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International Journal of Applied Mathematics and Computation
Volume 2(2),pp 11–33, 2010
http://ijamc.psit.in

Convergence theorems of common elements for mixed equilibrium problems, variational inequality problems and fixed point problems in Banach spaces

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Abstract:

In this paper, a hybrid iterative scheme is introduced for the approximation method for finding a common element of the set of mixed equilibrium problems, variational inequality problems and fixed point problems of two quasi-ϕ-nonexpansive mappings in a real uniformly convex and uniformly smooth Banach space. Then, we obtain a strong convergence theorem for the sequence generated by those process in Banach spaces. Moreover, we obtain new result for finding a zero point of maximal monotone operators in a Banach space. Our results improve and extend the corresponding results announced by Takahashi and Zembayashi [Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, Nonlinear Anal. 70 (2009) 45–57.] Qin et al. [Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math., 225 (2009), 20–30.] Wattanawitoon and Kumam [Strong convergence theorems by a new hybrid projection algorithm for fixed point problems and equilibrium problems of two relatively quasi-nonexpansive mappings, Nonlinear Anal: Hybrid Systems. 3 (2009) 11–20.] Cholamjiak [A hybrid iterative scheme for equilibrium problems, variational inequality problems and fixed point problems in Banach spaces, Fixed Point Theory Appl., (2009), Article ID 719360, 18 pages] and many authors.

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Keywords: Hybrid iterative; strong convergence; mixed equilibrium problem; variational inequality problem; quasi-ϕ-nonexpansive mapping.

1 Introduction

Let E be a real Banach space, E^* the dual space of E. Let C be a nonempty closed convex subset of E and Θ a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of numbers and $\varphi : C \to \mathbb{R}$ be a real-valued functionand $B : C \longrightarrow E^*$ be a nonlinear mapping. The *generalized mixed equilibrium problem*, is to find $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions to (1.1) is denoted by Ω, i.e.,

$$\Omega = \{x \in C : \Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}. \tag{1.2}$$

If $B = 0$, the problem (1.1) reduce into the *mixed equilibrium problem for* Θ, denoted by $MEP(\Theta, \varphi)$, is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

If $\varphi \equiv 0$, the problem (1.1) reduce into the *equilibrium problem for* Θ, denoted by $EP(\Theta)$, is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

If $\Theta \equiv 0$, the problem (1.1) reduce into the *minimize problem*, denoted by $Argmin(\varphi)$, is to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The above formulation (1.4) was shown in [6] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an $EP(\Theta)$. In other words, the $EP(\Theta)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of $EP(\Theta)$; see, for example [6, 12, 19, 34] and references therein. In 2005, Combettes and Hirstoaga [10] introduced an iterative scheme of finding the best approximation to the initial data when $EP(\Theta)$ is nonempty and they also proved a strong convergence theorem. Some solution methods have been proposed to solve the $EP(\Theta)$; see, for example, [10, 12, 14, 15, 29, 30, 31, 34] and references therein.

Let $A : C \longrightarrow E^*$ be an operator. The classical *variational inequality*, denoted by $VI(A, C)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.6}$$

Recall that let $A : C \longrightarrow E^*$ be a mapping. Then A is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \text{for all} \quad x, y \in C,$$

(ii) $\alpha-$inverse$-$strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \text{for all} \quad x, y \in C.$$

Let E be a real Banach space with norm $\|\cdot\|$ and let J be the *normalized duality mapping* from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ the generalized duality pairing between E and E^*. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

Consider the functional ϕ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{1.7}$$

Observe that, in a Hilbert space H, (1.7) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. The *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x) \tag{1.8}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [1, 2, 8, 13, 32]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.9}$$

Let C be a closed convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [26] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$. A mapping T from C into itself is said to be *relatively nonexpansive* [20, 28, 40] if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [5, 7]. T is said to be ϕ-nonexpansive, if $\phi(Tx, Ty) \leq \phi(x, y)$ for $x, y \in C$. T is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and $p \in F(T)$.

Remark 1.1 *The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [5, 7, 17] which requires the strong restriction: $F(T) = \widetilde{F(T)}$.*

Recall that an operator T in a Banach space is call closed, if $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$, then $Tx = y$.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Matsushita and Takahashi [18] introduced the following iteration sequence $\{x_n\}$: defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n) \tag{1.10}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0,1]$, J is the duality mapping on E, T is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

In 2005, Matsushita and Takahashi [17] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.11}$$

They proved that the $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

In 2007, Plubtieng and Ungchittrakool [23] proved the new generalized processes of two relatively nonexpansive mappings in a Banach space. Let C be a closed convex subset of a Banach space E and $S, T : C \longrightarrow C$ two relatively nonexpansive mappings such that $F := F(S) \cap F(T) \neq \emptyset$. Define $\{x_n\}$ in the following ways:

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n), \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \end{cases} \tag{1.12}$$

where $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ and $\{\beta_n^{(3)}\}$ are sequences in $[0,1]$ with $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ for all $n \in \mathbb{N} \cup \{0\}$. They proved that the $\{x_n\}$ converges strongly to $\Pi_F x$.

