

A comparative study of Haar Wavelet Method and Homotopy Perturbation Method for solving one-dimensional Reaction-Diffusion Equations

G.Hariharan and K.Kannan

Department of Mathematics, SASTRA University, Thanjavur-613 401,

Tamilnadu, INDIA

Email: nethra07@yahoo.co.in and kkannan@maths.sastra.edu

Abstract:

In this paper, we introduce a homotopy perturbation method to obtain exact solutions to some linear and nonlinear one-dimensional reaction-diffusion equations. This method is a powerful device for solving a wide variety of problems. Using the homotopy perturbation method, it is possible to find the exact solution or an approximate solution of the problem. The power of this manageable method is confirmed. An attempt is made to combine the advantages of the Homotopy Perturbation Method (HPM) and Haar wavelets. Good agreement with the exact solution has been observed. The results reveal that the HPM and HWM are very effective, convenient and quite accurate to systems of partial differential equations. It is predicted that the HWM and HPM can be found widely applicable in engineering.

Key words: Haar wavelets, one-dimensional reaction-diffusion equation, Homotopy Perturbation Method, computationally attractive

1 Introduction

Many reaction-diffusion problems in biology and chemistry are modeled by partial differential equations (PDEs). These problems have been extensively studied in the literature and their numerical solution can be accurately computed provided the diffusion coefficients, reaction excitations, initial and boundary data are given in a deterministic way. However, modeling real-life reaction-diffusion systems is complicated by the high heterogeneity of the diffusion process combined with insufficient information characterizing the kinetic reactions. An example concerns the spatio-temporal pattern formation in cell metabolism where the intact living cell is based on a highly complex spatial organization of its constituents. The reactants mediating, and processed by the chemical pathways of cell are heterogeneously distributed through the cytoplasm and cell membranes. The diffusion of reactant species among localized reaction regions within the cell is therefore a central feature of biochemistry. For more details, we refer to [29] and further references are cited therein.

Reaction-diffusion equations are used to simulate a variety of different phenomena, from physics and engineering [2] to mathematical biology [26]. In the last decades, there have been great advances in the development of finite difference, finite element, spectral techniques, adaptive and non-adaptive algorithms and finite volume methods for the partial differential equations, especially

for those of the advection-diffusion-reaction type. Reaction-diffusion equations also lead to many other interesting phenomena, such as, pulse splitting and shedding, reactions and competitions in excitable systems, and stability issues. Stable schemes for one-dimensional reaction-diffusion equation have demonstrated by Joao Teixeira [30]. Examples of this type of applications include numerical weather prediction and climate models [3] where the time step and grid sizes are imposed from large-scale flow considerations, atmospheric chemistry models or reactive flows in engineering [27]. Ramos [28] used a finite volume method for one-dimensional reaction-diffusion problems. Krishnan et al. [19] established Bifurcation analysis of nonlinear reaction-diffusion problems using wavelet-based reduction techniques. Wazwaz [32] presented the Adomain decomposition method for solving Fisher's equation.

This paper is devoted to study the linear single one-dimensional kinetic reaction-diffusion

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial U}{\partial x} \right) - \lambda U, \quad (1.1)$$

where t and x denote the time and spatial coordinate, respectively, U is the dependent variable and k is a constant diffusion coefficient.

It is somewhat surprising that among different solution techniques the wavelet method has not attained much attention. We found only one paper [31] in which the wavelet method is applied for solving singularly perturbed reaction-diffusion problems; for this purpose the cubic spline adaptive wavelet functions are used. Lepik [[20],[21]] had solved higher order as well as nonlinear ODEs and some nonlinear evolution equations by Haar wavelet method. There are two possibilities for getting out of this situation. One way is to regularize the Haar wavelets with interpolating splines (e.g. B-splines or Deslaurier-Dabuc interpolating wavelets). This approach has been applied by Cattani [4], but the regularization process considerably complicates the solution and the main advantage of the Haar wavelets-the simplicity gets to some extent lost. Hariharan et al. [[11],[12]] had solved linear and nonlinear PDEs.

Among the different wavelet families mathematically most simple are the Haar wavelets [10]. Due to simplicity the Haar wavelets are very effective for solving ordinary differential and partial differential equations. Therefore the idea, to apply Haar wavelet technique also for solving one-dimensional linear reaction-diffusion problem, arises. This is the main aim of the present paper. The method with far less degrees of freedom and with smaller CPU time provides better solutions than classical ones. The accuracy and effectiveness of the method are analyzed; the results obtained are compared with the results of other authors (using classical numerical techniques) and with the exact solution, evaluating the error.

