International Journal of Applied Mathematics and Computation
Volume 2(1),pp 1–10, 2010
http://ijamc.darbose.com

Solution of linear Volterra integro-differential equations via Sinc functions

M. Zarebnia[1], **Z. Nikpour**[2]

[1]*Department of Mathematics, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran*
Email: zarebnia@uma.ac.ir

[2]*Department of Mathematics, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran*
Email: zeinab.nikpur@yahoo.com

Abstract:
In this paper, Sinc-collocation method is developed to approximate of the second order linear Volterra integro-differential equations with boundary conditions. Properties of the Sinc-collocation method required for our subsequent development are given and utilized to reduce the computation of linear second order boundary value problems to some algebraic equations. The method is computationally attractive, and applications are demonstrated through illustrative examples.
Keywords: Boundary value problems, Volterra, Integro-differential, Sinc function.

1 Introduction

We consider the second order linear Volterra integro-differential equations of the form:

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x) + \lambda \int_a^x K(x,t)y(t)dt,\ x \in [a,b], \tag{1.1}$$

with boundary conditions

$$y(a) = y_a, \qquad y(b) = y_b, \tag{1.2}$$

where $K(x,t)$, $f(x)$, $y(x)$, and $p(x)$, $q(x)$, are analytic funvtions and λ is a parameter and y_a and y_b are real constants. $y(x)$ is the solution to be determined. Numerical methods for solution of linear Volterra integro-differential equations have been studied by the authors[$1-5$]. There have been considerable interest in solving integro-differential (1). Theorems which list the conditions for the existence and uniqueness of solutions

Corresponding author: **M. Zarebnia**

of such problems are contained in a book by Agarwal [1]. Two point boundary value problem for integro-differential equation of second order is discussed by J. Morchalo in [2]. Also, J. Morchalo [3] studied two point boundary value problem for integro-differential equation of higher order. A reliable algorithm for solving boundary value problems for higher-order integro-differential equation has been proposed by A. M-Wazwaz [4]. The aim of mentioned paper is to present an efficient analytical and numerical procedure for solving boundary value problems for integro-differential equations. E. Babolian et al [5], applied operational matrices of piecewise constant orthogonal functions for solving Volterra integral and integro-differential equations. They first obtained Laplace transform of the problem and then found numerical inversion of Laplace transform by operational matrices. Sinc methods have increasing been recognized as powerful tools for attacking problems in applied physics and engineering. The books [6, 7] provide excellent overviews of methods based on Sinc functions for solving ordinary and partial differential equations and integro-differential equations. In [8, 9], the Sinc collocation procedures for the eigenvalue problems are presented. M. Ng in [10], employed the preconditioned conjugate gradient method with boundary value problem using Sinc-Galerkin method. A block matrix formulation is presented for the Sinc-Galerkin technique applied to the Wind-driven current problem form oceanography [11]. In [12, 13], we used Sinc methods for numerical solutions of integral equations. Approximation by Sinc functions are typified by errors of the form $O(exp(-k/h))$, where $k > 0$ is a constant and h is a step size.

In this paper, a collocation procedure for the solution of the equation 1.1 with boundary conditions 1.2 using the Sinc functions is developed. Our method consists of reducing the solving of 1.1 to a set of algebraic equations. The properties of the Sinc functions are then utilized to evaluate the unknown coefficients.

The article is organized as follows. In section 2, we will give some preliminary definitions and theorems in [6, 7] that are employed to derive the formulations in section 3. Finally numerical examples are given in section 4 presents a method to treat nonhomogeneous boundary conditions. Finaly numerical examples are given in section 5 to illustrate the efficiency of the presented method.

2 A survey of some properties of the Sinc function

The sinc function is defined on the whole real line by

$$Sinc(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{2.1}$$

For any $h > 0$, the translated Sinc functions with evenly space nodes are given as follows:

$$S(j,h)(x) = Sinc(\frac{x - jh}{h}), \quad j = 0, \pm 1, \pm 2, \cdots, \tag{2.2}$$

which are called jth Sinc function. The Sinc function form for the interpolating point $x_k = kh$ is given by

$$S(j,h)(kh) = \delta^{(0)}_{jk} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{2.3}$$

