

Solvability of n^{th} order nonlinear eigenvalue problem

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Abstract:

We determine the values of a parameter λ for which there exist positive solutions to the n -th order nonlinear differential equation $y^{(n)}(t) + \lambda f(t, y(t)) = 0$, $t \in [a, b]$, satisfying the boundary conditions $y^{(i)}(a) = 0$, $0 \leq i \leq n-2$, $y^{(p)}(b) = 0$, $(1 \leq p \leq n-1, \text{ but fixed})$ where $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Cone theory and Krasnosel'skii fixed point theorem will be applied.

Keywords: Boundary value problem, eigenvalue interval, positive solution, fixed point, cone.

1 Introduction

The study of finding eigenvalues and intervals of eigenvalues, λ , where positive solutions exist was first employed by Erbe and Wang [6] when they worked to establish the existence of positive solutions in a cone for boundary value problems (BVPs) for second order ordinary differential equations. Continuing in a similar manner, Erbe, Wang and Hu [7] along with Elloe and Henderson [4] obtained further results. In addition, other extension of [6] can be viewed in the following studies [2, 3, 11, 12, 16]. In this paper we are concerned with the existence of eigenvalues for the n -th order differential equation

$$y^{(n)}(t) + \lambda f(t, y(t)) = 0, \quad t \in [a, b], \quad (1.1)$$

satisfying the two-point boundary conditions

$$y^{(i)}(a) = 0, \quad 0 \leq i \leq n-2, \quad (1.2)$$

$$y^{(p)}(b) = 0, \quad (1 \leq p \leq n-1, \text{ but fixed}) \quad (1.3)$$

where $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous with respect to y .

There has been much interest recently in this area of obtaining optimal eigenvalue intervals of boundary value problems, often using the Krasnosel'skii [13] fixed point theorems to obtain intervals based on positive solutions inside a cone. A few papers along these lines are Agarwal, Bohner, and Wong [1], Davis, Henderson, Prasad, and Yin [2], Elloe and Henderson [3, 4, 5], Erbe, Hu, and Wang [6], Henderson and Wang [10], Shahed [15].

We define the positive extended real numbers $f_0, f^0, f_\infty, f^\infty$ by

$$f_0 = \lim_{y \rightarrow 0^+} \min_{t \in [a, b]} \frac{f(t, y)}{y}, \quad (1.4)$$

$$f^0 = \lim_{y \rightarrow 0^+} \max_{t \in [a, b]} \frac{f(t, y)}{y}, \quad (1.5)$$

$$f_\infty = \lim_{y \rightarrow \infty} \min_{t \in [a, b]} \frac{f(t, y)}{y}, \quad (1.6)$$

$$f^\infty = \lim_{y \rightarrow \infty} \max_{t \in [a, b]} \frac{f(t, y)}{y} \quad (1.7)$$

and assume that they will exist.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous BVP corresponding to 1.1-1.3 and estimate the bounds for the Green's function. In Section 3, we present a lemma which is needed in the main result and establish a criteria to determine eigenvalue intervals for which the BVP 1.1-1.3 has at least one positive solution, by using the Krasnosel'skii fixed point theorem.

2 The Green's Function and Bounds

In this section, first we construct the Green's function for homogeneous two-point BVP $-y^{(n)} = 0$ with the boundary conditions 1.2-1.3 and estimate the existence of bounds for the Green's function.

Theorem 1. *The Green's function $G(t, s)$ for the homogeneous BVP*

$$-y^{(n)} = 0,$$

with the boundary conditions 1.2, 1.3 is given by

$$G(t, s) = \begin{cases} \frac{(t-a)^{n-1}(b-s)^{n-p-1}}{(n-1)! (b-a)^{n-p-1}}, & t \leq s \\ \frac{(t-a)^{n-1}(b-s)^{n-p-1}}{(n-1)! (b-a)^{n-p-1}} - \frac{(t-s)^{n-1}}{(n-1)!}, & s \leq t \end{cases}$$

for all $(t, s) \in [a, b] \times [a, b]$.