In 2009, Qin et al. [25] modified the Halpern-type iteration algorithm for closed quasi-

ϕ-nonexpansive mappings defined by:

$$\begin{cases} x_0 \in E \ \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J(x_1) + (1-\alpha_n) JT(x_n)), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1. \end{cases} \tag{1.13}$$

Then they proved that under appropriate control conditions the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$.

In 2009, Qin et al. [24] proved that $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 \in E \ \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n JT x_n + \gamma_n JS x_n), \\ u_n \in C \ \textit{such that} \ \Theta(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.14}$$

where J is the duality mapping on E. Then $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap F(T)\cap EP(\Theta)} x_0$.

Recently, Takahashi and Zembayashi [35], proposed the following modification of iteration process (1.10) for a relatively nonexpansive mapping:

$$\begin{cases} x_0 = x \in C, \quad C_0 = C, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) JS x_n), \\ u_n \in C \ \textit{such that} \ \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases} \tag{1.15}$$

where J is the duality mapping on E, and $\Pi_{C_n\cap Q_n}$ is the generalized projection from E onto $C_n \cap Q_n$. They proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap EP(\Theta)} x_0$.

Very recently, Cholamjiak [9] introduce process for finding common elements of set of equilibrium problems, set of variatinal inequality problems and the set of the fixed points for quasi-ϕ-nonexpansive mappings in Banach spaces

$$\begin{cases} x_0 \in C, \quad x_1 = \Pi_{C_1} x_0, \quad C_1 = C \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n JT x_n + \gamma_n JS z_n), \\ u_n \in C \ \textit{such that} \ \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{1.16}$$

then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = \Pi_F x_0$, where $F = F(T) \cap F(S) \cap EP(\Theta) \cap VI(A, C)$.

Motivated and inspired by the work of Plubtieng and Ungchittrakool [23], Qin et al. [24], Takahashi and Zembayashi [35], Wattanawitoon and Kumam [37] and Cholamjiak [9], we introduced the hybrid projection iterative scheme (so-call the CQ method) define by (3.1) below for finding a common element of the set of solutions of an equilibrium problem, set of solution of the variational inequality and the set of a common fixed points of two quasi-ϕ-nonexpansive mappings in the framework Banach spaces. Moreover, we obtain new result for finding a zero point of maximal monotone operators in a Banach space. The results obtained in this paper improve and extend the recent ones announced by Takahashi and Zembayashi [35], Cholamjiak [9] and many authors.

2 Preliminaries

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$.

The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x + y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}. \tag{2.1}$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$; see [3, **?**] for more details. Observe that every p-uniform convex Banach space is uniformly convex. One should note that no a Banach space is p-uniform convex for $1 < p < 2$. It is well known that a Hilbert space is *2*-uniformly convex and uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\} \tag{2.2}$$

for all $x \in E$. In particular, $J = J_2$ is called *the normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

We know the following (see [**?**]):

(1) If E is smooth, then J is single-valued;

(2) if E is strictly convex, then J is one-to-one and $\langle x - y, x^* - y^*\rangle > 0$ holds for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$;

(3) if E is reflexive, then J is surjective;

(4) if E is uniformly convex, then it is reflexive;

(5) if E^* is uniformly convex, then J is uniformly norm-to-norm continuous on each bounded subset of E.

The duality J from a smooth Banach space E into E^* is said to be *weakly sequentially continuous* [**?**] if $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup^* Jx$, where $\rightharpoonup^*$ implies the weak* convergence.

Lemma 2.1 ([4, 38]). *If E be a 2-uniformly convex Banach space. Then for all $x, y \in E$,*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ in Lemma is called the *2*-uniformly convex constant of E; see [3].

Lemma 2.2 ([4, 39]). *If E be a p-uniformly convex Banach space and let p be a given real number with $p \geq 2$. Then for all $x, y \in E$, $J_x \in J_p(x)$ and $J_y \in J_p(y)$*

$$\langle x - y, Jx - Jy\rangle \geq \frac{c^p}{2^{p-2}p}\|x - y\|^p,$$

where J_p is the generalized duality mapping of E and $\frac{1}{c}$ is the p-uniformly convexity constant of E.

It is obvious from the definition of ϕ that $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$. We also know that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \tag{2.3}$$

for all $x, y \in E$.

Lemma 2.3 (*Kamimura and Takahashi* [13]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\lim_{n\longrightarrow\infty} \phi(x_n, y_n) = 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$.*

Lemma 2.4 (*Alber* [1]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.5 (*Alber* [1]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.6 (*Qin et al.* [24, Lemma 2.4.]). *Let E be a uniformly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a closed and quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.7 (*Cho et al.* [11]). *Let X be a uniformly convex Banach space and $B_r(0)$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|),$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.8 (*Kamimura and Takahashi* [13]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r$.*

We make use of the following mapping V studied in Alber[1]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \tag{2.4}$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.9 (*Alber* [1]). *Let E be a reflexive, strictly convex and smooth Banach space and let V be as in* (2.4). *Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

A set valued mapping $T : H \longrightarrow 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \longrightarrow 2^H$ is *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is know that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - h\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into E^* is said to be hemicontinuous if for all $x, y \in C$, the mapping F of $[0, 1]$ into E^* defined by $F(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ the normal cone for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0, \quad \forall y \in C\}. \tag{2.5}$$

Theorem 2.10 (*Rockafellar* [27]). *Let C be a nonempty, closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & otherwise. \end{cases} \tag{2.6}$$

Then T is maximal monotone and $T^{-1}0 = VI(A, C)$.