Let

$$\sigma_{kj} = \int_0^{k-j} \frac{sin(\pi t)}{\pi t} dt, \tag{2.4}$$

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \sigma_{kj}, \tag{2.5}$$

then define a matrix whose (k, j)th entry is given by $\delta_{kj}^{(-1)}$ as $I^{(-1)} = [\delta_{kj}^{(-1)}]$. If f is defined on the real line, then for $h > 0$ the series

$$C(f, h)(x) = \sum_{j=-\infty}^{\infty} f(hk) Sinc(\frac{x - kh}{h}), \tag{2.6}$$

is called the Whittaker cardinal expansion of f, whenever this series converges. But in practical, we need to use some specific numbers of terms in the above series such as $j = -N, ..., N$, where N is the number of Sinc grid points. They are based in the infinite strip D_d in the complex plane

$$D_d = \{w = u + iv : |v| < d \le \frac{\pi}{2}\}. \tag{2.7}$$

To construct approximation on the interval $\Gamma = [a, b]$, we consider the conformal map

$$\phi(z) = \ln\left(\frac{z - a}{b - z}\right). \tag{2.8}$$

The map ϕ carries the eye-shaped region

$$D = \left\{z = x + iy : |arg(\frac{z - a}{b - z})| < d \le \frac{\pi}{2}\right\}. \tag{2.9}$$

For the Sinc method, the basis functions on the interval $\Gamma = [a, b]$ for $z \in D$ are derived from the composite translated sinc functions,

$$S_j(z) = S(j, h) \circ \phi(z) = Sinc\left(\frac{\phi(z) - jh}{h}\right). \tag{2.10}$$

The function

$$z = \phi^{-1}(w) = \frac{a + be^w}{1 + e^w}, \tag{2.11}$$

is an inverse mapping of $w = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \{\psi(y) = \phi^{-1}(y) \in D : -\infty < y < \infty\} = [a, b]. \tag{2.12}$$

The Sinc grid points $z_k \in \Gamma$ in D will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \quad k = \pm 1, \pm 2, \ldots. \tag{2.13}$$

For further explanation of the procedure, the important class of functions is denoted by $L_\alpha(D)$. The properties of functions in $L_\alpha(D)$ and detailed discussions are given in [6, 7]. We recall the following definition and theorems for our purpose.

Definition 2.1 *Let $L_\alpha(D)$ be the set of all analytic functions y in D, for which there exists a constant C such that*

$$|y(z)| \leq C\frac{|\rho(z)|^\alpha}{[1+|\rho(z)|]^{2\alpha}}, \quad z \in D, \quad 0 < \alpha \leq 1, \tag{2.14}$$

where $\rho(z) = e^{\phi(z)}$.

Theorem 2.1 *Let $y \in L_\alpha(D)$, let N be positive integer, and let h be selected by the formula $h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$, then there exists positive constant, C_1, independent of N, such that*

$$\sup_{x\in\Gamma}\left| y(x) - \sum_{j=-N}^{N} y(x_j)S(j,h)\circ\phi(x)\right| \leq C_1 e^{-(\pi d\alpha N)^{\frac{1}{2}}}. \tag{2.15}$$

Theorem 2.2 *Let $\frac{y}{\phi'} \in L_\alpha(D)$, with $0 < \alpha \leq 1$, and let $\delta_{kj}^{(-1)}$ be defined as in 2.5, let N be a positive integer, and let h be selected by the formula*

$$h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}, \tag{2.16}$$

then there exists constant C_2, which is independent of N, such that

$$\left|\int_a^{x_k} y(t)dt - h\sum_{k=-N}^{N}\delta_{kj}^{-1}\frac{y(t_k)}{\phi'(t_k)}\right| \leq C_2 e^{-(\pi d\alpha N)^{\frac{1}{2}}}. \tag{2.17}$$

The nth derivative $y(x)$ at some points x_k can be approximated using finite number of terms as

$$y^{(n)}(x_k) \approx h^{-n}\sum_{j=-N}^{N}\delta_{jk}^{(n)}y_k, \tag{2.18}$$

where

$$\delta_{jk}^{(n)} = h^n\frac{d^n}{d\phi^n}S(j,h)\circ\phi(x)|_{x=x_k}. \tag{2.19}$$

In particular

$$\delta_{jk}^{(0)} = S(j,h)\circ\phi(x)|_{x=x_k} = \begin{cases} 1, & k=j, \\ 0, & k\neq j, \end{cases} \tag{2.20}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}S(j,h)\circ\phi(x)|_{x=x_k} = \begin{cases} 0, & k=j, \\ \frac{(-1)^{(k-j)}}{(k-j)}, & k\neq j, \end{cases} \tag{2.21}$$

$$\delta_{jk}^{(2)} = h^2 S(k,h)\circ\phi(x)|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & k=j, \\ \frac{-2(-1)^{(k-j)}}{(k-j)^2}, & k\neq j. \end{cases} \tag{2.22}$$