We introduce a Haar wavelet method for solving one-dimensional linear reaction-diffusion equations, which will exhibit several advantageous features:

- i) Very high accuracy fast transformation and possibility of implementation of fast algorithms compared with other known methods.
- ii) The simplicity and small computation costs, resulting from the sparsity of the transform matrices and the small number of significant wavelet coefficients.
- iii) The method is also very convenient for solving the boundary value problems, since the boundary conditions are taken care of automatically.

In this paper, we solve one-dimensional reaction-diffusion equations via Homotopy perturbation method (HPM). HPM introduced by He [13] has been used by many mathematicians and engineers to solve various functional equations. In this method the solution is considered as the sum of an infinite series which converges rapidly to the accurate solutions[[1],[14],[15],[16],[22],[23],[25],[24],[7],[8],[9]]. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$ which is considered as a small parameter. The numerical results have been compared with Haar wavelet solutions.

This paper is arranged in the following manner. In Section 2, we present the basic ideas of Homotopy Perturbation Method(HPM). In Section 3 we illustrate the Haar Wavelet System. In Section 4, we present the integration of wavelets. In Section 5, function approximation is presented. In Section

6, the Homotopy Perturbation Method(HPM) for solving one-dimensional reaction-diffusion equation is established. Haar Wavelet Method(HWM) for solving one-dimensional reaction-diffusion equation is presented in section 7. In section 8, 4 test problems are discussed. Concluding remarks are given in section 8.

2 Homotopy Perturbation Method(HPM)

To illustrate basic ideas of this method, we consider the following equation:

$$A(u) - f(r) = 0, r \in \Omega, \quad (2.1)$$

with the boundary condition

$$B \left\{ u, \frac{\partial u}{\partial n} = 0 \right\}, t \in \Gamma, \quad (2.2)$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω . A can be divided into two parts which are L and N , where L is linear and N is nonlinear. Eq.(2) can therefore be written as follows:

$$L(u) + N(u) - f(r) = 0, r \in \Omega, \quad (2.3)$$

By the homotopy technique, we construct a homotopy $V(r, p) : \Omega \times [0, 1] \rightarrow \Re$, which satisfies:

$$H(V, p) = (1 - p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, \quad (2.4)$$

$p \in [0, 1], r \in \Omega$, or

$$H(V, p) = L(V) - L(u_0) + pL(u_0) + p[N(V) - f(r)] = 0. \quad (2.5)$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (2), which satisfies the boundary conditions. Obviously, from Eqs.(5) or (6) we will have

$$H(V, 0) = L(V) - L(u_0) = 0 \quad (2.6)$$

$$H(V, 1) = A(V) - f(r) = 0 \quad (2.7)$$

The changing process of p from zero to unity is just that of $V(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called homotopy. According to the HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs.(5) or (6) can be written as a power series in p :

$$V = V_0 + pV_1 + p^2V_2 + p^3V_3 + \dots, \quad (2.8)$$

and the exact solution is obtained as follows:

$$u = \lim_{p \rightarrow 1} V = \lim_{p \rightarrow 1} (V_0 + pV_1 + p^2V_2 + p^3V_3 + \dots) = \sum_{j=0}^{\infty} V_j. \quad (2.9)$$

The series (10) is convergent for most cases, and the rate of convergence depends on $L(u)$ [16].

3 The Haar System

The Haar wavelets of higher resolution levels are based on the mother wavelet. For each level of resolution the number of wavelets is doubled while the domain of each is halved. The magnitude of each function is modified so that the inner product of each wavelet function with itself is one. The inner product of any wavelet coefficient with any other wavelet coefficient, at any resolution level, or with the scaling function, is 0.

In 1910 Alfred Haar [10] introduced a function, which presents a rectangular pulse pair. After that various generalizations were proposed (a state-of-the-art about Haar transforms can be found in [10]). In the 1980s it turned out that the Haar function was in fact the Daubechies wavelet of order 1. It is the simplest orthonormal wavelet with compact support. An essential shortcoming of the Haar wavelets is that they are not continuous. The derivatives do not exist in the points of discontinuity; therefore it is not possible to apply the Haar wavelets directly to solving differential equations. There are at least two possibilities of ending this impasse. First, the piecewise constant Haar functions can be regularized with interpolation splines; this technique has been applied by Cattani [4]. This greatly complicates the solution process and the main advantage of the Haar wavelets - their simplicity - gets lost. Another possibility was proposed by Chen and Hsiao [5]. They recommended to expand into the Haar series not the function itself, but its highest derivative appearing in the differential equation; the other derivatives (and the function) are obtained through integrations. All these ingredients are then incorporated into the whole system, discretized by the Galerkin or collocation method. Chen and Hsiao [5] demonstrated the possibilities of their method by solving linear systems of ordinary differential equations (ODEs) and partial differential equations (PDEs). In [6] an optimal control problem with the quadratic performance index is discussed. In [18] Hsiao and Wang applied this method to solving singular bilinear and nonlinear systems. Haar functions appear very attractive in many applications as for example, image coding, edge extraction, and binary logic design.