For solving the mixed equilibrium problem, let us assume that the bifunction $\Theta : C \times C \to \mathbb{R}$ and $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous satisfies the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y);$$

(A4) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex and lower semi-continuous.

Lemma 2.11 (*Blum and Oettli* [6]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$ and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$\Theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

The following lemma was also given by Combettes in [10].

Lemma 2.12 (*Takahashi and Zembayashi* [36]). *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E, and let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$. For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \{z \in C : \Theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\},$$

for all $x \in C$. Then the following hold:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) *$F(T_r) = EP(\Theta)$;*

(4) *$EP(\Theta)$ is closed and convex.*

Lemma 2.13 (*Takahashi and Zembayashi* [36]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$, and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Lemma 2.14 (*Zhang* [41]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $B : C \longrightarrow E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous and Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$\Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $K_r : C \longrightarrow C$ as follows:

$$K_r(x) = \{u \in C : \Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C\} \tag{2.7}$$

for all $x \in E$. Then, the followings hold:

1. *K_r is single-valued;*

2. *K_r is firmly nonexpansive, i.e., for all $x, y \in E$, $\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle$;*

3. *$F(K_r) = \Omega$;*

4. *Ω is closed and convex.*

5. *$\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, z)$ $\forall p \in F(K_r)$, $z \in E$.*

Remark 2.15 (Zhang *[41]). It follows from Lemma 2.12 that the mapping $K_r : C \longrightarrow C$ defined by* (2.7) *is a relatively nonexpansive mapping. Thus, it is quasi-ϕ-nonexpansive.*

3 Strong convergence theorems

In this section, using the CQ hybrid method, we prove a strong convergence theorem for finding a common element of the set of solutions of a mixed equilibrium problem, the set of solutions of the variational inequality problem and the set of fixed points of quasi-ϕ-nonexpansive mappings in a Banach space.

Theorem 3.1 *Let C be a nonempty closed convex subset of a smooth and 2-uniformly convex Banach space E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function, let A be an α-inverse-strongly monotone operator of C into E^* and let $T, S : C \to C$ be closed quasi-ϕ-nonexpansive mappings such that $F := F(T) \cap F(S) \cap VI(A, C) \cap MEP(\Theta, \varphi) \neq \emptyset$ and $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in F$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 = x \in C, \\ w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ u_n \in C \;\; \text{such that} \;\; \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases} \tag{3.1}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ satisfying the restrictions:

(i) $\limsup_{n\longrightarrow\infty} \alpha_n < 1$;

(ii) $\beta_n + \gamma_n + \delta_n = 1$;

(iii) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\liminf_{n\to\infty} \beta_n\delta_n > 0$;

(iv) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;

(v) $\{\lambda_n\} \subset [d, e]$ *for some* d, e *with* $0 < d < e < \frac{c^2\alpha}{2}$, *where* $\frac{1}{c}$ *is the 2-uniformly convex constant of* E

Then $\{x_n\}$ *converges strongly to* $p \in F$, *where* $p = \Pi_F x$.

***Proof*.** We first show that $C_n \cap Q_n$ is closed and convex for each $n \geq 0$. It is obvious that C_n is closed and Q_n is closed and convex. Since

$$\phi(z, u_n) \leq \phi(z, x_n)$$

is equivalent to

$$2\langle z, Ju_n\rangle - 2\langle z, Jx_n\rangle \leq \|u_n\|^2 - \|x_n\|^2,$$

C_n is convex. So, $C_n \cap Q_n$ is closed and convex subset of E for all $n \in \mathbb{N} \cup \{0\}$.
Put $v_n = J^{-1}(Jx_n - \lambda_n Ax_n)$. We observe that $u_n = K_{r_n} y_n$ for all $n \geq 1$ and let $p \in F$, it follows from the definition of quasi-ϕ-nonexpansive that

$$\begin{aligned}
\phi(p, u_n) &= \phi(p, K_{r_n} y_n) \\
&\leq \phi(p, y_n) \\
&= \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)Jz_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)Jz_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\langle p, Jz_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|z_n\|^2 \\
&= \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n), \qquad (3.2)
\end{aligned}$$

and then

$$\begin{aligned}
\phi(p, z_n) &= \phi(p, J^{-1}(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n)) \\
&= \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2\gamma_n\langle p, JTx_n\rangle - 2\delta_n\langle p, Jwx_n\rangle \\
&\quad + \|\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n\|^2 \\
&\leq \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2\gamma_n\langle p, JTx_n\rangle - 2\delta_n\langle p, JSw_n\rangle \\
&\quad + \beta_n\|Jx_n\|^2 + \gamma_n\|JTx_n\|^2 + \delta_n\|JSw_n\|^2 \\
&= \beta_n\phi(p, x_n) + \gamma_n\phi(p, Tx_n) + \delta_n\phi(p, Sw_n) \\
&\leq \beta_n\phi(p, x_n) + \gamma_n\phi(p, x_n) + \delta_n\phi(p, w_n). \qquad (3.3)
\end{aligned}$$