3 The Sinc-collocation method

By considering theorem 2 the solution of second order Volterra integro-differential equations

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x) + \lambda \int_a^x K(x,t)y(t)dt, \;\; x \in [a,b], \tag{3.1}$$

is approximated by the following linear combination of the sinc functions:

$$y(x) \approx y_n(x) = \sum_{j=-N}^{N} y_j S(j,h) \circ \phi(x), \qquad n = 2N+1. \tag{3.2}$$

For the second term on the right-hand side of 3.1, we assume that $\frac{K(x,.)}{\phi'} \in L_\alpha(D)$, then by setting $x = x_k$ and using theorem 2, we obtain:

$$\int_a^{x_k} K(x,t)y(t)dt \approx h \sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{K(x_k,t_j)}{\phi'(t_j)} y_j, \tag{3.3}$$

where y_j denotes an approximate value of $y(x_j)$.
Having replaced the second term on the right-hand side of 3.1 with the right hand side of 3.3 and having substituted $x = x_k$ for $k = -N, ..., N$, that x_k are Sinc grid points, also by replacing $y(x)$ by $y_n(x)$ as in 3.2 we get the collocation result:

$$y_n''(x_k) + p(x_k)y_n'(x_k) + q(x_k)y_n(x_k) = f(x_k) + \lambda \sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{K(x_k,t_j)}{\phi'(t_j)} y_j. \tag{3.4}$$

Where

$$y_n(x) = \sum_{j=-N}^{N} y_j [S(j,h) \circ \phi(x)], \tag{3.5}$$

$$y_n'(x) = \sum_{j=-N}^{N} y_j [S(j,h) \circ \phi(x)]', \tag{3.6}$$

$$y_n''(x) = \sum_{j=-N}^{N} y_j [S(j,h) \circ \phi(x)]''. \tag{3.7}$$

Now, by using relations 2.18-refeq24, we have:

$$[S(j,h) \circ \phi(x)]'|_{x=x_k} = \phi' \frac{d}{d\phi}[S(j,h) \circ \phi(x)]|_{x=x_k} = \phi'(x_k) h^{-1} \delta_{jk}^{(1)}, \tag{3.8}$$

$$[S(j,h) \circ \phi(x)]''|_{x=x_k} = [\phi\prime \frac{d}{d\phi}[S(j,h) \circ \phi(x)]'|_{x=x_k} = \phi'' \frac{d}{d\phi}[S(j,h) \circ \phi]$$

$$+\phi\prime \frac{d^2}{d\phi^2}[S(j,h) \circ \phi]|_{x=x_k} = \phi\prime\prime(x_k) h^{-1} \delta_{jk}^{(1)} + [\phi\prime(x_k)]^2 h^{-2} \delta_{jk}^{(2)}. \tag{3.9}$$

By using the properties of the Sinc function and replacing 3.5-3.9 in the equation 3.4, we rewrite 3.4 as:

$$\sum_{j=-N}^{N}[\phi\prime(x_k)^2\frac{\delta_{jk}^{(2)}}{h^2}+[\phi''(x_k)+p(x_k)\phi'(x_k)]\frac{\delta_{jk}^{(1)}}{h},+q(x_k)\delta_{jk}^{(0)}-\lambda h\delta_{kj}^{(-1)}\frac{K(x_k,t_j)}{\phi\prime(t_j)}]y_j$$
$$=f(x_k) \tag{3.10}$$

Having multiplied the resulting equations by $h^2/[\phi\prime(x_k)]^2$, we have

$$\sum_{j=-N}^{N}[\delta_{jk}^{(2)}+h(\frac{\phi''x_k}{\phi'(x_k)}+\frac{p(x_k)}{\phi\prime(x_k)})\delta_{jk}^{(1)}+h^2\frac{q(x_k)}{[\phi'(x_k)]^2}\delta_{jk}^{(0)}-\lambda h^3\frac{K(x_k,t_j)}{[\phi'(x_k)]^2\phi'(t_j)}\delta_{kj}^{(-1)}]y_j$$
$$=h^2\frac{f(x_k)}{[\phi'(x_k)]^2}. \tag{3.11}$$

