the diagonal matrix $(I - \bar{D})^{-1}$ and in this case the computational cost is high. To overcome on this problem we use the preconditioner Q as a right preconditioner for the system (2.1) and neglect the diagonal preconditioner $(I - \bar{D})^{-1}$. In this case the proposed preconditioned system may be written as

$$PAQy = Pb, \qquad x = Qy. \tag{2.2}$$

In the case that the coefficient matrix A of the original system is symmetric, we have $Q = P^T$ and the coefficient matrix of the system (2.2) would be symmetric. This means that the proposed preconditioner preserves the symmetry.

Table 1: Numerical results for Example 1.

n	No Precon.	$PAx = Pb$	$PAQy = Pb,\ x = Qy$
2500	14(0.27)	12(0.28)	7(0.19)
10000	39(2.75)	25(2.41)	14(1.81)
22500	61(10.16)	46(9.88)	22(5.25)
40000	72(20.95)	67(23.75)	31(13.16)

3 Numerical experiments

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes. In all the experiments, vector $b = A(1, 1, \dots, 1)^T$ was taken to be the right-hand side of the linear system and a null vector as an initial guess. The stopping criterion used was always

$$\frac{\|b - Ax_k\|_2}{\|b\|_2} < 10^{-10}.$$

We present two examples to compare the numerical results of the GMRES(m) [9] for solving Eqs. (1.1), (1.3) and (2.2).

Example 1. We consider the two dimensional convection-diffusion equation (see [11])

$$-(u_{xx} + u_{yy}) + 2e^{x+y}(xu_x + yu_y) = f(x, y), \quad \text{in} \quad \Omega = (0, 1) \times (0, 1),$$

with the homogeneous Dirichlet boundary conditions. Discretization of this equation on a $(p+1) \times (p+1)$ grid, by using the second order centered differences for the second and first order differentials gives a linear system of equations of order $n = p^2$ with n unknowns.

We first assume $p = 30$. In this case, the coefficient matrix of the obtained system is of dimension $n = 900$. In Figure 1, we depict the eigenvalues of A, $(I + L)A$ and $(I + L)A(I + U)$. This figure shows that the spectrum of the preconditioned matrix $(I + L)A(I + U)$ is more clustered than those of the matrices A and $(I + L)A$.

We also set $p = 50, 100, 150, 200$ which yield four matrices of order $n = 2500$, 10000, 22500, and 40000, respectively. Number of iterations for the convergence of the GMRES(30) method for solving Eqs. (1.1), (1.3) and (2.2) are given Table 1. We mention that, the CPU times (in seconds) are reported in the parenthesis. As we observe our preconditioner is more effective than the preconditioner $P = I + L$.

Example 2. (see [2]) In this example, we consider the three-dimensional convection-diffusion equation

$$-(u_{xx} + u_{yy} + u_{zz}) + q(u_x + u_y + u_z) = f(x, y, z),$$

on the unit cube $\Omega = [0, 1] \times [0, 1] \times [0, 1]$, with constant q and subject to Dirichlet-type boundary conditions. Discretizing this equation with seven-point finite difference (the centered differences to the diffusive terms and first-order upwind approximation to the convective terms) and assuming the numbers of grid points in all three directions are the