From Lemma 2.5 and Lemma 2.9

$$\begin{aligned}
\phi(p, w_n) &= \phi(p, \Pi_C v_n) \\
&\leq \phi(p, v_n) = \phi(p, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&\leq V(p, Jx_n - \lambda_n Ax_n + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - p, \lambda_n Ax_n\rangle \\
&= V(p, Jx_n) - 2\lambda_n\langle v_n - p, Ax_n\rangle \\
&= \phi(p, x_n) - 2\lambda_n\langle x_n - p, Ax_n\rangle + 2\langle v_n - x_n, -\lambda_n Ax_n\rangle. \qquad (3.4)
\end{aligned}$$

Since $p \in VI(A, C)$ and A is α-inverse-strongly monotone, we have

$$\begin{aligned}
-2\lambda_n\langle x_n - p, Ax_n\rangle &= -2\lambda_n\langle x_n - p, Ax_n - Ap\rangle - 2\lambda_n\langle x_n - p, Ap\rangle \\
&\leq -2\alpha\lambda_n\|Ax_n - Ap\|^2, \qquad (3.5)
\end{aligned}$$

and we obtain

$$\begin{aligned} 2\langle v_n - x_n, -\lambda_n A x_n\rangle &= 2\langle J^{-1}(Jx_n - \lambda_n A x_n) - x_n, -\lambda_n A x_n\rangle \\ &\le 2\|J^{-1}(Jx_n - \lambda_n A x_n) - x_n\|\|\lambda_n A x_n\| \\ &= \frac{4}{c^2}\|Jx_n - \lambda_n A x_n - Jx_n\|\|\lambda_n A x_n\| \\ &= \frac{4}{c^2}\lambda_n^2\|Ax_n\|^2 \\ &\le \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2. \end{aligned} \tag{3.6}$$

Replacing (3.5) and (3.6) into (3.4), we get

$$\begin{aligned} \phi(p, w_n) &\le \phi(p, x_n) - 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Ax_n - Ap\|^2 \\ &\le \phi(p, x_n). \end{aligned} \tag{3.7}$$

From (3.2), (3.3) and (3.7), we have

$$\phi(p, u_n) \le \phi(p, x_n). \tag{3.8}$$

Hence, we have $p \in C_n$. This implies that

$$F \subset C_n, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.9}$$

Next, we show by induction that $F \subset C_n \cap Q_n$ for all $n \in \mathbb{N} \cup \{0\}$. From $Q_0 = C$, we have $F \subset C_0 \cap Q_0$. Suppose that $F \subset C_k \cap Q_k$ for some $k \in \mathbb{N} \cup \{0\}$. Then there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = \Pi_{C_k \cap Q_k} x$. From the definition of x_{k+1}, we have, for all $z \in C_k \cap Q_k$,

$$\langle x_{k+1} - z, Jx - Jx_{k+1}\rangle \ge 0. \tag{3.10}$$

Since $F \subset C_k \cap Q_k$, we have

$$\langle x_{k+1} - p, Jx_0 - Jx_{k+1}\rangle \ge 0, \quad \forall p \in F, \tag{3.11}$$

and hence $p \in Q_{k+1}$. So, we have

$$F \subset Q_{k+1}. \tag{3.12}$$

Hence by (3.9) and (3.12) we obtain

$$F \subset C_{k+1} \cap Q_{k+1}.$$

So, we have that $F \subset C_k \cap Q_k$ for all $n \in \mathbb{N} \cup \{0\}$. This means that $\{x_n\}$ is well defined.

Using $x_n = \Pi_{Q_n} x$, from Lemma 2.5, one has

$$\phi(x_n, x) = \phi(\Pi_{Q_n} x, x) \le \phi(p, x) - \phi(p, x_n) \le \phi(p, x),$$

for each $p \in F \subset Q_n$ and $x_n = \Pi_{Q_n} x$. Thus $\phi(x_n, x)$ is bounded. Then $\{x_n\}$, $\{Sw_n\}$ and $\{Tx_n\}$ are bounded.

