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Some Properties of the First Order Polarization Tensor for 3-D Domains

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Abstract In this paper, numerical approximation and two properties of first order Polarization Tensor for 3-D domain are highlighted. Based on the definition of the generalized polarization tensor by integral equation, the quadrature method is adapted to simplify the approximation of the first order polarization tensor. Furthermore, the softwares MATLAB and Netgen Mesh Generator are used for computation purpose. Here, the numerical results obtained by following the method show reasonable convergence to the exact first order polarization tensor but at several known factors. Finally, these results are further discussed to demonstrate the transformation and types of the first order polarization tensor for few different objects.

Keywords Matrix; Mesh; Quadrature

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1 Introduction

During advance development of mathematical research on acoustics and electromagnetism, the concept of polarization tensors (PT) arise for numerous problems and applications in science and engineering. At earlier stage, the first order PT or is known as the virtual mass is proposed by Pólya and Szegő [1] to geometrically describe or to gain other relevant information about a particular solid in their studies on hydrodynamics and electrostatics. Many authors then extensively use and generalize this idea especially when dealing with problems involving electrical potentials [2]. Furthermore, instead of using the original definition, Ammari and Kang [2] introduce a new technique to determine the PT which is by solving a boundary integral operator in two or higher dimensions. This approach is much more easier than the previous definition to be fastly computed by the means of numerical method and through computers.

Therefore, the main purpose of this study is to review and discuss some properties of PT as documented in [2] to increase the understanding about this promising terminology. Here, the steps and procedures taken to numerically calculate the 3D PT specifically the first order one by numerical integration approach will firstly be presented. This report is hopefully able to become a useful outline and reference to systematically implement the first order PT for any relevant applications in the future.

2 Background of the Study

This study originated from a transmission problem discussed by many literatures (for examples in [2], [3] and [4]) which will be stated first here. Following from [2], consider a

Lipschitz bounded domain B in $\mathbb{R}^3$ such that the origin O is in B and let the conductivity of B be equal to k where $0 < k \neq 1 < +\infty$. Suppose that H be a harmonic function in $\mathbb{R}^3$ and u be the solution to the following problem.

$$\begin{cases} \text{div}(1 + (k-1)\chi(B)\text{grad}(u)) = 0 \text{ in } \mathbb{R}^3 \\ u(x) - H(x) = O(1/|x|^2) \text{ as } |x| \to \infty \end{cases} \tag{1}$$

where χ denotes the characteristic function of B. The mathematical formulation (1) actually appears in many industrial applications such as medical imaging, landmine detector and material sciences ([1–4]).

The main idea of this study which are the polarization tensors is defined by [2] through the following far-field expansion of u as follows.

$$(u - H)(x) = \sum_{|i|,|j|=1}^{+\infty} \frac{(-1)^{|i|}}{i!j!} \partial_x^i \Gamma(x) M_{ij}(k, B) \partial^j H(0) \text{ as } |x| \to +\infty \tag{2}$$

for $i = (i_1, i_2, i_3)$, $j = (j_1, j_2, j_3)$ multi indices, Γ is a fundamental solution of the Laplacian and $M_{ij}(k, B)$ is the generalized polarization tensor (GPT). The GPT is usually referred as the dipole in electromagnetic applications by physicists because it shows the distribution of the conductivity of the object.

Furthermore, Ammari and Kang [2] extend the definition of GPT in (2) through an integral equation over the boundary of B by

$$M_{ij} = \int_{\partial B} y^j \phi_i(y) d\sigma(y) \tag{3}$$

where $\phi_i(y)$ is given by

$$\phi_i(y) = (\lambda I - \mathcal{K}_B^*)^{-1}(v_x \cdot \nabla x^i)(y) \tag{4}$$

for $x, y \in \partial B$ with v_x is the outer unit normal vector to the boundary ∂B at x and λ is defined by $\lambda = (k+1)/2(k-1)$. $\mathcal{K}_B^*$ is a singular integral operator defined with Cauchy principal value $P.V.$ by

$$\mathcal{K}_B^* \phi(x) = \frac{1}{4\pi} P.V. \int_{\partial B} \frac{\langle x - y, v_x \rangle}{|x - y|^3} \phi(y) d\sigma(y). \tag{5}$$

Finding PT by using (3), (4) and (5) is the main interest and preferable in this study because of their advantage that depend only on the shape of B and the conductivity instead of any other parameters of the problem.