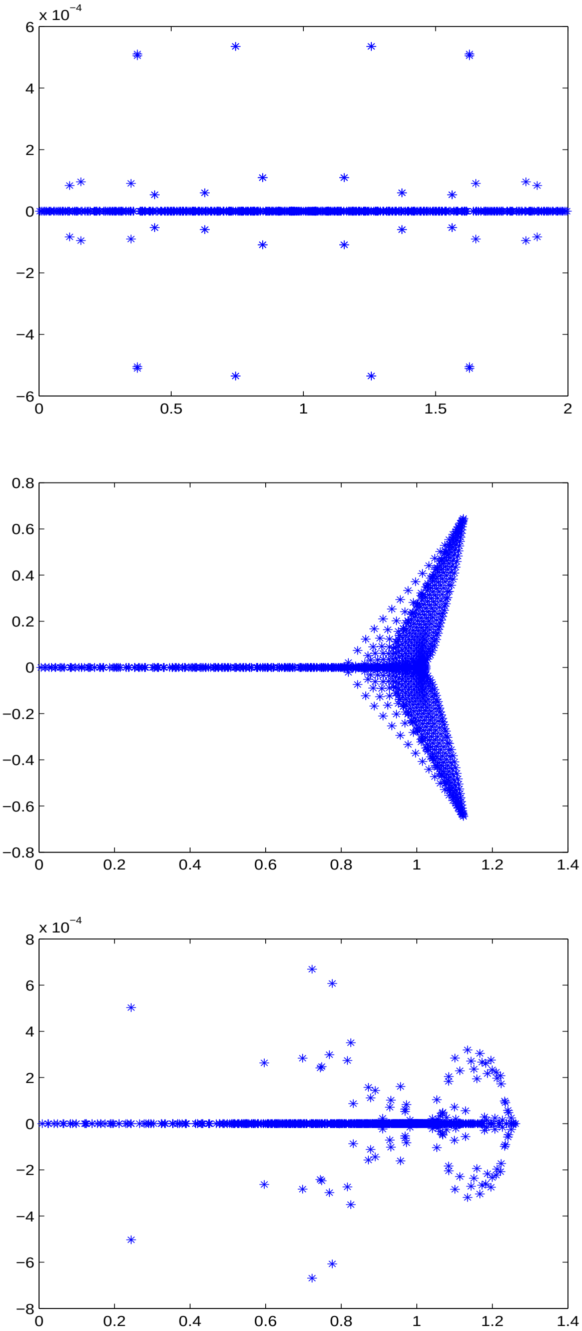

Figure 1: Spectra of A (top), spectra of $(I+L)A$ (middle) and spectra of $(I+L)A(I+U)$ (down) for Example 1.

Table 2: Numerical results for Example 2.

n	No Precon.	$PAx = Pb$	$PAQy = Pb,\ x = Qy$
15625	33(1.05)	27(1.28)	12(0.70)
27000	44(2.52)	36(2.84)	15(1.52)
42875	58(5.41)	46(5.89)	18(2.89)
64000	73(10.08)	57(10.67)	24(6.06)

same and equal to $p+1$, we obtain a system of linear equations with coefficient matrix A of order $n = p^3$. The matrix A can be written as

$$A = T_x \otimes I_p \otimes I_p + I_p \otimes T_y \otimes I_p + I_p \otimes I_p \otimes T_z,$$

where

$$T_x = \text{tridiag}(t_2, t_1, t_3), \quad T_y = \text{tridiag}(t_2, 0, t_3), \quad T_z = \text{tridiag}(t_2, 0, t_3),$$

in which

$$t_1 = 6 + 6r, \quad t_2 = -1 - 2r, \quad t_3 = -1,$$

with $r = (qh)/2$. We set $q = 100$.

The spectrum pictures of the matrices A, $(I+L)A$ and $(I+L)A(I+U)$ for $n = 10^3 = 1000$ are plotted in Figure 2. As we observe the eigenvalues of the original matrix A are real (imaginary parts of the eigenvalues are almost equal to zero). Whereas, several eigenvalues of the matrix $(I+L)A$ are complex but all of the eigenvalues of the preconditioned matrix $(I+L)A(I+U)$ are all real. Moreover, the spectrum of the preconditioned matrix $(I+L)A(I+U)$ is more clustered than those of the matrices A and $(I+L)A$.

We also set $p = 25, 30, 35$ and 40. Therefore four matrices of dimensions $n = 15625, 27000, 42875$ and 64000 are obtained. Numerical results are given in Table 2. All the notations and assumptions are as before. Here we mention that we have used the GMRES(10) iterative method. As we observe our preconditioner is more effective than the preconditioner $P = I + L$.