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x \in C_n \cap Q_n$ and $x_n = \Pi_{Q_n} x$, we have

$$\phi(x_n, x) \le \phi(x_{n+1}, x), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, $\{\phi(x_n, x)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x)\}$ exists. By the construction of Q_n, we have $Q_m \subset Q_n$ and $x_m = \Pi_{Q_m} x \in Q_n$ for any positive integer $m \ge n$. It follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{Q_n} x) \\ &\le \phi(x_m, x) - \phi(\Pi_{Q_n} x, x) \\ &= \phi(x_m, x) - \phi(x_n, x). \end{aligned} \tag{3.13}$$

Letting $m, n \longrightarrow \infty$ in (3.13), we have $\phi(x_m, x_n) \longrightarrow 0$ as $n \longrightarrow \infty$. It follows from Lemma 2.3, that $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \longrightarrow \hat{x} \in C$ as $n \longrightarrow \infty$. Since

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{Q_n} x) \le \phi(x_{n+1}, x) - \phi(\Pi_{Q_n} x, x) = \phi(x_{n+1}, x) - \phi(x_n, x),$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n \longrightarrow \infty} \phi(x_{n+1}, x_n) = 0$. From $x_{n+1} = \Pi_{C_n \cap Q_n} x \in C_n$, we have

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, we also have

$$\lim_{n \longrightarrow \infty} \phi(x_{n+1}, u_n) = 0.$$

Since E is uniformly convex and smooth, we have from Lemma 2.3 that

$$\lim_{n \longrightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \longrightarrow \infty} \|x_{n+1} - u_n\| = 0.$$

So, we have

$$\lim_{n \longrightarrow \infty} \|x_n - u_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n \longrightarrow \infty} \|Jx_n - Ju_n\| = 0.$$

Since E is uniformly smooth Banach spaces, one knows that E^* is a uniformly convex Banach apace. Let $r = \sup_{n \in \mathbb{N} \cup \{0\}} \{\|x_n\|, \|Tx_n\|, \|Sw_n\|\}$. From Lemma 2.7 and (3.7), we have

$$\begin{aligned} \phi(p, z_n) &= \phi(p, J^{-1}(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n)) \\ &= \|p\|^2 - 2\beta_n \langle p, Jx_n \rangle - 2\gamma_n \langle p, JTx_n \rangle - 2\delta_n \langle p, JSw_n \rangle \\ &\quad + \|\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n\|^2 \\ &\le \|p\|^2 - 2\beta \langle p, Jx_n \rangle - 2\gamma_n \langle p, JTx_n \rangle - 2\delta_n \langle p, JSw_n \rangle \\ &\quad + \beta_n \|x_n\|^2 + \gamma_n \|Tx_n\|^2 + \delta_n \|Sw_n\|^2 - \beta_n \gamma_n g(\|JTx_n - Jx_n\|) \\ &= \beta_n \phi(p, x_n) + \gamma_n \phi(p, Tx_n) + \delta_n \phi(p, Sw_n) - \beta_n \gamma_n g(\|JTx_n - Jx_n\|) \\ &\le \phi(p, x_n) - \beta_n \gamma_n g(\|JTx_n - Jx_n\|) - 2\lambda_n (\alpha - \frac{2}{c^2} \lambda_n) \delta_n \|Ax_n - Ap\|^2. \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.2), we have

$$\begin{aligned}\phi(p,u_n) &\leq \alpha_n\phi(p,x_n)+(1-\alpha_n)[\phi(p,x_n)-\beta_n\gamma_n g(\|JTx_n-Jx_n\|)\\ &\quad -2\lambda_n(\alpha-\frac{2}{c^2}\lambda_n)\delta_n\|Ax_n-Ap\|^2]\\ &\leq \alpha_n\phi(p,x_n)+(1-\alpha_n)\phi(p,x_n)-(1-\alpha_n)\beta_n\gamma_n g(\|JTx_n-Jx_n\|)\\ &\quad -2\lambda_n(1-\alpha_n)(\alpha-\frac{2}{c^2}\lambda_n)\delta_n\|Ax_n-Ap\|^2.\end{aligned} \tag{3.15}$$

Therefore, we have

$$(1-\alpha_n)\beta_n\gamma_n g(\|JTx_n-Jx_n\|)\leq\phi(p,x_n)-\phi(p,u_n).$$

On the other hand, we have

$$\begin{aligned}\phi(p,x_n)-\phi(p,u_n) &= \|x_n\|^2-\|u_n\|^2-2\langle p,Jx_n-Ju_n\rangle\\ &\leq \|x_n-u_n\|(\|x_n\|+\|u_n\|)+2\|p\|\|Jx_n-Ju_n\|.\end{aligned}$$

It follow from $\|x_n-u_n\|\longrightarrow 0$ and $\|Jx_n-Ju_n\|\longrightarrow 0$ that

$$\lim_{n\longrightarrow\infty}(\phi(p,x_n)-\phi(p,u_n))=0. \tag{3.16}$$

Observing that assumption $\liminf_{n\longrightarrow\infty}\beta_n\gamma_n>0$ and by Lemma 2.8, we also

$$\lim_{n\longrightarrow\infty} g\|Jx_n-JTx_n\|=0.$$

It follows from the property of g that

$$\lim_{n\longrightarrow\infty}\|Jx_n-JTx_n\|=0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we see that

$$\lim_{n\longrightarrow\infty}\|x_n-Tx_n\|=0. \tag{3.17}$$

Similarly, one can obtain

$$\lim_{n\longrightarrow\infty}\|x_n-Sw_n\|=0. \tag{3.18}$$

By (3.15), we have

$$2\lambda_n(\alpha-\frac{2}{c^2}\lambda_n)\delta_n\|Ax_n-Ap\|^2\leq\phi(p,x_n)-\phi(p,u_n),$$

