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Numerical Solutions of Linear Fredholm Integral Equations Using Half-Sweep Arithmetic Mean Method

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Abstract:

The main aim of this paper is to examine the effectiveness of the Half-Sweep Arithmetic Mean (HSAM) iterative method in solving dense linear systems generated from the discretization of the first and second kind linear Fredholm integral equations. The formulation and implementation of the proposed method is also presented. Numerical tests and comparisons with other existing methods are given to illustrate the effectiveness of the proposed method.
Keywords: Linear Fredholm equations, half-sweep iteration, quadrature, Arithmetic Mean.

1 Introduction

Studies on iterative methods play an important role to accelerate convergence rate in solving any system of linear equations generated by discretizing mathematical models in science and engineering problems. The discovery of the half-sweep iteration method has been inspired by Abdullah [2] via the Explicit Decoupled Group (EDG) iterative method for solving a sparse linear system obtained from the discretization of the two-dimensional Poisson equations. Following to that, an application of the half-sweep iteration concept with iterative methods has been extensively studied by many researchers. For instance, further studies of the half-sweep iteration concept, especially with EDG method in solving partial differential equations problems have been conducted, see [3, 7, 19].

Besides EDG method, effectiveness of the half-sweep iteration concept with the two-stage iterative methods in solving the sparse linear system has been also studied. Two-stage iterative method also called as inner and outer iteration schemes have been proposed widely to be one of the feasible and successful classes of numerical algorithms for solving any linear system. In 2004, the standard Arithmetic Mean (AM) method [14] which is one of the two-stage iterative methods has been modified by combining the concept of the half-sweep iteration and then called as the Half-Sweep Arithmetic Mean (HSAM) method to solve two-point boundary value problems [15]. Standard AM method can be also named as the Full-Sweep Arithmetic Mean (FSAM) method. From the perspective of HSAM method, many researches have been carried out to examine the effectiveness of the HSAM method [16, 17]. However, in this paper, the application of the HSAM iterative method for solving first and second kind linear integral equations type of Fredholm are examined.

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The remainder of this paper is organized in following way. In next section, the formulation of the full- and half-sweep quadrature approximation equations will be elaborated. The latter sections of this paper will discuss the formulations of the FSAM and HSAM iterative methods in solving linear systems generated from discretization of the Eq. (2.1) and then some numerical results will be shown to assert the performance of the proposed method. Besides that, analysis on computational complexity is also included and the concluding remarks are given in final section.

2 Quadrature Approximation Equations

Generally, linear integral equations of Fredholm type in the standard form can be defined as follows

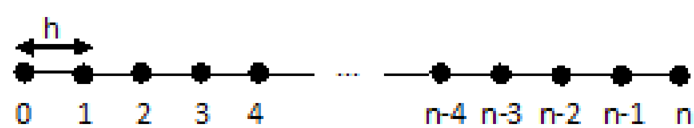
$$\lambda y\left(x\right)+\int_{\Gamma}K\left(x,t\right)y\left(t\right)dt=f\left(x\right),\Gamma=[a,b] \tag{2.1}$$

where the parameter λ, kernel $K\left(x,t\right)\in L_2[a,b]\times[a,b]$ and free term $f\left(x\right)\in L_2[a,b]$ are given, and $y\left(x\right)$ is the unknown function to be determined. $K\left(x,t\right)$ is called Fredholm kernel if the kernel in Eq. (2.1) is continuous on the square $S=\{a\le x\le b,\ a\le t\le b\}$ or at least square integrable on this square [11]. By referring Eq. (2.1), linear Fredholm integral equations can be classified as the first and second kind whenever $\lambda=0$ and $\lambda\ne0$ respectively.