Before this, Capdeboscq et al. [5] has successfully developed algorithm to numerically compute GPT but unfortunately for two dimensional domain only. Similarly, in their publications, although Ammari and Kang [2] describe and implement GPT for many applications, most of the studies focus on two dimensional cases. Therefore, this study will focus on understanding the computation and properties of PT for three dimensional domain in order to properly use them in future relevant real world physical problems.

2.1 First Order PT

At this stage, this study will only focus on discussion of the first order PT and some of its properties. The first order PT can be evaluated by using (3) for $i, j = (1,0,0), (0,1,0)$ and $(0,0,1)$ only so that $|i| = i_1 + i_2 + i_3 = 1 = |j|$. By combining all possible values of i and j, the first order PT is a real 3×3 matrix in the form

$$M = \begin{bmatrix} M_{(1,0,0)(1,0,0)} & M_{(1,0,0)(0,1,0)} & M_{(1,0,0)(0,0,1)} \\ M_{(0,1,0)(1,0,0)} & M_{(0,1,0)(0,1,0)} & M_{(0,1,0)(0,0,1)} \\ M_{(0,0,1)(1,0,0)} & M_{(0,0,1)(0,1,0)} & M_{(0,0,1)(0,0,1)} \end{bmatrix}. \tag{6}$$

M also is symmetry and this has been proven in [2]. The procedure to numerically calculate (6) by quadrature rule of numerical integration will be explained in Section 3 of this article.

2.2 Explicit Formulae of First Order PT for Ellipsoid and Sphere

Suppose that B is an ellipsoid centered at origin in the Cartesian coordinate system represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ where a, b and c each is the length of semi principal axes of B such that $0 < c \leq b \leq a$. The first order PT of B when the conductivity is k as derived in [2] is given by

$$M(k,B) = (k-1)|B| \begin{bmatrix} \frac{1}{(1-P)+kP} & 0 & 0 \\ 0 & \frac{1}{(1-Q)+kQ} & 0 \\ 0 & 0 & \frac{1}{(1-R)+kR} \end{bmatrix} \tag{7}$$

where $|B|$ is the volume of B, P, Q and R are constants defined by

$$P = \frac{bc}{a^2} \int_1^{+\infty} \frac{1}{t^2 \sqrt{t^2 - 1 + (\frac{b}{a})^2} \sqrt{t^2 - 1 + (\frac{c}{a})^2}} dt,$$

$$Q = \frac{bc}{a^2} \int_1^{+\infty} \frac{1}{(t^2 - 1 + (\frac{b}{a})^2)^{\frac{3}{2}} \sqrt{t^2 - 1 + (\frac{c}{a})^2}} dt,$$

$$R = \frac{bc}{a^2} \int_1^{+\infty} \frac{1}{\sqrt{t^2 - 1 + (\frac{b}{a})^2} (t^2 - 1 + (\frac{c}{a})^2)^{\frac{3}{2}}} dt.$$

The first order PT for the sphere is then again formulated by [2] by setting $a = b = c$ in (7) which in the form

$$M(k,B) = (k-1)|B| \begin{bmatrix} \frac{3}{2+k} & 0 & 0 \\ 0 & \frac{3}{2+k} & 0 \\ 0 & 0 & \frac{3}{2+k} \end{bmatrix}. \tag{8}$$

2.3 Positivity and Transformation of First Order PT

This study will only discuss two properties of first order PT as stated in the title of the section. These properties are already proven by Ammari and Kang [2]. They are only restated here in the next two theorems.

Theorem 1 *If $k > 1$ then the GPT is positive definite while it is negative definite if $0 < k < 1$.*

Theorem 2 *Let R be a unitary matrix transformation of a domain B and R^T is the transpose of R such that $B' = RB$. If $M(k,B)$ and $M(k,B')$ are the first order PT associated to domains B and B' respectively for a conductivity $0 < k \neq 1 < +\infty$ then $M(k,B') = RM(k,B)R^T$.*