which yield that

$$\lim_{n\longrightarrow\infty}\|Ax_n-Ap\|=0. \tag{3.19}$$

From Lemma 2.5, Lemma 2.9, and (3.6), we have

$$\begin{aligned}\phi(x_n,w_n)=\phi(x_n,\Pi_C v_n) &\leq \phi(x_n,v_n)\\ &= \phi(x_n,J^{-1}(Jx_n-\lambda_nAx_n))\\ &= V(x_n,Jx_n-\lambda_nAx_n)\\ &\leq V(x_n,(Jx_n-\lambda_nAx_n)+\lambda_nAx_n)-2\langle J^{-1}(Jx_n-\lambda_nAx_n)-x_n,\lambda_nAx_n\rangle\\ &= \phi(x_n,x_n)+2\langle v_n-x_n,\lambda_nAx_n\rangle\\ &= 2\langle v_n-x_n,\lambda_nAx_n\rangle\\ &\leq 2\lambda_n^2\|Ax_n-Ap\|^2.\end{aligned}$$

From Lemma 2.3 and (3.19), we have

$$\lim_{n\longrightarrow\infty}\|x_n - w_n\| = 0. \tag{3.20}$$

Since J is also uniformly norm-to-norm continuous on bounded sets, we see that

$$\lim_{n\longrightarrow\infty}\|Jx_n - Jw_n\| = 0. \tag{3.21}$$

By (3.18) and (3.20), we obtain

$$\lim_{n\longrightarrow\infty}\|Sw_n - w_n\| = 0. \tag{3.22}$$

From (3.20), we have

$$\lim_{n\longrightarrow\infty}\|Sx_n - x_n\| = 0.$$

Since S and T are closed operators and $x_n \longrightarrow \hat{x}$, hence $\hat{x}$ is a common fixed point of S and T, i.e., $\hat{x} \in F(T) \cap F(S)$.

Next, we show that $\hat{x} \in MEP(\Theta, \varphi)$. Since $u_n = K_{r_n} y_n$. From Lemma 2.13, we have

$$\begin{aligned}
\phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \\
&\leq \phi(\hat{x}, y_n) - \phi(\hat{x}, K_{r_n} y_n) \\
&\leq \phi(\hat{x}, x_n) - \phi(\hat{x}, K_{r_n} y_n) \\
&= \phi(\hat{x}, x_n) - \phi(\hat{x}, u_n) \\
&= \|x_n\|^2 - \|u_n\|^2 - 2\langle \hat{x}, Jx_n - Ju_n\rangle \\
&\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|\hat{x}\|\|Jx_n - Ju_n\|.
\end{aligned}$$

It follows from $\|x_n - u_n\| \longrightarrow 0$ and $\|Jx_n - Ju_n\| \longrightarrow 0$ that

$$\phi(u_n, y_n) \to 0, \quad as\ n \to \infty.$$

and so

$$\lim_{n\longrightarrow\infty}\|u_n - y_n\| = 0. \tag{3.23}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\longrightarrow\infty}\|Ju_n - Jy_n\| = 0. \tag{3.24}$$

From (3.1) and (A2), we also have

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq \Theta(y, u_n), \quad \forall y \in C.$$

Hence,

$$\varphi(y) - \varphi(u_{n_i}) + \langle y - u_{n_i}, \frac{Ju_{n_i} - Jy_{n_i}}{r_{n_i}}\rangle \geq \Theta(y, u_{n_i}), \quad \forall y \in C.$$

From $\|x_n - u_n\| \longrightarrow 0$, we get $u_{n_i} \longrightarrow \hat{x}$. Since $\dfrac{Ju_{n_i} - Jy_{n_i}}{r_{n_i}} \longrightarrow 0$, it follows by (A4) and the weakly lower semicontinuous of φ that

$$\Theta(y, \hat{x}) + \varphi(\hat{x}) - \varphi(y) \leq 0, \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)u$. Since $y \in C$ and $\hat{x} \in C$, we have $y_t \in C$ and hence $\Theta(y_t, \hat{x}) + \varphi(\hat{x}) - \varphi(y_t) \leq 0$. So, from (A1), (A4) and the convexity of φ, we have

$$\begin{aligned} 0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, u) + t\varphi(y) + (1-t)\varphi(y) - \varphi(y_t) \\ &\leq t(\Theta(y_t, y) + \varphi(y) - \varphi(y_t)). \end{aligned}$$

Dividing by t, we get $\Theta(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0$. From (A3) and the weakly lower semi-continuity of φ, we have $\Theta(\hat{x}, y) + \varphi(y) - \varphi(\hat{x}) \geq 0$ for all $y \in C$ implies $\hat{x} \in MEP(\Theta, \varphi)$.