Now, since $\delta_{jk}^{(0)}=\delta_{kj}^{(0)}$, $\delta_{jk}^{(1)}=-\delta_{kj}^{(1)}$, $\delta_{jk}^{(2)}=\delta_{kj}^{(2)}$, and since $\frac{\phi''(x_k)}{[\phi'(x_k)]^2}=-(\frac{1}{\phi'(x_k)})'$, we get the collocation result as:

$$\sum_{j=-N}^{N}\{\delta_{kj}^{(2)}+h[(\frac{1}{\phi'(x_k)})'-\frac{p(x_k)}{\phi\prime(x_k)}]\delta_{kj}^{(1)}+h^2\frac{q(x_k)}{[\phi'(x_k)]^2}\delta_{kj}^{(0)}-\lambda h^3\frac{K(x_k,t_j)}{[\phi'(x_k)]^2\phi'(t_j)}\delta_{kj}^{(-1)}\}y_j$$
$$=h^2\frac{f(x_k)}{[\phi'(x_k)]^2}. \tag{3.12}$$

We set $I^{(m)}=[\delta_{kj}^{(m)}]$, $m=-1,0,1,2$, where $\delta_{kj}^{(m)}$ denote the (k,j) element of the matrix $I^{(m)}$. Also, we denote $\mathbf{K}=[\frac{K(x_k,t_j)}{[\phi'(x_k)]^2\phi'(t_j)}]$, and $D(1/\phi')=diag(1/\phi'(x_{-N}),\ldots,1/\phi'(x_N))$. $\mathbf{K}$ and $I^{(m)}$, $m=-1,0,1,2$ are square matrices of order $(2N+1)\times(2N+1)$, then the system of 3.12 can be given in the matrix form as:

$$AY=P, \tag{3.13}$$

where

$$A=I^{(2)}+hD[(\frac{1}{\phi'})'-\quad\frac{p}{\phi\prime}]I^{(1)}+\quad h^2D(\frac{q}{\phi'^2})I^{(0)}\quad-\lambda h^3(\mathbf{K}\circ I^{(-1)}), \tag{3.14}$$

$$P=h^2[f(x_{-N})/[\phi'(x_{-N})]^2,\ldots,f(x_N)/[\phi'(x_N)]^2]^T, \tag{3.15}$$

$$Y=[y_{-N},\ldots,y_N]^T. \tag{3.16}$$

The notation " $\circ$ " denotes the Hadamard matrix multiplication. The above linear system containing $(2N+1)$ equation with $(2N+1)$ unknown coefficient $\{y_j\}_{j=-N}^{N}$. Solving this linear system, we can obtain the approximate solution of Volterra integro-differential equation 1.1 as follows:

$$y_n(x)=\sum_{j=-N}^{N}y_j[S(j,h)\circ\phi(x)]. \tag{3.17}$$

4 boundary conditions

In the previous section, we developed the Sin-collocation method for Volterra integro-differential equation with homogeneous boundary conditions, but in case if boundary conditions are non-homogeneous that is:

$$y(a) = y_a, \qquad (b) = y_b, \tag{4.1}$$

where y_a and y_b are not equal to zero, that we use the change of variable as

$$u(x) = y(x) - \frac{(b-x)}{(b-a)}y_a - \frac{(x-a)}{(b-a)}y_b. \tag{4.2}$$

So that by using the above change of variable yields the Volterra integro-differential equation

$$u''(x) + p(x)u'(x) + q(x)u(x) = R(x) + \lambda \int_a^x K(x,t)u(t)d, \qquad x \in \Gamma = [a,b],$$

$$u(a) = 0, \qquad u(b) = 0, \tag{4.3}$$

$$R(x) = f(x) - \frac{(y_b - y_a)}{(b-a)}p(x) - \frac{(y_b - y_a)x - ay_b + by_a}{(b-a)}q(x)+$$

$$\lambda \int_a^x K(x,t)\frac{(y_b - y_a)t + y_a b - ay_b}{2(b-a)}dt. \tag{4.4}$$