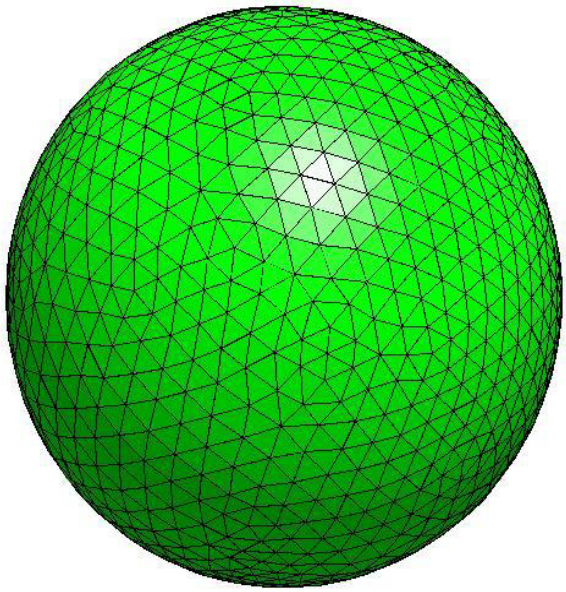


Figure 1: Triangularization of a Sphere with 2480 Elements

3 Methodology

The procedure on how to numerically calculate the PT for 3D domain in this report will be explained in three seperated steps. However, it is also essential to firstly discuss the discretization of the boundary of a 3D object into triangular elements in order to develop precise and efficient numerical method to achieve the purpose of this research. Besides, some properties of elementary linear algebra which will be used to describe positivity and transformation of first order PT are also briefly stated in this section.

3.1 Triangularization of an Object

The boundary of a 3D object is actually a surface. For the purpose of computation, this surface is discritized into few triangular planes by a software called as Netgen Mesh Generator (see Figure 1 as example). This software will automatically generate triangular meshes or elements for any desired surfaces and the information about these triangles will then be used to calculate the first order PT of the surface by numerical method in this study.

In order to get a better approximation of the surface, the total mesh can be freely refined by the software with some limitations, for example user can not exactly set the total elements that he want (see [6] for detail explanation). Table 1 shows the convergence of the surface area of a triangularized sphere with radius 1 to the exact surface area of the sphere as the total elements increase. This suggest that a good approximation of the surface by the triangularized one must be used during computation to obtain accurate result.

Table 1: The Convergence of a Triangularized Sphere

Total Elements	Approximated Surface Area
620	12.4405
2480	12.5347
9920	12.5584

3.2 Procedures to Numerically Calculate 3D Polarization Tensor

Since computing the PT involves an integral operator with singularity, it is best to turn to the numerical method to reduce difficulties when solving these mathematical expressions. While there are variety procedures of numerical calculation for this purpose, only the simple quadrature rule with piecewise constant basis functions will be considered here. The three steps to compute the PT will be discussed now and simplified for the first order case in the next three sections.

3.2.1 Formulating $\mathcal{K}_B^*$

Suppose that the boundary ∂B of B is triangularized into N triangular planes (or elements). By adapting quadrature method for Fredholm integral equation of the first kind in [7] with piecewise constant basis functions (see [8]), (5) can now be approximated by

$$\mathcal{K}_B^* = \frac{1}{4\pi} \sum_{t=1}^{N} w_t \frac{\langle x - y_t, v_x \rangle}{|x - y_t|^3} \tag{9}$$

for selected abscissas t with corresponds weight w_t is the surface area and y_t is the barycentre (or average of the vertices) of the t-th elements. Following collocation requirement, (9) is then extended to the following system of relation

$$\mathcal{K}_B^* = \frac{1}{4\pi} \left(w_t \frac{\langle x_s - y_t, v_{x_s} \rangle}{|x_s - y_t|^3} \right), \text{ for } s = 1, ..., N, \ t = 1, ..., N, \tag{10}$$

so that x_s and v_{x_s} each becomes the barycentre and the outer unit normal vector for the s-th element respectively.

Since $x, y \in \partial B$, to avoid confusion in computation, (10) is then rewritten as

$$\mathcal{K}_B^* = \frac{1}{4\pi} \left(w_t \frac{\langle x_s - x_t, v_{x_s} \rangle}{|x_s - x_t|^3} \right), \text{ for } s = 1, ..., N, \ t = 1, ..., N. \tag{11}$$

Thus, $\mathcal{K}_B^*$ is $N \times N$ matrix in the form

$$\mathcal{K}_B^* = \frac{1}{4\pi} \begin{bmatrix} w_1 K(x_1, x_1) & \dots & w_N K(x_1, x_N) \\ \vdots & \ddots & \vdots \\ w_1 K(x_N, x_1) & \cdots & w_N K(x_N, x_N) \end{bmatrix} \tag{12}$$

where $K(x_s, x_t) = \frac{\langle x_s - x_t, v_{x_s} \rangle}{|x_s - x_t|^3}$ for $s = 1, ..., N, t = 1, ..., N$. However, due to the singularity of $\mathcal{K}_B^*$, formula (12) can not be used to calculated $K(x_s, x_t)$ for every $s = t$. Therefore,

an analytic procedure is used to approximate $K(x_s, x_t)$ for $s = t$ which is by solving the original integral equation of $\mathcal{K}_B^*$ over a plane triange. This approach will actually result all diagonal elements of (12) to be equal to 0 and is briefly explained through the next lemma.