Next, we show that $\hat{x} \in VI(A, C)$. Define $T \subset E \times E^*$ be as in (2.6). By Theorem 2.10, T is maximal monotone and $T^{-1}0 = VI(A, C)$. Let $(v, w) \in G(T)$. Since $w \in Tv = Av + N_C(v)$, we get $w - Av \in N_C(v)$. From $w_n \in C$, we have

$$\langle v - w_n, w - Av \rangle \geq 0. \tag{3.25}$$

On the other hand, since $w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then by Lemma 2.4, we have

$$\langle v - w_n, Jw_n - (Jx_n - \lambda_n Ax_n) \rangle \geq 0,$$

thus

$$\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n \rangle \leq 0. \tag{3.26}$$

It follows from (3.25) and (3.26) that

$$\begin{aligned} \langle v - w_n, w \rangle &\geq \langle v - w_n, Av \rangle \\ &\geq \langle v - w_n, Av \rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n \rangle \\ &= \langle v - w_n, Av - Ax_n \rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} \rangle \\ &= \langle v - w_n, Av - Aw_n \rangle + \langle v - w_n, Aw_n - Ax_n \rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} \rangle \\ &\geq -\|v - w_n\| \tfrac{\|w_n - x_n\|}{\alpha} - \|v - w_n\| \frac{\|Jx_n - Jw_n\|}{b} \\ &\geq -M(\frac{\|w_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jw_n\|}{b}), \end{aligned}$$

where $M = \sup_{n \geq 1}\{\|v - w_n\|\}$. From (3.20) and (3.21), we obtain $\langle v - \hat{x}, w \rangle \geq 0$. By the maximality of T, we have $\hat{x} \in T^{-1}0$ and hence $\hat{x} \in VI(A, C)$. Hence $\hat{x} \in F := VI(C, A) \cap T^{-1}(0) \cap MEP(\Theta, \varphi)$.

Finally, we prove that $\hat{x} = \Pi_F x_0$. From $x_n = \Pi_{C_n \cap Q_n} x$, we have

$$\langle Jx - Jx_n, x_n - z \rangle \geq 0, \quad \forall z \in C_n \cap Q_n.$$

Since $F \subset C_n \cap Q_n$, we also have

$$\langle Jx - Jx_n, x_n - p \rangle \geq 0, \quad \forall p \in F. \tag{3.27}$$

By taking limit in (3.27), one has

$$\langle Jx - J\hat{x}, \hat{x} - p\rangle \geq 0, \quad \forall p \in F.$$

At this point, in view of Lemma 2.4, one sees that $\hat{x} = \Pi_F x_0$. This completes the proof. $\square$

Corollary 3.2 *Let C be a nonempty closed convex subset of a smooth and 2-uniformly convex Banach space E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function, let $T, S : C \to C$ be closed quasi-ϕ-nonexpansive mappings such that $F := F(T) \cap F(S) \cap MEP(\Theta, \varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 = x \in C, \\ z_n = J^{-1}(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSx_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ u_n \in C \ \text{ such that } \ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases} \tag{3.28}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ satisfying the restrictions:

(i) $\limsup_{n\longrightarrow\infty} \alpha_n < 1$;

(ii) $\beta_n + \gamma_n + \delta_n = 1$;

(iii) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\liminf_{n\to\infty} \beta_n\delta_n > 0$;

(iv) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x$.

Proof. In Theorem 3.1 if $A \equiv 0$, then (3.1) reduced to (3.28). $\square$

Since every relatively nonexpansive mapping is a quasi-ϕ-nonexpansive mapping, we obtain the following result.

Corollary 3.3 *Let C be a nonempty closed convex subset of a smooth and 2-uniformly convex Banach space E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function, let A be an α-inverse-strongly monotone operator of C into E^* and let $T : C \to C$ be closed relatively nonexpansive mappings such that $F := F(T) \cap F(S) \cap VI(A, C) \cap MEP(\Theta) \neq \emptyset$ and $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in F$. Let $\{x_n\}$ be a sequence generated by the* *following manner:*

$$\begin{cases} x_0 = x \in C, \\ w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ u_n \in C \quad \text{such that} \quad \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases} \tag{3.29}$$

for every $n \in \mathbb{N} \cup \{0\}$, *where* J *is the duality mapping on* E. *Assume that* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *are sequences in* $[0, 1]$ *satisfying the restrictions:*

(i) $\limsup_{n\longrightarrow\infty} \alpha_n < 1$;

(ii) $\beta_n + \gamma_n + \delta_n = 1$;

(iii) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\liminf_{n\to\infty} \beta_n\delta_n > 0$;

(iv) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;

(v) $\{\lambda_n\} \subset [d, e]$ *for some* d, e *with* $0 < d < e < \dfrac{c^2\alpha}{2}$, *where* $\dfrac{1}{c}$ *is the 2-uniformly convex constant of* E

Then $\{x_n\}$ *converges strongly to* $p \in F$, *where* $p = \Pi_F x$.

4 Applications

4.1 A zero point of monotone operator

Next, we consider the problem of finding a zero point of an inverse-strongly monotone operator of E into E^*.