Recently, Haar wavelets have been applied extensively for signal processing in communications and physics research, and have proved to be a wonderful mathematical tool. After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system. The previous work in system analysis via Haar wavelets was led by Chen and Hsiao [5], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the application for the Haar analysis into the dynamical systems. Then, the pioneer work in state analysis of linear time delayed systems via Haar wavelets was laid down by Hsiao [17], who first proposed a Haar product matrix and a coefficient matrix.

In this, we discuss the wavelet approximation of a given function $f \in L^2(\mathbb{R})$ in the Haar wavelet system. The Haar wavelet family for $t \in [0, 1]$ is defined as follows.

$$h_i(t) = \begin{cases} 1 & \text{for } t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ -1 & \text{for } t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0 & \text{elsewhere} \end{cases} \quad (3.1)$$

Haar scaling function is given by

$$\phi(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1, \\ 0 & \text{otherwise} \end{cases} \quad (3.2)$$

Let f be a function in $L^2(\mathbb{R})$ and $I_{jk} = [k2^{-j}, (k+1)2^{-j}]$.

We can define piecewise constant approximation f_j of f at scale 2^{-j} by

For all $x \in I_{j,k}$, $k \in Z$

$$f_j(x) = 2^j \int_{I_{j,k}} f(t) dt, \quad (3.3)$$

ie., f is approximatex by its mean value on each interval $I_{j,k}$, $k \in Z$.

4 Integration of Wavelets

In Eq.(11) integer $m = 2^j$ ($j = 0, 1, 2, \dots, J$) indicates the level of the wavelet; $k = 0, 1, 2, \dots, m-1$ is the translation parameter. Maximal level of resolution is J . The index i is calculated according to the formula $i = m + k + 1$; in the case of minimal values $m=1, k=0$, we have $i=2$, the maximal value of i is $2M = 2^{(J+1)}$. It is assumed that the value $i=1$ corresponds to the scaling function for which $h_1 \equiv 1$ in $[0, 1]$. Let us define the collocation points $t_l = (l - 0.5) / 2M$, $(1, 2, \dots, 2M)$ and discretise the Haar function $h_i(t)$: In this way we get the coefficient matrix $H(i, l) = (h_i(t_l))$, which has the dimension $2M \times 2M$. Each integer i has a unique decomposition into the two integers l and k . Sample computations are shown in the following table.

Table 1: Index computation for Haar basis functions

l	0	1	1	2	2	2	2	3	3	3	3	3	...
k	0	0	1	0	1	2	3	0	1	2	3	4	...
$i = 2^l + k + 1$	2	3	4	5	6	7	8	9	10	11	12	13	...

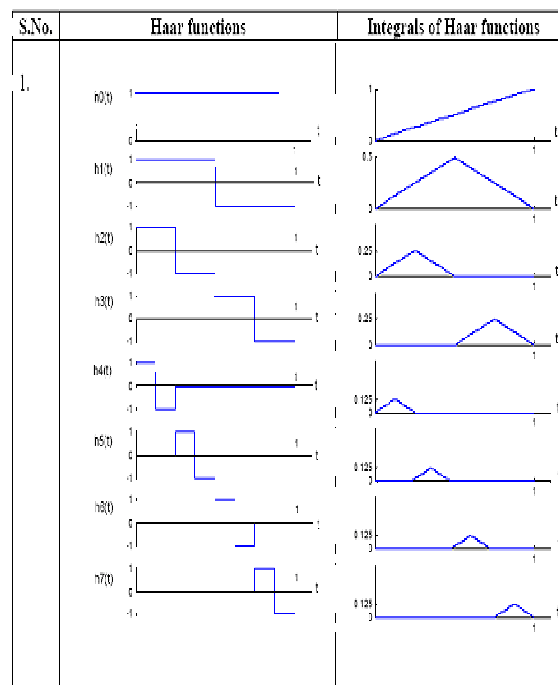


Figure 1: First eight Haar scaling functions and their integrals.

The orthogonal set of Haar wavelets $h_0(t)$ to $h_7(t)$ is shown in Figure 1, which contains a family of single square wavelets. The first basis $h_0(t)$ is called the scaling function, which is equal

to 1 for the whole unit time interval. The second basis $h_1(t)$ is the fundamental square wave. The others, $h_2(t)$ to $h_7(t)$ are generated from $h_1(t)$ via two operations: translation and dilation.