Then by using 3.13 in 4.3, we obtain the approximate solution as:

$$u_n(x) = \sum_{j=-N}^{N} u_j S(j,h) \circ \phi(x). \tag{4.5}$$

5 Numerical examples

In order to illustrate the performance of the Sinc-collocation method for the Volterra integro-differential equations and justify the accuracy and efficiency of the method, we consider the following examples. The examples have been solved by presented method with different values of N and α, $0 < \alpha \leq 1$. In all examples we take $\alpha = 1$ and $d = \frac{\pi}{2}$, which yields $h = \pi(\frac{1}{2N})^{\frac{1}{2}}$. The errors are reported on the set of Sinc grid points

$$S = \{x_{-N}, \ldots, x_0, \ldots, x_N\},$$

$$x_k = \frac{e^{kh}}{1+e^{kh}}, \quad k = -N, \ldots, N. \tag{5.1}$$

The maximum error on the Sinc grid points is

$$\|E_S(h)\|_\infty = \max_{-N \leq j \leq N} |y(x_j) - y_n(x_j)|. \tag{5.2}$$

The numerical results are tabulated in Tables 1 and 2.

Example 5.1. We consider the Volterra integro-differential equation

$$y'' - \frac{1}{x}y' + y = f(x) + \int_0^x K(x,t)y(t)dt, \quad y(0) = 0, \quad y(1) = 0, \tag{5.3}$$

with exact solution $y(x) = x(1 - x^2)$.
Where

$$K(x,t) = \frac{e^t sint}{xt}, \quad f(x) = -x^3 - x(2 + e^x \frac{1}{\sqrt{2}} sin(x - \frac{\pi}{4})) - \frac{1}{x}(xcosx - sinx).$$

The example has been solved for differential values of N and $h = \pi(\frac{1}{2N})^{1/2}$. The maximum of absolute errors on the Sinc grid S are tabulated in Table 1. For example 5.1, the graph of the exact and approximate solutions are shown in Figure 1, including the approximations for $N = 2$ and $N = 7$. For large number of N the approximate is indistinguishable from the exact solution.

Table 1: Results for example 5.1.

N	h	$\|E_s\|$
5	0.993458	5.92003×10^{-3}
10	0.702481	9.65024×10^{-4}
15	0.573573	2.23322×10^{-4}
20	0.496729	6.30761×10^{-5}
25	0.444288	2.04361×10^{-5}
30	0.405577	7.32063×10^{-6}
35	0.375492	2.83447×10^{-6}
40	0.354240	1.166816×10^{-7}

Example 5.2. Consider the following Volterra integro-differential equation with the exact solution $y(x) = e^x$.

$$y'' - \frac{1}{1-x}y = f(x) + \int_0^x K(x,t)y(t)dt, \quad y(0) = 1, \ \ y(1) = e, \tag{5.4}$$

where

$$f(x) = e^x(1 - \frac{1}{(1-x)}) - \frac{7(-2 + 2e^x + x^2)cosx}{16(e + x)},$$

$$K(x,t) = \frac{7}{8}\frac{(e^t + t)}{x + e}cosx.$$

We solved example 5.2 for differential values of N and $h = \pi(\frac{1}{2N})^{1/2}$. The maximum of absolute errors on the Sinc grid S are tabulated in Table 2. This Table indicates that

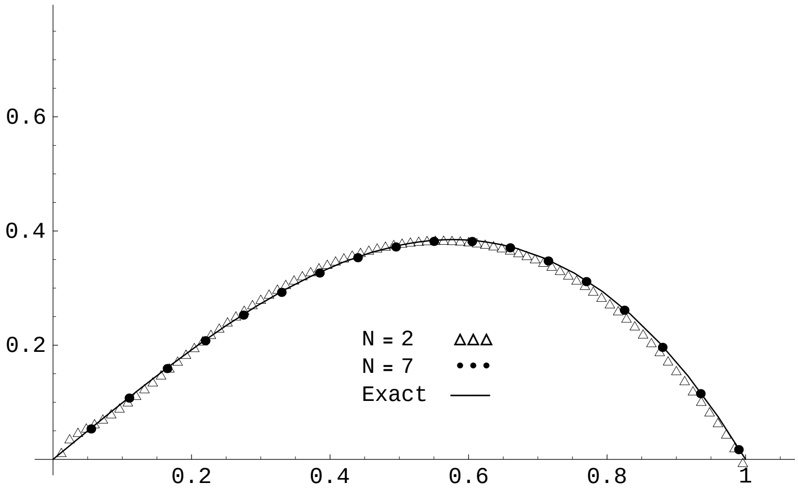

Figure 1: Exact and approximate solution for Example 5.1,($N = 2, 7$)

as N increases the error is decreased more rapidly. The exact and approximate solutions for example 5.2 are shown in Figure 2, including the approximations for N=2, and 7. For $N \geq 7$, the approximate solutions are indistinguishable(on this scale) from the exact solution.