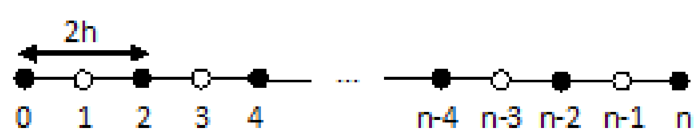
By considering numerical technique, there are many methods can be used to discretize the linear Fredholm integral equations into linear systems such as quadrature [1, 4, 8] and projection [6, 9, 10, 13] methods. However, in this paper discretization scheme based on quadrature method was used to discretize the linear Fredholm integral equations of the first and second kind. Basically, quadrature method can be defined as follows

$$\int_a^b y\left(t\right)\,dt=\sum_{j=0}^{n}A_j y\left(t_j\right)+\varepsilon_n\left(y\right) \tag{2.2}$$

where t_j $(j=0,\,1,\,2,\,\cdots,\,n)$ is the abscissas of the partition points of the integration interval, A_j $(j=0,\,1,\,2,\,\cdots,\,n)$ is numerical coefficients that do not depend on the function $y\left(t\right)$ and $\varepsilon_n\left(y\right)$ is the truncation error of Eq. (2.2). Meanwhile, Fig. 1 shows the finite grid networks in order to form the full- and half-sweep quadrature approximation equations.



(a)



(b)

Figure 1: a) and b) show distribution of uniform node points for the full- and half-sweep cases respectively

Based on Fig. 1, the full- and half-sweep iterative methods will compute approximate values onto node points of type only until the convergence criterion is reached. Then, other approximate solutions at remaining points (points of the different type,) can be computed using the direct method [2, 3, 7, 19].

By applying Eq. (2.2) into Eq. (2.1) and neglecting the error, $\varepsilon_n(y)$, a system of linear equations can be formed for approximation values of $y(t)$. The following linear system generated using quadrature method can be easily shown in matrix form as follows

$$M \underset{\sim}{y} = \underset{\sim}{f} \tag{2.3}$$

where

$$M = \begin{bmatrix} \lambda + A_0 K_{0,0} & A_p K_{0,p} & A_{2p} K_{0,2p} & \cdots & A_n K_{0,n} \\ A_0 K_{p,0} & \lambda + A_p K_{p,p} & A_{2p} K_{p,2p} & \cdots & A_n K_{p,n} \\ A_0 K_{2p,0} & A_p K_{2p,p} & \lambda + A_{2p} K_{2p,2p} & \cdots & A_n K_{2p,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_0 K_{n,0} & A_p K_{n,p} & A_{2p} K_{n,2p} & \cdots & \lambda + A_n K_{n,n} \end{bmatrix}_{((\frac{n}{p})+1)x((\frac{n}{p})+1)},$$

$$\underset{\sim}{y} = \left[y_0 \quad y_p \quad y_{2p} \quad \cdots \quad y_{n-2p} \quad y_{n-p} \quad y_n \right]^T,$$

and

$$\underset{\sim}{f} = \left[f_0 \quad f_p \quad f_{2p} \quad \cdots \quad f_{n-2p} \quad f_{n-p} \quad f_n \right]^T.$$

In order to facilitate the formulation of the full- and half-sweep quadrature approximation equations for problem (2.1), further discussion will be restricted onto repeated trapezoidal (RT) scheme, which is based on linear interpolation formula with equally spaced data. Based on RT scheme, numerical coefficient A_j will satisfy the following relation

$$A_j = \begin{cases} \frac{1}{2} ph, & j = 0, n \\ ph, & otherwise \end{cases} \tag{2.4}$$

where the constant step size, h is defined as follows

$$h = \frac{b-a}{n} \tag{2.5}$$

and n is the number of subintervals in the interval $[a, b]$. Meanwhile, the value of p, which corresponds to 1 and 2, represents the full- and half-sweep cases respectively.

3 Arithmetic Mean Methods

As stated in previous section, AM methods are one of the two-stage iterative methods and the iterative process involves of solving two independent systems such as $\underset{\sim}{y}^{\,1}$ and $\underset{\sim}{y}^{\,2}$. To develop the formulation of AM methods, express the coefficient matrix M as the matrix sum

$$M = L + D + U \tag{3.1}$$

where L, $D$and U are the strictly lower triangular, diagonal and strictly upper triangular matrices respectively. Thus, by adding positive acceleration parameter, ω the general scheme for FSAM and HSAM methods is defined by

$$\left.\begin{aligned} (D + \omega L) \underset{\sim}{y}^{\,1} &= ((1-\omega) D - \omega U) \overset{(k)}{\underset{\sim}{y}} + \omega f \\ (D + \omega U) \underset{\sim}{y}^{\,2} &= ((1-\omega) D - \omega L) \overset{(k)}{\underset{\sim}{y}} + \omega f \\ \overset{(k+1)}{\underset{\sim}{y}} &= \frac{1}{2} \left(\underset{\sim}{y}^{\,1} + \underset{\sim}{y}^{\,2} \right) \end{aligned}\right\} \tag{3.2}$$

where $\underset{\sim}{\overset{(0)}{y}}$ is an initial vector approximation to the solution and $0 < \omega < 2$.