Haar wavelets have several useful properties such as,

- (1) the Haar set forms a local basis since each Haar function contains just one wavelet which nonzero over some subinterval and remains zero elsewhere in the interval $[0,1]$;
- (2) the Haar basis functions are orthogonal to one another;
- (3) the integration of Haar wavelets can be expandable into Haar series.

The operational matrix of integration P , which is a $2M$ square matrix, is defined by the equation

$$(PH)_{il} = \int_0^{t_l} h_i(t) dt \quad (4.1)$$

$$(QH)_{il} = \int_0^{t_l} dt \int_0^t h_i(t) dt \quad (4.2)$$

The elements of the matrices H, P and Q can be evaluated according to (11), (14) and (15).

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$P_2 = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$P_4 = \frac{1}{16} \begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$P_8 = \frac{1}{64} \begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chen and Hsiao [5] showed that the following matrix equation for calculating the matrix P of order m holds

$$P_{(m)} = \frac{1}{2m} \begin{bmatrix} 2mP_{(m/2)} & -H_{(m/2)} \\ H_{(m/2)}^{-1} & O \end{bmatrix}$$

where O is a null matrix of order $\frac{m}{2} \times \frac{m}{2}$

$$H_{m \times m} \triangleq [h_m(t_0) \ h_m(t_1) \ - \ - \ h_m(t_{m-1})] \quad (4.3)$$

and

$$\frac{i}{m} \leq t < i + \frac{1}{m},$$

$$H_{m \times m}^{-1} = \frac{1}{m} H_{m \times m}^T \text{diag}(r)$$

It should be noted that calculations for $P_{(m)}$ and $H_{(m)}$ must be carried out only once; after that they will be applicable for solving whatever differential equations. Since H and H^{-1} contain many zeros, this phenomenon makes the Haar transform much faster than the Fourier Transform and it is even faster than the Walsh transform. This is one of the reasons for rapid convergence of the Haar wavelet series.

5 Function Approximation

Any function $y(x) \in L^2[0, 1]$ can be decomposed as

$$y(x) = \sum c_n h_n(x) \quad (5.1)$$

where the coefficients c_n are determined by

$$c_n = 2^j \int_0^1 y(x) h_n(x) dx \quad (5.2)$$

where $n = 2^j + k, j \geq 0, 0 \leq k < 2^j$. Especially $c_0 = \int_0^1 y(x) dx$.

The series expansion of $y(x)$ contains infinite number of terms. If $y(x)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $y(x)$ will be terminated at finite terms, that is

$$y(x) = \sum_{n=0}^{m-1} c_n h_n(x) = c_{(m)}^T h_{(m)}(x) \quad (5.3)$$

where the coefficients $c_{(m)}^T$ and the Haar function vector $h_{(m)}(x)$ are defined as $c_{(m)}^T = [c_0, c_1, \dots, c_{m-1}]$ and $h_{(m)}(x) = [h_0(x), h_1(x), \dots, h_{m-1}(x)]^T$ where 'T' means transpose and $m = 2^j$

6 HPM for the one-dimensional reaction-diffusion equation

In this section, we consider the one-dimensional reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial u}{\partial x} \right) - \lambda u, \quad (6.1)$$

Subject to the initial condition

$$u(x, 0) = f(x) \quad (6.2)$$

Eq.(20) can be written as

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - \lambda u. \quad (6.3)$$

For solving Eq.(13), by homotopy perturbation method, we construct a homotopy as follows:

$$(1-p)\left[\frac{\partial V}{\partial t} - \frac{\partial u_0}{\partial t}\right] + p\left[\frac{\partial V}{\partial t} - k\frac{\partial^2 V}{\partial x^2} + \lambda V\right] = 0 \quad (6.4)$$

(or)

$$\frac{\partial V}{\partial t} - \frac{\partial u_0}{\partial t} - p\frac{\partial V}{\partial t} + p\frac{\partial u_0}{\partial t} + p\frac{\partial V}{\partial t} - k\frac{\partial^2 V}{\partial x^2} + \lambda V = 0 \quad (6.5)$$

$$\frac{\partial V}{\partial t} = \frac{\partial u_0}{\partial t} + p\left[k\frac{\partial^2 V}{\partial x^2} - \lambda V\right] = 0 \quad (6.6)$$