Lemma 1 Suppose that T is a plane triangle and v_x is the outward unit normal vector to T at x. Then for any $y \in T$

$$P.V. \int_T \frac{\langle x - y, v_x \rangle}{|x - y|^3} dA_T(y) = 0. \tag{13}$$

Proof In order to evaluate the Cauchy principal integral (13), consider a small ball with radius ϵ and centered at $x \in T$ denoted by $B_\epsilon(x)$. Then for any $y \in T$,

$$P.V. \int_T \frac{\langle x-y, v_x \rangle}{|x-y|^3} dA_T(y) = \lim_{\epsilon \to 0^+} \int_{T \cap B_\epsilon(x)'} \frac{\langle x-y, v_x \rangle}{|x-y|^3} dA_T(y)$$

where $B_\epsilon(x)'$ is the complement of $B_\epsilon(x)$. However, $|x-y| \neq 0$ but $\langle x-y, v_x \rangle = (x-y) \cdot v_x = 0$ for any $x, y \in T \cap B_\epsilon(x)'$. Therefore,

$$\lim_{\epsilon \to 0^+} \int_{T \cap B_\epsilon(x)'} \frac{\langle x-y, v_x \rangle}{|x-y|^3} dA_T(y) = \lim_{\epsilon \to 0^+} 0 = 0.$$

□

3.2.2 Identifying $\phi_i(y)$

$\mathcal{K}_B^*$ obtained from previous procedure is then used to calculate $\phi_i(y) = (\lambda I - \mathcal{K}_B^*)^{-1}(v_x \cdot \nabla x^i)(y)$ and since $\mathcal{K}_B^*$ is $N \times N$ matrix then I must be identity matrix of size N. $(v_x \cdot \nabla x^i)(y)$ is the derivative of every point $x \in \partial B$ to the power of index i in the direction of outer unit normal vector to ∂B at that point, v_x. This expression can be easily computed for the first order PT such that when $i = (1,0,0), (0,1,0)$ and $(0,0,1)$ by using property of index. This will only yield $(v_x \cdot \nabla x^i)(y)$ to be just the first, second and the third component of the vector v_x. Therefore, due to the triangularization of the surface, if given k to calculate λ then $\phi_i(y)$ for $i = (1,0,0), (0,1,0)$ and $(0,0,1)$ can be conveniently written as $N \times 3$ matrix with columns $(\lambda I - \mathcal{K}_B^*)^{-1}$ multiply by the first, second and third component of v_{x_t} for every $t = 1, ...N$ as follows

$$\phi_i(y) = \left[(\lambda I - \mathcal{K}_B^*)^{-1}(v_{x_t}^{1\text{st}}) \quad \vdots \quad (\lambda I - \mathcal{K}_B^*)^{-1}(v_{x_t}^{2\text{nd}}) \quad \vdots \quad (\lambda I - \mathcal{K}_B^*)^{-1}(v_{x_t}^{3\text{rd}}) \right]. \tag{14}$$

3.2.3 Evaluating $M(k, B)$

By applying the same quadrature rule now to (3) then the PT of an object B with conductivity k can be approximated by the following

$$M(k, B) = \sum_{t=1}^{N} w_t y_t^j \phi_i(y). \tag{15}$$

However, in order to obtain the first order PT, this equation can be further simplified by considering all combination between i and j for $i, j = (1,0,0), (0,1,0)$ and $(0,0,1)$. Notice that $y_t^{(1,0,0)}$, $y_t^{(0,1,0)}$ and $y_t^{(0,0,1)}$ each is actually the x-coordinate, y-coordinate and

z-coordinate of the barycentre of the t-th element. So, by expanding (15) and using (14), (15) can just be obtained through matrix multiplication as follows

$$M(k, B) = [\phi_i(y)]^T D(w)B(y) \tag{16}$$

where $[\phi_i(y)]^T$ a $3 \times N$ matrix is the transpose of (14), $D(w)$ is a diagonal $N \times N$ matrix with diagonal elements are the surface area w_t and $B(y)$ is a matrix of the size $N \times 3$ such that each of its column is the x-coordinate, y-coordinate and z-coordinate of the barycentre of the t-th element respectively. This will imply (16) is exactly in the same form as (6).

3.3 Positive Definite and Rotation Matrix

In order to describe some properties of first order PT, two definitions about matrices from linear algbera that will be used later on are recalled and stated here.