The AM methods require a slight additional computational effort of the sum of two matrices at each iteration k, but its rate of convergence is relatively insensitive to the exact choice of the parameter ω [14]. Practically, the value of ω will be determined by implementing some computer programs and then choose one value of ω, where its number of iterations is the smallest. By determining values of matrices L, $D$and U as stated in Eq. (3.1), the general algorithm for FSAM and HSAM iterative methods to solve problem (2.1) would be generally described in Algorithm 1.
Algorithm 1: FSAM and HSAM methods

1. Level (2.1)

For $i = 0,\ p,\ 2p,\ \cdots, n-2p,\ n-p,\ n$ and $j = 0,\ p,\ 2p,\ \cdots, n-2p,\ n-p,\ n$, Calculate

$$y_i^1 \leftarrow \begin{cases} (1-\omega)\, y_i^{(k)} + \left(\omega \sum_{j=p}^{n} A_j K_{ij} y_j^{(k)} + \omega\, f_i\right) \Big/ (\lambda + A_i K_{ii}) & , i = 0 \\ (1-\omega)\, y_i^{(k)} + \left(\omega \sum_{j=0}^{n-p} A_j K_{ij} y_j^{1} + \omega\, f_i\right) \Big/ (\lambda + A_i K_{ii}) & , i = n \\ (1-\omega)\, y_i^{(k)} + \left(\omega \sum_{j=0}^{i-p} A_j K_{ij} y_j^{1} + \omega \sum_{j=i+p}^{n} A_j K_{ij} y_j^{(k)} + \omega\, f_i\right) \Big/ (\lambda + A_i K_{ii}) & , others \end{cases}$$

1. Level (2.2)

For $i = n,\ n-p,\ n-2p\ \cdots,\ 2p,\ p,\ 0$ and $j = 0,\ p,\ 2p,\ \cdots, n-2p,\ n-p,\ n$, Calculate

$$y_i^2 \leftarrow \begin{cases} (1-\omega)\, y_i^{(k)} + \left(\omega \sum_{j=p}^{n} A_j K_{ij} y_j^{2} + \omega\, f_i\right) \Big/ (\lambda + A_i K_{ii}) & , i = 0 \\ (1-\omega)\, y_i^{(k)} + \left(\omega \sum_{j=0}^{n-p} A_j K_{ij} y_j^{(k)} + \omega\, f_i\right) \Big/ (\lambda + A_i K_{ii}) & , i = n \\ (1-\omega)\, y_i^{(k)} + \left(\omega \sum_{j=0}^{i-p} A_j K_{ij} y_j^{(k)} + \omega \sum_{j=i+p}^{n} A_j K_{ij} y_j^{2} + \omega\, f_i\right) \Big/ (\lambda + A_i K_{ii}) & , others \end{cases}$$

1. For $i = 0,\ p,\ 2p,\ \cdots, n-2p,\ n-p,\ n$

Calculate

$$y_i^{(k+1)} \leftarrow \frac{1}{2}\left(y_i^1 + y_i^2\right)$$

The FSAM and HSAM algorithms are explicitly performed by using all equations at level (2.1) and (2.2) alternatively until the specified convergence criterion is satisfied. Generally, the basic idea for the convergence analysis of the AM methods has been proven by [14].

4 Numerical Simulations

In order to compare the performances of the iterative methods described in the previous section, several experiments were carried out on the following well-posed Fredholm integral equations problems and case where $a = 0$ and $b = 1$. In comparison, the Full-Sweep Gauss-Seidel (FSGS) method acts as the control of comparison of numerical results. Three criteria such as number of iterations, execution time and maximum absolute error will be considered in comparison for FSGS, FSAM and HSAM iterative methods. For the following examples, interval $[a, b]$ will be uniformly divided into $n = 2^q, q \geq 2$ and convergence test for the implementation of the iterative methods considered the tolerance error, $\varepsilon = 10^{-10}$.
Example 1 [12]
Consider the Fredholm integral equations of the first kind