Suppose the solution of Eq.(16) has the form

$$V(x, t) = V_0(x, t) + pV_1(x, t) + p^2V_2(x, t) + \dots = \sum_{i=0}^{\infty} p^i V_i(x, t) \quad (6.7)$$

$$p^0 : \frac{\partial V_0}{\partial t} = \frac{\partial u_0}{\partial t}, \quad (6.8)$$

$$V_0(x, 0) = f(x).$$

$$p^1 : \frac{\partial V_1}{\partial t} = \frac{\partial V_0}{\partial x^2} - \lambda V_0 - \frac{\partial u_0}{\partial t}, \quad (6.9)$$

$$V_1(x, 0) = 0.$$

$$p^{k+1} : \frac{\partial V_{k+1}}{\partial t} = \frac{\partial V_k}{\partial x^2} - \lambda V_k, \quad (6.10)$$

$$V_{k+1}(x, 0) = 0, k \geq 1.$$

Considering $u_0(x, t) = u(x, 0) = f(x)$, we have

$$V_0(x, t) = u_0(x, t) = f(x), \quad (6.11)$$

$$V_{k+1}(x, t) = \int_0^t \left\{ \frac{\partial^2 V_k(x, s)}{\partial x^2} - \lambda V_k(x, s) \right\} ds \quad (6.12)$$

where $k \geq 0$. Note that in this case $\frac{\partial u_0}{\partial t} = 0$.

7 Haar Wavelet (HW) solution for One-dimensional reaction-diffusion equation

Consider the linear kinetic one-dimensional reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial u}{\partial x} \right) - \lambda u, \quad (7.1)$$

with the initial condition $u(x, 0) = f(x)$, $0 \leq x \leq 1$ and the boundary conditions

$$u(0, t) = g_0(t), u(1, t) = g_1(t), 0 < t \leq T.$$

Let us divide the interval $(0,1]$ into N equal parts of length $\Delta = (0,1]/N$ and denote $t_s = (s-1)\Delta, s = 1, 2, \dots, N$. We assume that $\dot{u}''(x, t)$ can be expanded in terms of Haar wavelets as formula

$$\dot{u}''(x, t) = \sum_{n=0}^{m-1} c_s(n) h_n(x) = c_{(m)}^T h_{(m)}(x), \quad (7.2)$$

where \cdot and \cdot' means differentiation with respect to t and x respectively, the row vector $c_{(m)}^T$ is constant in the subinterval $t \in (t_s, t_{s+1}]$.

Integrating formula (33) with respect to t from t_s to t and twice with respect to x from 0 to x , we obtain

$$u''(x, t) = (t - t_s) c_{(m)}^T h_{(m)}(x) + u''(x, t_s), \quad (7.3)$$

$$u(x, t) = (t - t_s) c_{(m)}^T Q_{(m)} h_{(m)}(x)$$

$$+ u(x, t_s) - u(0, t_s) + x[u'(0, t) - u'(0, t_s)] + u(0, t) \quad (7.4)$$

$$\dot{u}(x, t) = c_{(m)}^T Q_{(m)} h_{(m)}(x) + x \dot{u}'(0, t) + \dot{u}(0, t) \quad (7.5)$$

Using the boundary conditions, we obtain

$$\begin{aligned} u(0, t_s) &= g_0(t_s), \\ u(1, t_s) &= g_1(t_s), \\ \dot{u}(0, t) &= g'_0(t), \\ \dot{u}(1, t) &= g'_1(t). \end{aligned}$$

Putting $x = 1$ in formulae (35) and (36), we have

$$\begin{aligned} u'(0, t) - \dot{u}'(0, t) &= -(t - t_s) c_{(m)}^T Q_{(m)} h_{(m)} \\ &+ g_1(t) - g_0(t) - g_1(t_s) + g_0(t_s), \end{aligned} \quad (7.6)$$

$$\dot{u}'(0, t) = g'_1(t) - c_{(m)}^T Q_{(m)} h_{(m)} - g'_0(t) \quad (7.7)$$

Substituting formulae (37) and (38) into formulae (34)-(36), and discretizing the results by assuming $x \rightarrow x_l, t \rightarrow t_{s+1}$ we obtain

$$u''(x_l, t_{s+1}) = (t_{s+1} - t_s) c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + u''(x_l, t_s) \quad (7.8)$$

$$\begin{aligned} u(x_l, t_{s+1}) &= (t_{s+1} - t_s) c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + u(x_l, t_s) - g_0(t_s) + g_0(t_{s+1}) x_l [-(t_{s+1} - t_s) C_{(m)}^T P_{(m)} f + \\ &+ g_1(t_{s+1}) - g_0(t_{s+1}) - g_1(t_s) + g_0(t_s)] \\ \dot{u}(x_l, t_{s+1}) &= c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + g'_0(t_{s+1}) \end{aligned}$$

$$+ x_l [-c_{(m)}^T P_{(m)} f + g'_1(t_{s+1}) - g'_0(t_{s+1})] \quad (7.9)$$

where the vector f is defined as $f = \begin{bmatrix} 1, & \underbrace{0, \dots, 0}_{(m-1) \text{ elements}} \end{bmatrix}^T$

In the following scheme

$$\dot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}) + k u''(x_l, t_{s+1}) - \lambda u(x_l, t_{s+1}) \quad (7.10)$$

which leads us from the time layer t_s to t_{s+1} is used.