4 Conclusion

We have proposed a two-sided preconditioner for M-matrices. We have shown that our preconditioner improves the convergence rate of the AOR iterative method. Numerical results have been presented to show the effectiveness of the preconditioner. The presented numerical results show that the proposed preconditioner is more effective than that of an available preconditioner.

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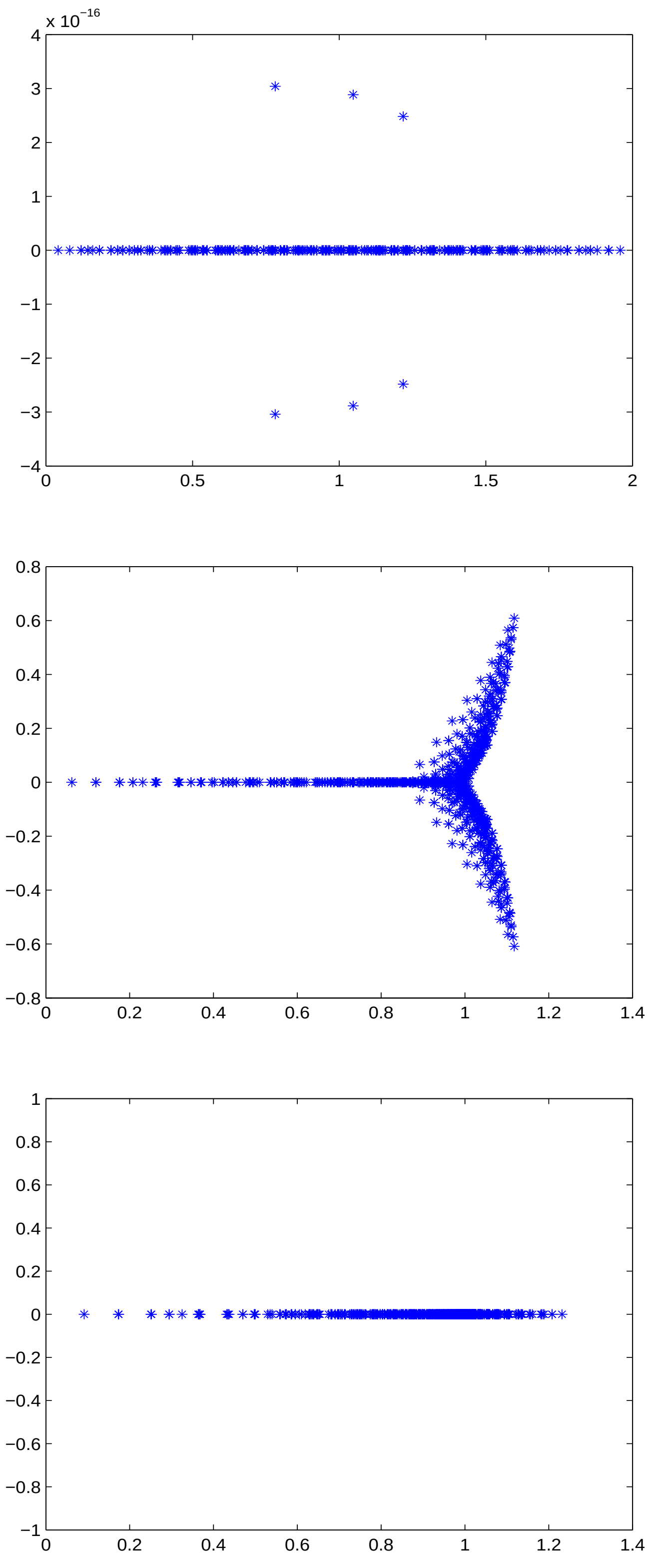

Figure 2: Spectra of A (top), spectra of $(I+L)A$ (middle) and spectra of $(I+L)A(I+U)$ (down) for Example 2.

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