$$\int_0^1 K(x,t)\, y(t)\, dt = e^x + (1-e)x - 1, 0 < x < 1 \tag{4.1}$$

with kernel

$$K(x,t) = \begin{cases} t(x-1), & t \leq x \\ x(t-1), & x < t \end{cases}$$

and the exact solution of the problem (4.1) is given by

$$y(x) = e^x.$$

Example 2 [5]
Consider the Fredholm integral equations of the first kind

$$\int_0^1 K(x,t)\, y(t)\, dt = \frac{1}{6}(x^3 - x), 0 < x < 1 \tag{4.2}$$

with kernel

$$K(x,t) = \begin{cases} t(x-1), & t < x \\ x(t-1), & x \leq t \end{cases}.$$

Exact solution of the problem (4.2) is

$$y(x) = x.$$

Results of numerical simulations, which were obtained from implementations of the FSGS, FSAM and HSAM iterative methods for Examples 1 and 2 have been recorded in Tables 1 and 2 respectively.

Table 1: Comparison of a number of iterations, execution time and maximum absolute error for the iterative methods at optimum value of ω (Example 1)

	Number of iterations				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	394	479	568	667	778
FSAM	143	144	145	146	149
HSAM	140	143	144	145	146
	Execution time (seconds)				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	3.64	17.31	81.49	382.33	1792.86
FSAM	2.56	10.33	41.62	167.14	682.84
HSAM	0.64	2.60	10.39	41.78	171.22
	Maximum absolute error				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	8.6244 E-7	2.1571 E-7	5.5889 E-8	2.5713 E-8	4.2551 E-8
FSAM	8.6107 E-7	2.1456 E-7	5.3522 E-8	1.3573 E-8	6.8853 E-9
HSAM	8.5921 E-6	2.1528 E-6	5.3797 E-7	1.3413 E-7	3.3494 E-8

Example 3 [18]
Consider the Fredholm integral equation of the second kind

$$y(x) = x + \int_0^1 (4x\,t - x^2) y(t)\, dt, 0 \leq x \leq 1 \tag{4.3}$$

and the exact solution is given by

$$y(x) = 24x - 9x^2.$$

Example 4 [11]

Table 2: Comparison of a number of iterations, execution time and maximum absolute error for the iterative methods at optimum value of ω (Example 2)

	Number of iterations				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	381	461	550	646	753
FSAM	135	135	137	139	142
HSAM	131	135	135	137	139
	Execution time (seconds)				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	3.48	16.67	79.55	379.97	1774.25
FSAM	2.75	11.04	44.42	181.69	742.33
HSAM	0.95	3.92	15.69	63.70	258.07
	Maximum absolute error				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	6.8225 E-10	8.3429 E-10	8.4449 E-10	9.7143 E-10	9.7966 E-10
FSAM	1.2307 E-9	1.4924 E-9	1.1483 E-9	6.0293 E-10	1.0732 E-9
HSAM	1.8384 E-9	1.2307 E-9	1.4924 E-9	1.1483 E-9	6.0293 E-10

Consider the Fredholm integral equation of the second kind

$$y(x) = x^6 - 5x^3 + x + 10 + \int_0^1 (x^{\,2} + t^2) y(t) dt, 0 \leq x \leq 1 \tag{4.4}$$

with the exact solution

$$y(x) = x^6 - 5x^3 + \frac{1045}{28}x^2 + x + \frac{2141}{84}.$$

For Examples 3 and 4, numerical results of the iterative methods have recorded in Table 3 and 4 respectively.

Meanwhile, reduction percentage of the number of iterations and execution time for the FSAM and HSAM methods compared with FSGS method have been summarized in Table 5.

5 Computational Complexity Analysis

In order to measure the computational complexity of the FSAM and HSAM iterative methods, an estimation amount of the computational work required for both methods have been conducted. The computational work is estimated by considering the arithmetic operations performed per iteration. Based on Algorithm 1, it can be observed that there are $\frac{2n}{p}+5$ additions/subtractions (ADD/SUB) and $\frac{4n}{p}+9$ multiplications/divisions (MUL/DIV) in computing a value for each node point in the solution domain for first kind linear Fredholm integral equations. Meanwhile, for second kind linear Fredholm integral equations, it can concluded that there are $\frac{2n}{p}+7$ additions/subtractions (ADD/SUB) and $\frac{4n}{p}+9$ multiplications/divisions (MUL/DIV) operations. From the order of the coefficient matrix, M in Eq. (2.3), the total number of arithmetic operations per iteration for the FSAM and HSAM iterative methods in solving first and second kinds Fredholm integral equations has been summarized in Table 6.