Lemma 1. *For $(t, s) \in [a, b] \times [a, b]$, we have*

$$G(t, s) \leq G(b, s). \quad (2.1)$$

Proof. For $a \leq t \leq s \leq b$, we have

$$\begin{aligned} G(t, s) &= \frac{(t-a)^{n-1}(b-s)^{n-p-1}}{(n-1)! (b-a)^{n-p-1}} \leq \frac{(b-a)^{n-1}(b-s)^{n-p-1}}{(n-1)! (b-a)^{n-p-1}} \\ &= G(b, s). \end{aligned}$$

Similarly, for $a \leq s \leq t \leq b$, we have $G(t, s) \leq G(b, s)$. Thus, we have

$$G(t, s) \leq G(b, s), \text{ for all } (t, s) \in [a, b] \times [a, b].$$

□

Lemma 2. *Let $I = \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]$. For $(t, s) \in I \times [a, b]$, we have*

$$G(t, s) \geq \gamma G(b, s). \quad (2.2)$$

Proof. The Green's function $G(t, s)$ for the homogeneous BVP corresponding to 1.1-1.3 is clearly shows that

$$G(t, s) > 0, \quad \text{on } (a, b) \times (a, b). \quad (2.3)$$

For $a \leq t \leq s \leq b$ and $t \in I$, we have

$$\frac{G(t, s)}{G(b, s)} = \left(\frac{t-a}{b-a} \right)^{n-1} \geq \frac{1}{4^{n-1}}$$

Similarly, for $a \leq s \leq t \leq b$ and $t \in I$, we have

$$\begin{aligned} \frac{G(t, s)}{G(b, s)} &= \frac{(t-a)^{n-1}(b-s)^{n-p-1} - (t-s)^{n-1}(b-a)^{n-p-1}}{(b-a)^{n-1}(b-s)^{n-p-1} - (b-s)^{n-1}(b-a)^{n-p-1}} \\ &\geq \frac{(t-a)^{n-p-1}(b-s)^{n-p-1}[(t-a)^p - (t-s)^p]}{(b-a)^{n-1}(b-s)^{n-p-1} - (b-s)^{n-1}(b-a)^{n-p-1}} \\ &\geq \frac{1}{p} \left(\frac{t-a}{b-a} \right)^{n-2} \geq \frac{1}{p} \left(\frac{t-a}{b-a} \right)^{n-1} \geq \frac{1}{p \cdot 4^{n-1}} \end{aligned} \quad (2.4)$$

Therefore

$$G(t, s) \geq \gamma G(b, s), \quad \text{for } (t, s) \in I \times [a, b],$$

where $\gamma = \left\{ \frac{1}{4^{n-1}}, \frac{1}{p \cdot 4^{n-1}} \right\}$ □

3 Intervals of Positive Solutions

In this section, first we prove a lemma which is needed in our main result and establish a criteria to determine eigenvalue intervals for which there exist at least one positive solution of the BVP 1.1-1.3.

Let $y(t)$ be the solution of the BVP 1.1-1.3, and is given by

$$y(t) = \lambda \int_a^b G(t, s) f(s, y(s)) ds, \quad \text{for all } t \in [a, b]. \quad (3.1)$$

Define

$$X = \{ u : u \in C[a, b] \},$$

with norm

$$\| u \| = \max_{t \in [a, b]} | u(t) |.$$

Then $(X, \| \cdot \|)$ is a Banach space. Define a set κ by

$$\kappa = \left\{ u \in X : u(t) \geq 0, \quad \text{on } [a, b] \quad \text{and} \quad \min_{t \in I} u(t) \geq \gamma \| u \| \right\}, \quad (3.2)$$

then it is easy to see that κ is a positive cone in X . Now we define the operator $T : \kappa \rightarrow X$ by

$$Ty(t) = \lambda \int_a^b G(t, s) f(s, y(s)) ds, \quad \text{for all } t \in [a, b]. \quad (3.3)$$

If $y \in \kappa$ is a fixed point of T , then y satisfies 3.1 and hence y is a positive solution of the BVP 1.1-1.3. We seek a fixed point of the operator T in the cone κ .

Lemma 3. *The operator T defined in 3.3 is a self map on κ .*

Proof. Let $y \in \kappa$. From 2.3, we have $Ty(t) \geq 0$, for all $t \in [a, b]$ and

$$\begin{aligned} Ty(t) &= \lambda \int_a^b G(t, s) f(s, y(s)) ds \\ &\leq \lambda \int_a^b G(b, s) f(s, y(s)) ds, \end{aligned}$$

so that

$$\|Ty\| \leq \lambda \int_a^b G(b, s) f(s, y(s)) ds.$$

Next, if $y \in \kappa$, then the inequality above we have

$$\begin{aligned} Ty(t) &= \lambda \int_a^b G(t, s) f(s, y(s)) ds \\ &\geq \gamma \lambda \int_a^b G(b, s) f(s, y(s)) ds, \\ &\geq \gamma \|Ty\|, \quad t \in I. \end{aligned}$$

Hence $T : \kappa \rightarrow \kappa$.