Substituting the equations (39) and (40) into the equation (41), we gain

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l \left[-c_{(m)}^T p_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1}) \right] + g_0'(t_{s+1}) \\ = u''(x_l, t_{s+1}) + k u''(x_l, t_{s+1}) - \lambda u(x_l, t_{s+1})$$

From the above formula, the wavelet coefficients $c_{(m)}^T$ can be successively calculated.

In the following section we provide some examples and calculate the absolute errors by using the formula

$$E_W = |u_{exact} - u_{Haar}|$$

and

$$\delta ex = \|u(x, t) - u_{ex}(x, t)\| / 2M.$$

8 Test Problems

Problem:1

In Eq.(22) , we set $u(x, 0) = f(x) = \lambda$, where λ is a constant. By Eqs.(21) and (22) we have

$$V_0(x, t) = \lambda$$

$$V_1(x, t) = \lambda^2 t$$

$$V_2(x, t) = \lambda^3 t^2$$

$V_3(x, t) = \lambda^4 t^3$ and so on for other components. The solution in a closed form is given by

$$u(x, t) = \sum_{k=0}^{\infty} V_k$$

$$= \lambda + \lambda^2 t + \lambda^3 t^2 + \lambda^4 t^3 + \dots$$

$$= \lambda [1 + \lambda t + \lambda^2 t^2 + \lambda^3 t^3 + \dots]$$

$$= \lambda [1 - \lambda t]^{-1}$$

$= \frac{\lambda}{1 - \lambda t}$ Using HPM, the exact solution in a closed form of Eq.(22) can be compared with Haar solutions. More accurate results can be obtained by using a larger m .(See table.II)

Problem:2

In Eq.(22), we set $u(x, 0) = x$. By Eqs.(21) and (22) we have $V_0(x, t) = x$

$$V_1(x, t) = \lambda x t$$

$$V_2(x, t) = \lambda^2 t^2 x$$

$V_3(x, t) = \lambda^3 t^3 x$ and so on for other components. The solution in a closed form is given by

$$u(x, t) = \sum_{k=0}^{\infty} V_k$$

$$= x + \lambda x t + \lambda^2 t^2 x + \lambda^3 t^3 x + \dots$$

$$= x [1 + \lambda t + \lambda^2 t^2 + \lambda^3 t^3 + \dots]$$

$$= x [1 - \lambda t]^{-1}$$

$$= \frac{x}{1 - \lambda t}$$

Problem:3

If we take $k = 1$ and $\lambda = -1$ in the equation (22), we obtain the linear heat equation, namely

$$u_t = u_{xx} + u \tag{8.1}$$

We impose the initial condition

$$u(x, 0) = \cos(\pi x) \tag{8.2}$$

and boundary conditions

$$u(0, t) = e^{(1 - \pi^2)t}, u_x(0, t) = 0. \tag{8.3}$$

Using HPM, an exact solution in a closed form is given by

$$u(x, t) = e^{(1-\pi^2)t} \cos(\pi x). \quad (8.4)$$

In the following scheme

$$\dot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}) + u(x_l, t_{s+1}) \quad (8.5)$$

which leads us from the time layer t_s to t_{s+1} is used.

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l + x_l [-c_{(m)}^T P_{(m)} \cos(\pi x) + g_1'(t_{s+1}) - g_0'(t_{s+1})] + g_0'(t_{s+1})) \\ = u''(x_l, t_{s+1}) + u(x_l, t_{s+1})$$

From the above formula the wavelet coefficients can be successively calculated. This process is started with

$$\begin{aligned} u(x_l, t_s) &= -\pi \sin(\pi x) \\ u'(x_l, t_s) &= -\pi^2 \cos(\pi x) \\ u''(x_l, t_s) &= \pi^3 \sin(\pi x) \end{aligned}$$

Problem:4

Consider the Fisher's equation

$$u_t = u_{xx} + u(1 - u) \quad (8.6)$$

We impose the initial condition

$$u(x, 0) = f(x) = \lambda \quad (8.7)$$

Here λ is a constant.

By Eqs.(30) and (31) we have $V_0(x, t) = \lambda$,
 $V_1(x, t) = \lambda(1 - \lambda)t$,
 $V_2(x, t) = \lambda(1 - \lambda)(1 - 2\lambda)\frac{t^2}{2!}$, and so on for other components. The solution in a closed form is given by

$$\frac{\lambda e^t}{1 - \lambda + \lambda e^t} \quad (8.8)$$

The Haar wavelet scheme is given by $c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l [-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1})] + g_0'(t_{s+1})$

$$= u''(x_l, t_{s+1}) + u(x_l, t_{s+1}) [1 - u(x_l, t_{s+1})]. \quad (8.9)$$

From the above formula $c_{(m)}^T$ can be calculated successively. The HPM results can be compared with the Haar wavelet solutions.