Table 2: Results for example 5.2.

N	h	$\|E_s\|$
5	0.993458	2.91837×10^{-3}
10	0.702481	4.81675×10^{-4}
15	0.573573	1.11592×10^{-4}
20	0.496729	3.15332×10^{-5}
25	0.444288	1.02171×10^{-5}
30	0.405577	3.66018×10^{-6}
35	0.375492	1.41721×10^{-6}
40	0.354240	5.84069×10^{-7}

6 Conclusion

The Sinc-collocation method is used to solve the second order linear Volterra integro-differential equations with boundary conditions. The numerical examples show that the accuracy improve with increasing the number of Sinc grid points N.

References

[1] R. P. Agarwal, Boundary value problems for higher order integro-differential equations, Nonlinear Analysis, Theory, Meth. Appl., 7, pp. 259-270, (1983)

[2] J. Morchalo, On two point boundary value problem for integro-differential equation of second order, Fasc. Math., 9, pp. 51-56, (1975)

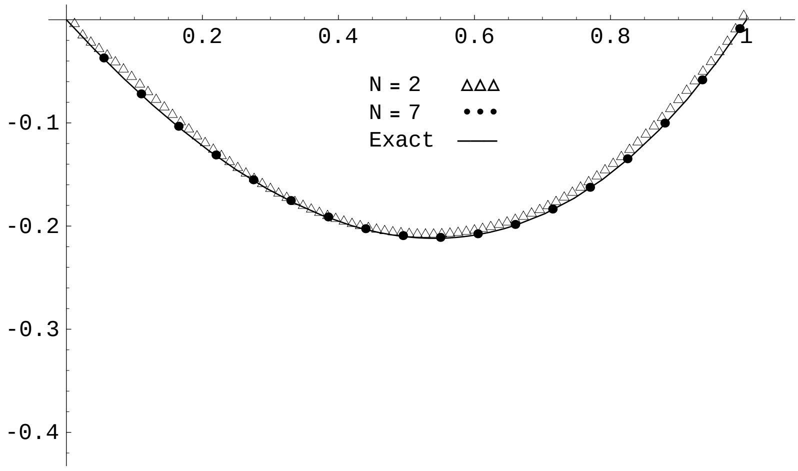

Figure 2: Exact and approximate solution for Example 5.2,($N = 2, 7$)

[3] J. Morchalo, On two point boundary value problem for integro-differential equation of higher order, Fasc. Math., 9, pp. 77-96, (1975)

[4] A. Wazwaz, A reliable algorithm for solving boundary value problems for higher-order integro-differential equations, Appl. Math. Comp., 118, pp. 327-342, (2001)

[5] E. Babolian, A. Salimi Shamloo, Numerical solution of Volterra integral and integro-differential equations of convolution type by using operational matrices of piecewise constant orthogonal functions, J. comput. Appl. Math., 214, pp. 495-508, (2008)

[6] F. Stenger, Numerical Methods Based on Sinc and Analytic Functions, Springer-Verlag, New York, 1993

[7] J. Lund, K. Bowers, Sinc Methods For Quadrature and Differential Equations, SIAM, Philadelphia, PA, 1992

[8] J. Lund, B. V. Rilay, A Sinc-collocation method for the computation of the eigenvalues of the radial Schrodinger equation, IMA J. Numer. Anal., 4, pp. 83-98, (1984)

[9] N. Eggert, M. Jarratt, J. Lund, Sinc function computations of the eigenvalues of Sturm-Liouville problems, J. Comput. Phys., 69, pp. 209-229, (1987)

[10] M. Ng, Fast iterative methods for symmetric sinc-Galerkin system, IMA J. Numer. Anal., 19, pp. 357-373, (1999)

[11] S. Koonpraset, K. L. Bowers, Block matrix Sinc-Galerkin solution of the wind-driven current problem, Appl. Math. Comput., 155, pp. 607-635, (2004)

[12] J. Rashidinia, M. Zarebnia, Solution of a Volterra integral equation by the Sinc-Collocation method, J. Comput. Appl. Math., 206, pp. 801-813, (2007)

[13] J. Rashidinia, M. Zarebnia, Convergence of approximate solution of system of Fredholm integral equations, J. Math. Anal. Appl., 333, pp. 1216-1227, (2007)