Standard arguments involving the Arzela-Ascoli theorem shows that T is completely continuous operator. \square

To establish eigenvalue intervals we will employ the following fixed point theorem due to Krasnosel'skii [13].

Theorem 2. *Let X be a Banach space, $K \subseteq X$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K$ is completely continuous operator such that either*

- (i) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$, or
- (ii) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$

holds. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 3. *For each λ satisfying*

$$\frac{1}{[\gamma^2 \int_{s \in I} G(b, s) ds] f_\infty} < \lambda < \frac{1}{[\int_a^b G(b, s) ds] f^0}, \quad (3.4)$$

there exists at least one positive solution of the BVP 1.1-1.3 in κ .

Proof. Let λ be given as in 3.4. Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{[\gamma^2 \int_{s \in I} G(b, s) ds] (f_\infty - \epsilon)} \leq \lambda \leq \frac{1}{[\int_a^b G(b, s) ds] (f^0 + \epsilon)}.$$

Let T be the cone preserving, completely continuous operator defined as in 3.3. By the definition of f^0 , there exists an $H_1 > 0$ such that

$$\max_{t \in [a, b]} \frac{f(t, y)}{y} \leq (f^0 + \epsilon), \quad \text{for } 0 < y \leq H_1.$$

It follows that, $f(t, y) \leq (f^0 + \epsilon)y$, for $0 < y \leq H_1$. So choosing $y \in \kappa$ with $\|y\| = H_1$, then from 2.1, we have

$$\begin{aligned}
 Ty(t) &= \lambda \int_a^b G(t, s) f(s, y(s)) ds \\
 &\leq \lambda \int_a^b G(b, s) f(s, y(s)) ds \\
 &\leq \lambda \int_a^b G(b, s) (f^0 + \epsilon) y(s) ds \\
 &\leq \lambda \int_a^b G(b, s) (f^0 + \epsilon) \|y\| ds \\
 &\leq \|y\|.
 \end{aligned} \tag{3.5}$$

Consequently, $\|Ty\| \leq \|y\|$. So, if we define

$$\Omega_1 = \{y \in X : \|y\| < H_1\},$$

then

$$\|Ty\| \leq \|y\|, \text{ for } y \in \kappa \cap \partial\Omega_1. \tag{3.6}$$

By the definition of f_∞ , there exists an $\overline{H}_2 > 0$ such that

$$\min_{t \in [a, b]} \frac{f(t, y)}{y} \geq (f_\infty - \epsilon), \text{ for } y \geq \overline{H}_2.$$

It follows that,

$$f(t, y) \geq (f_\infty - \epsilon)y, \text{ for } y \geq \overline{H}_2.$$

Let

$$H_2 = \max \left\{ 2H_1, \frac{\overline{H}_2}{\gamma} \right\},$$

and let

$$\Omega_2 = \{y \in X : \|y\| < H_2\}.$$

Now choose $y \in \kappa \cap \partial\Omega_2$ with $\|y\| = H_2$, so that

$$\min_{t \in I} y(t) \geq \gamma \|y\| \geq \overline{H}_2.$$

Consider,

$$\begin{aligned}
 Ty(t) &= \lambda \int_a^b G(t, s) f(s, y(s)) ds \\
 &\geq \lambda \int_a^b \gamma G(b, s) f(s, y(s)) ds \\
 &\geq \gamma \lambda \int_a^b G(b, s) (f_\infty - \epsilon) y(s) ds \\
 &\geq \gamma^2 \lambda \int_a^b G(b, s) (f_\infty - \epsilon) \|y\| ds \\
 &\geq \|y\|
 \end{aligned} \tag{3.7}$$

Thus,

$$\|Ty\| \geq \|y\|, \text{ for } y \in \kappa \cap \partial\Omega_2. \tag{3.8}$$

Applying (i) of Theorem 2 to 3.6 and 3.8 yields that T has a fixed point $y(t) \in \kappa \cap (\overline{\Omega}_2 \setminus \Omega_1)$. This fixed point is a positive solution of the BVP 1.1-1.3 for the given λ . \square

Theorem 4. For each λ satisfying

$$\frac{1}{[\gamma^2 \int_{s \in I} G(b, s) ds] f_0} < \lambda < \frac{1}{[\int_a^b G(b, s) ds] f^\infty}, \quad (3.9)$$

there exists at least one positive solution of the BVP 1.1-1.3 in κ .