Table 3: Comparison of a number of iterations, execution time and maximum absolute error for the iterative methods at optimum value of ω (Example 3)

	Number of iterations				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	194	194	195	195	195
FSAM	84	84	84	84	84
HSAM	84	84	84	84	84
	Execution time (seconds)				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	2.20	8.75	35.06	140.31	560.50
FSAM	1.82	7.30	28.91	116.74	465.32
HSAM	0.40	1.56	6.17	24.62	98.41
	Maximum absolute error				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	4.6922 E-4	1.1730 E-4	2.9325 E-5	7.3307 E-6	1.8321 E-6
FSAM	4.6922 E-4	1.1730 E-4	2.9325 E-5	7.3311 E-6	1.8326 E-6
HSAM	1.8771 E-3	4.6922 E-4	1.1730 E-4	2.9325 E-5	7.3311 E-6

6 Conclusions

In the previous section, it has shown that the formulation of full- and half-sweep quadrature approximation equations based on repeated trapezoidal scheme can easily generate a linear system as shown in Eq. (2.3). Through numerical results obtained in Tables 1-4, clearly show that by applying the AM methods can reduce number of iterations compared to the FSGS method. Through the observation in Tables 1-4, found that application of the half-sweep iteration concept reduce the execution time of the iterative method. Since the implementation of the half-sweep iteration only considers approximately half of all interior node points in a solution domain. In terms of accuracy, approximate solutions for the FSAM and HSAM methods are in good agreement compared to the FSGS method. Overall, the numerical results show that the HSAM method is a better method compared to the FSGS and FSAM methods in the sense of number of iterations and execution time.

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Table 4: Comparison of a number of iterations, execution time and maximum absolute error for the iterative methods at optimum value of ω (Example 4)

	Number of iterations				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	56	56	56	56	56
FSAM	32	32	32	32	32
HSAM	32	32	32	32	32
	Execution time (seconds)				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	0.74	2.86	11.50	46.73	185.58
FSAM	0.55	2.00	8.01	32.00	127.41
HSAM	0.14	0.57	2.24	8.93	35.75
	Maximum absolute error				
Methods	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	4.7770 E-4	1.1942 E-4	2.9856 E-5	7.4639 E-6	1.8659 E-6
FSAM	4.7770 E-4	1.1942 E-4	2.9856 E-5	7.4639 E-6	1.8659 E-6
HSAM	2.0482 E-3	5.1267 E-4	1.2824 E-4	3.2071 E-5	8.0188 E-6

Table 5: Reduction percentage of the number of iterations and execution time

Example	**Methods**	**Number of iterations**	**Execution time**
1	**FSAM**	63.71 – 80.85%	29.67 – 61.91%
	HSAM	64.47 – 81.23%	82.42 – 90.45%
2	**FSAM**	64.57 – 81.14%	20.98 – 58.16%
	HSAM	65.62 – 81.54%	72.70 – 85.45%
3	**FSAM**	56.70 – 56.92%	16.57 – 17.54%
	HSAM	56.70 – 56.92%	81.82 – 82.45%
4	**FSAM**	42.86%	25.68 – 31.52%
	HSAM	42.86%	80.07 – 81.08%

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Table 6: Total number of arithmetic operations per iteration for FSAM and HSAM methods.

First kind linear Fredholm integral equations		
Methods	**Arithmetic Operation**	
	ADD/SUB	**MUL/DIV**
FSAM	$2n^2 + 7n + 5$	$4n^2 + 13n + 9$
HSAM	$\frac{n^2}{2} + \frac{7n}{2} + 5$	$n^2 + \frac{13n}{2} + 9$
Second kind linear Fredholm integral equations		
Methods	**Arithmetic Operation**	
	ADD/SUB	**MUL/DIV**
FSAM	$2n^2 + 9n + 7$	$4n^2 + 13n + 9$
HSAM	$\frac{n^2}{2} + \frac{9n}{2} + 7$	$n^2 + \frac{13n}{2} + 9$

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