Table 2: The absolute errors at different times and space locations for problem.3

(x, t)	HPM solution	Haar solution	
		$m = 16$	$m = 32$
(0.25,0.1)	6.9388E-016	3.2570E-005	2.3455E-005
(0.5,1.0)	1.3322E-015	5.5452E-005	0.2282E-005
(0.75,1.0)	1.8873E-015	4.3456E-005	0.2122E-005
(0.25,0.5)	1.2679E-009	1.9567E-005	1.1102E-016
(0.5,0.5)	2.4570E-009	5.4572E-004	2.2204E-016
(0.75,0.5)	3.4933E-009	7.7654E-004	2.7755E-016
(0.25,1.0)	6.1931E-007	1.6562E-004	8.3266E-017
(0.5,1.0)	1.2001E-006	3.7982E-003	1.3877E-016
(0.75,1.0)	1.7063E-006	4.4324E-003	2.7755E-016

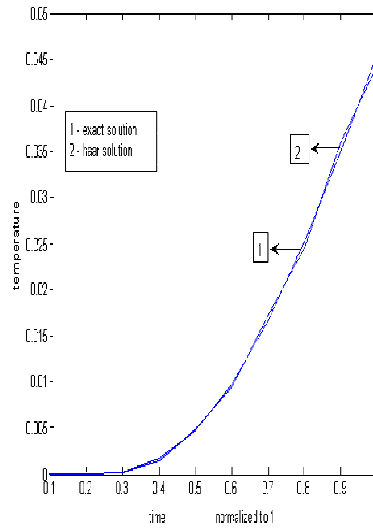


Figure 2: Comparison between exact and Haar solution of one-dimensional reaction-diffusion equation $x=20, k=12.5$ and $m=16$.

Using Homotopy Perturbation Method (HPM), the exact solution in a closed form is given by $u(x, t) = e^{(1-\pi^2)t} \cos(\pi x)$ can be compared with the Haar solution. The accuracy of the results is estimated by the error function. In the case of error estimates, if the exact solution of the problem $u = u(x, t)$ is known we shall calculate the differences $\Delta_{ex}(l) = u(x_l, t_{s+1}) - u_{ex}(x_l, t_{s+1})$, $l = 1, 2, \dots, 2M$ and we define the error estimates as $\delta_{ex} = \max_l |\Delta_{ex}(l)|$ (local estimate) (or) $\delta_{ex} = \frac{1}{2M} \|u(x, t) - u_{ex}(x, t)\|$ (global estimate). The convergence of Haar method is fast and its accuracy is high, as numerical examples show error [21].

Computer simulation was carried out in the cases $m = 16$, the computed results were compared with the exact solution, more accurate results can be obtained by using a larger m . The method with far less degrees of freedom and with a smaller CPU time provides better solutions than classical ones.

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System Vostro 1400 Processor x86 Family 6 Model 15 Stepping 13 Genuine Intel 1596 Mhz.

9 Conclusion

The goal to obtain exact and Haar solutions for one-dimensional reaction-diffusion problems has been achieved. The theoretical elegance of the Haar wavelet approach can be appreciated from the simple mathematical relations and their compact derivations and proofs. It has been well demonstrated that while applying the nice properties of Haar wavelets, the partial differential equations can be solved conveniently and accurately by using Haar wavelet method systematically.

The homotopy perturbation method deforms a complex problem under study to a simple problem routinely. If initial guess is suitably chosen, one iteration is enough, making the method a most attractive one. The method is of remarkable simplicity, while the obtained results are of utter accuracy on the whole solution domain. The method can be applied to various other nonlinear problems without any difficulty. Both the methods are wholly competitive for solving linear and nonlinear partial differential equations. An authentic conclusion can be drawn from the numerical results that the Haar Wavelet Method (HWM) provides more accurate numerical solutions than Homotopy Perturbation Method (HPM). In my opinion the Haar wavelet method is

wholly competitive in comparison with the classical methods.

Future work will involve the extension of the scheme to two and three dimensions and to the advection-diffusion equation.