Theorem 4.1 *Let* E *be a smooth and 2-uniformly convex Banach space. Let* Θ *be a bifunction from* $E \times E$ *to* $\mathbb{R}$ *satisfying (A1)-(A4) and let* $\varphi : E \longrightarrow \mathbb{R}$ *be a proper lower semicontinuous and convex function, let* A *be an* α*-inverse-strongly monotone operator of* E *into* E^* *and* $A^{-1}0 = \{u \in E : Au = 0\} \neq \emptyset$ *and let* $T, S : E \to E$ *be closed quasi-*ϕ*-nonexpansive mappings such that* $F := F(T) \cap F(S) \cap A^{-1}0 \cap MEP(\Theta, \varphi) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by the following manner:*

$$\begin{cases} x_0 = x \in E, \\ w_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ u_n \in E \quad \text{such that} \quad \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in E, \\ C_n = \{z \in E : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases}$$

$$(4.1)$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the conditions (i)-(v) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x$.

Proof. Setting $C = E$ in Theorem 3.1, we have $\Pi_E = I$. We also have $VI(A, E) = A^{-1}0$ then the condition $\|Ay\| \leq \|Ay - Au\|$ holds for all $y \in E$ and $u \in A^{-1}0$. So, we obtain the result. □

4.2 A zero of maximal monotone operator

Let B be a multivalued operator from E to E^* with domain $D(B) = \{z \in E : Az \neq \emptyset\}$ and range $R(B) = \cup\{Bz : z \in D(B)\}$. An operator B is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(B)$ and $y_i \in Ax_i, i = 1, 2$. A monotone operator B is said to be maximal if its graph $G(B) = \{(x, y) : y \in Ax\}$ is not property contained in the graph of any other monotone operator. We know that if B is a maximal monotone operator, then $B^{-1}(0) = \{z \in D(B) : 0 \in Bz\}$ is closed and convex. Let E be a reflexive, strictly convex and smooth Banach space, and let B be a monotone operator from E to E^*, we know that B is maximal if and only if $R(J + rB) = E^*$ for all $r > 0$. Let $J_r : E \to D(B)$ defined by $J_r = (J + rB)^{-1}J$ and such a J_r is called the resolvent of B. We know that J_r is a relatively nonexpansive (closed relatively quasi-nonexpansive for example; see [24]); and $B^{-1}(0) = F(J_r)$ for all $r > 0$ (see [16, 21, 22, 32] for more details).

Theorem 4.2 *Let E be a smooth and 2-uniformly convex Banach space. Let Θ be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : E \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function, let A be an α-inverse strongly monotone of E into E^*. Let B be a maximal monotone operator of E into E^*, let J_r be a resolvent of B and a closed mapping and let $T, S : E \to E$ be two closed quasi-ϕ-nonexpansive mappings such that $F := B^{-1}(0) \cap VI(A, C) \cap MEP(\Theta, \varphi) \neq \emptyset$ and $\|Ay\| \leq \|Ay - Au\|$ for all $y \in E$ and $u \in VI(A, E)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 = x \in E, \\ w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n Jx_n + \gamma_n JJ_r x_n + \delta_n JJ_r w_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\ u_n \in E \quad \text{such that} \quad \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in E, \\ C_n = \{z \in E : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases} \tag{4.2}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the conditions (i)-(v) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x$.

Proof. Since J_r is a closed relatively nonexpansive mapping and $B^{-1}0 = F(J_r)$. So, we obtain the result. □

4.3 Complementarity problems

Let C be a nonempty, closed convex cone in E, A an operator of C into E^*. We define its *polar* in E^* to be the set

$$K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0, \forall x \in C\}. \tag{4.3}$$

Then the element $u \in C$ is called a solution of the *complementarity problem* if

$$Au \in K^*, \ \ \langle u, Au \rangle = 0. \tag{4.4}$$

The set of solutions of the complementarity problem is denoted by $CP(A, K)$.

Theorem 4.3 *Let K be a closed convex subset of a smooth and 2-uniformly convex Banach space E. Let Θ be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : K \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function and let A be an α-inverse strongly monotone of E into E^*. Let S and T be two closed quasi-ϕ-nonexpansive mappings of K into itself such that satisfies $F := F(S) \cap F(T) \cap MEP(\Theta, \varphi) \cap CP(A, K) \neq \emptyset$ and $\|Ay\| \leq \|Ay - Au\|$ for all $y \in K$ and $u \in CP(A, K)$. For an initial point $x_0 \in E$, define a sequence $\{x_n\}$ as follows:*

$$\begin{cases} x_0 = x \in E, \\ w_n = \Pi_K J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSw_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ u_n \in K \ \ \text{such that} \ \ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \ \ \forall y \in K, \\ C_n = \{z \in K : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in K : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases} \tag{4.5}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ satisfying the the conditions (i)-(v) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x$.

As in the proof Lemma 7.1.1 of Takahashi in [33], we have $VI(A, K) = CP(A, K)$. So, we obtain the desired result.

5 Acknowledgements

The authors would like to thank the Centre of Excellence in Mathematics, under the Commission on Higher Education, Ministry of Education, Thailand.

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