Proof. Let λ be given as in 3.9, and choose $\epsilon > 0$ such that

$$\frac{1}{[\gamma^2 \int_{s \in I} G(b, s) ds] (f_0 - \epsilon)} \leq \lambda \leq \frac{1}{[\int_a^b G(b, s) ds] (f^\infty + \epsilon)}.$$

Let T be the cone preserving, completely continuous operator that was defined by 3.3. By the definition of f_0 , there exists an $J_1 > 0$ such that

$$\min_{t \in [a, b]} \frac{f(t, y)}{y} \geq (f_0 - \epsilon), \quad \text{for } 0 < y \leq J_1.$$

It follows that,

$$f(t, y) \geq (f_0 - \epsilon)y, \quad \text{for } 0 < y \leq J_1.$$

So, choose $y \in \kappa$ with $\|y\| = J_1$, then

$$\begin{aligned} Ty(t) &= \lambda \int_a^b G(t, s) f(s, y(s)) ds \\ &\geq \lambda \int_a^b \gamma G(b, s) f(s, y(s)) ds \\ &\geq \gamma \lambda \int_a^b G(b, s) (f_0 - \epsilon) y(s) ds \\ &\geq \gamma^2 \lambda \int_a^b G(b, s) (f_0 - \epsilon) \|y\| ds \\ &\geq \|y\|. \end{aligned} \quad (3.10)$$

Consequently, $\|Ty\| \geq \|y\|$. So, if we define

$$\Omega_1 = \{y \in X : \|y\| < J_1\},$$

then

$$\|Ty\| \geq \|y\|, \quad \text{for } y \in \kappa \cap \partial\Omega_1. \quad (3.11)$$

It remains for us to consider f^∞ . By the definition of f^∞ , there exists an $\bar{J}_2 > 0$ such that

$$\max_{t \in [a, b]} \frac{f(t, y)}{y} \leq (f^\infty + \epsilon), \quad \text{for } y \geq \bar{J}_2.$$

It follows that

$$f(t, y) \leq (f^\infty + \epsilon)y, \quad \text{for } y \geq \bar{J}_2.$$

There are two cases.

Case(i). Suppose f is bounded. Then there is $L > 0$ such that $f(t, y) \leq L$, for all $0 < y < \infty$. Let

$$J_2 = \max \left\{ 2J_1, \lambda L \int_a^b G(b, s) ds \right\}.$$

Then, for $y \in \kappa$ with $\|y\| = J_2$, we have

$$\begin{aligned}
 Ty(t) &= \lambda \int_a^b G(t, s) f(s, y(s)) ds \\
 &\leq \lambda \int_a^b G(b, s) f(s, y(s)) ds \\
 &\leq \lambda L \int_a^b G(b, s) ds \\
 &\leq \|y\|
 \end{aligned} \tag{3.12}$$

so that $\|Ty\| \leq \|y\|$. So, if we define

$$\Omega_2 = \{y \in X : \|y\| < J_2\},$$

then

$$\|Ty\| \leq \|y\|, \quad \text{for } y \in \kappa \cap \partial\Omega_2. \tag{3.13}$$

Case(ii). Suppose f is unbounded. Let $J_2 > \max\{2J_1, \bar{J}_2\}$ be such that $f(t, y) \leq f(t, J_2)$, for $0 < y \leq J_2$. Let $y \in \kappa$ with $\|y\| = J_2$, then

$$\begin{aligned}
 Ty(t) &= \lambda \int_a^b G(t, s) f(s, y(s)) ds \\
 &\leq \lambda \int_a^b G(b, s) f(s, y(s)) ds \\
 &\leq \lambda \int_a^b G(b, s) f(s, J_2) ds \\
 &\leq \lambda \int_a^b G(b, s) (f^\infty + \epsilon) J_2 ds \\
 &\leq J_2 \\
 &= \|y\|.
 \end{aligned} \tag{3.14}$$

$$\tag{3.15}$$

Thus, $\|Ty\| \leq \|y\|$. For this case, if we define

$$\Omega_2 = \{y \in X : \|y\| < J_2\},$$

then

$$\|Ty\| \leq \|y\|, \quad \text{for } y \in \kappa \cap \partial\Omega_2. \tag{3.16}$$

Thus, in either of the cases, an application of part (ii) of Theorem 2 to 3.11, 3.13 and 3.16 yields that T has a fixed point $y(t) \in \kappa \cap (\bar{\Omega}_2 \setminus \Omega_1)$. This fixed point is a positive solution of the BVP 1.1-1.3 for the given λ . \square

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