References

- [1] S. Abbasbandy, Iterated He's homotopy perturbation method for quadratic Riccati differential equation, *Appl.Math. Comput.* 175 (2006) 581-589.
- [2] G.I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics*, Cambridge University Press, Cambridge, 1996.
- [3] A.C.M. Beljaars, Numerical schemes for parametrizations, in: *Proceedings on Numerical Methods in Atmospheric Models*, vol. 2, ECMWF, Reading, UK, 1991, p. 1.
- [4] C.Cattani, Haar wavelet spline, *J.Interdisciplinary Math.*4 (2001) 35-47.
- [5] C.F.Chen, C.H.Hsiao, Haar wavelet method for solving lumped and distributed parameter systems, *IEEE-Proc.: Control theory Appl.* 144(1997) 87-94.
- [6] D.K.Cheng, *Analysis of linear systems*, Addison-Wesley, Newyork, 1961.
- [7] M.S.H. Chowdhury, I. Hashim, Solutions of EmdenFowler equations by homotopy-perturbation method, *Nonlinear Analysis: Real World Applications* 10 (2009) 104115.
- [8] L. Cveticanin, Homotopy-perturbation method for pure nonlinear differential equation, *Chaos Solitons Fractals* 30 (2006) 12211230.
- [9] D.D. Ganji, A. Sadighi, Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations, *Int. J. Nonlinear Sci. Numer. Simulation* 7 (4) (2006) 411-418.
- [10] A.Haar, Zur theorie der orthogonalen Funktionsysteme. *Math. Annal* 1910:69:331-71.
- [11] G.Hariharan, K.Kannan, K.R.Sharma, Haar wavelet in estimating depth profile of soil temperature, *Appl. Math. Comput.* 210 (2009) 119-125.
- [12] G.Hariharan, K.Kannan, K.R.Sharma, Haar wavelet method for solving Fisher's equation, *Appl. Math. Comput.* 211 (2009) 284-292.
- [13] J.H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, *Appl. Math. Comput.* 151 (2004) 287-292.
- [14] J.H. He, Comparison of homotopy perturbation method and homotopy analysis method, *Appl. Math. Comput.* 156 (2004) 527539.
- [15] J.H. He, Homotopy perturbation method: A new nonlinear analytical technique, *Applied Mathematics and Computation* 135 (2003), pp. 7379.
- [16] J.H.He, A coupling method of a homotopy technique and a perturbation technique for nonlinear problems, *Int.J.Non-linear mech.*,35(1):(2000)37-43.
- [17] C.H.Hsiao, State analysis of linear time delayed systems via Haar wavelets, *Math.Comp.Simulat.*44, pp.457-470, 1997.
- [18] C.H.Hsiao, W.J.Wang, Haar wavelet approach to nonlinear stiff systems, *Math.Comp.Simulat.*44, pp.457-470, 1997.
- [19] J. Krishnan, O. Runborg, I. G. Kevrekidis, Bifurcation analysis of nonlinear reaction-diffusion problems using wavelet-based reduction techniques, *Computers and Chemical Engineering* 28 (2004) 557-574.
- [20] U.Lepik, Numerical solution of differential equations using Haar wavelets, *Math.and Comp. Simulation* 68(2005) 127-143.
- [21] U.Lepik, Numerical solution of evolution equations by the Haar wavelet method, *Appl. Math. Comput.* 185 (2007) 695-704.
- [22] L.-F. Mo, Variational approach to reaction-diffusion process, *Physics Letters A*, vol. 368, no. 3-4, pp. 263265, 2007.
- [23] S. Momani, Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, *Phys. Lett. A* 365 (2007) 345350.
- [24] Yu-Xi Wang, Hua-You Si, Lu-Feng Mo, Homotopy Perturbation Method for Solving Reaction-Diffusion Equations, *Math.Prob. in Engg.*, Vol. 2008, Article ID 795838, 5 pages
- [25] S. Momani, Z. Odibat, Comparison between the homotopy perturbation method and the variational iteration method for linear fractional partial differential equations, *Comput. Math. Appl.* (2007), in press (doi:10.1016/j.camwa.2006.12.037).
- [26] J.D. Murray, *Mathematical Biology*, Springer-Verlag, Berlin, 1993.

- [27] E.S. Oran, J.P. Boris, Numerical Simulation of Reactive Flow, Elsevier, New York, 1987.
- [28] J.I. Ramos, A finite volume method for one-dimensional reaction-diffusion problems, Appl. Math. Comput. 188 (2007) 739-748.
- [29] A. Stundzia, C. Lumsden, Stochastic simulation of coupled reaction-diffusion processes, J. Comput. Phys. 127 (1996) 196-207.
- [30] J.Teixeira, Stable schemes for partial differential equations: the one-dimensional reaction-diffusion equation, Math. Comput. in Simul. 64 (2004) 507-520.
- [31] Vivek Kumar and Mani Mehra, Cubic spline adaptive wavelet scheme to solve singularly perturbed reaction-diffusion problems, Intl. J. of Wavelets, Multiresol. Infor. Processing, 5, 2 (2007) 317-331.
- [32] A.M.Wazwaz, An analytical study of Fisher's equation by using Adomian decomposition method, Appl. Math. Comput. 154 (2004) 609-620.