Definition 1 Positive and Negative Definite Matrix [9, 10]
A square real matrix A is positive definite if for all non zero vector $\mathbf{v} \in \mathbb{R}^3$, $\mathbf{v}^T A\mathbf{v} > 0$. In contrast, A is negative definite if $\mathbf{v}^T A\mathbf{v} < 0$.

Definition 2 Rotation Matrix of a Surface [11]
The rotation matrix for a surface in Cartesian coordinate system which is rotated counter clockwise from positive z-axis by an angle θ is given by

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4 Results

In this section, first order PT of several shapes will be computed through MATLAB for various conductivities by the numerical procedures discussed in the previous section. The approximated first order PT for a sphere and an ellipsoid by the proposed method will be compared to the first order PT for sphere and ellipsoid evaluated through (7) and (8) to investigate the effectiveness of the method. These numerical results will also be used to discuss the transformation and positivity of the first order PT for every object.

4.1 Exact and Approximated First Order PT for a Sphere

The first order PT for the sphere $x^2 + y^2 + z^2 = 1$ is calculated and approximated by following the steps discussed in Section 3.1 and 3.2 for conductivities, k between 10^{-5} and 10^2. According to (8), the first order PT for a sphere is a diagonal matrix with all diagonal elements are equal so all diagonal elements of approximated first order PT for the sphere which are denoted as DM11, DM22, DM33 are plotted in the same graph with one of the diagonal element of the exact first order PT for the same sphere, EM11 against the log of conductivities, k. In addition, all non-diagonal elements of the approximated first order PT for this sphere are also plotted against the log of conductivities in the same graph to investigate whether they are all converge to 0. Figure 2, Figure 3 and Figure 4 each shows the related graph generated by MATLAB where the diagonal and non-diagonal elements

are plotted in part (a) and part (b) respectively. These figures will also actually tell the approximated first order PT for the sphere triangularized with different number of elements in order to demonstrate the convergence of the numerical method implemented.

4.2 Transformation of the First Order PT

Consider an ellipsoid, B in the Cartesian form as $(\frac{x}{3})^2+(\frac{y}{2})^2+z^2=1$ where its exact surface area is around 48.9719. The exact first order PT for B here is calculated with conductivity 1.5 by using (7) and is given by

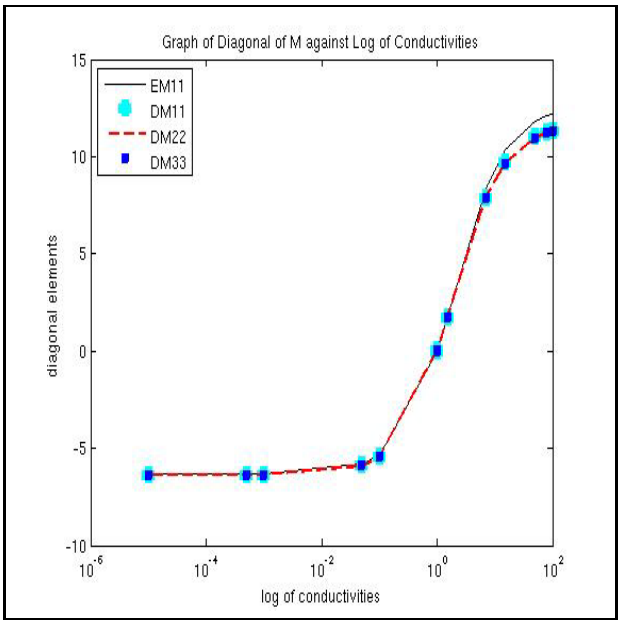
$$M(1.5,B)=\begin{bmatrix}11.6555 & 0 & 0\\ 0 & 11.0856 & 0\\ 0 & 0 & 9.7544\end{bmatrix}.$$

Suppose that now B is rotated three different times 90° counter clockwise with respect to every of its positive axis such that the Cartesian equation of B after rotation now becomes $(\frac{x}{2})^2+(\frac{y}{3})^2+z^2=1$, $x^2+(\frac{y}{2})^2+(\frac{z}{3})^2=1$ and $(\frac{x}{3})^2+y^2+(\frac{z}{2})^2=1$. Table 2 shows the first order PT for each rotated B at conductivity 1.5 computed numerically by the same proposed procedures. In this case, each rotated B is firstly triangularized by Netgen Mesh Generator with the 'fine' meshing option with N number of elements and $\hat{A}$ is the approximated surface area.

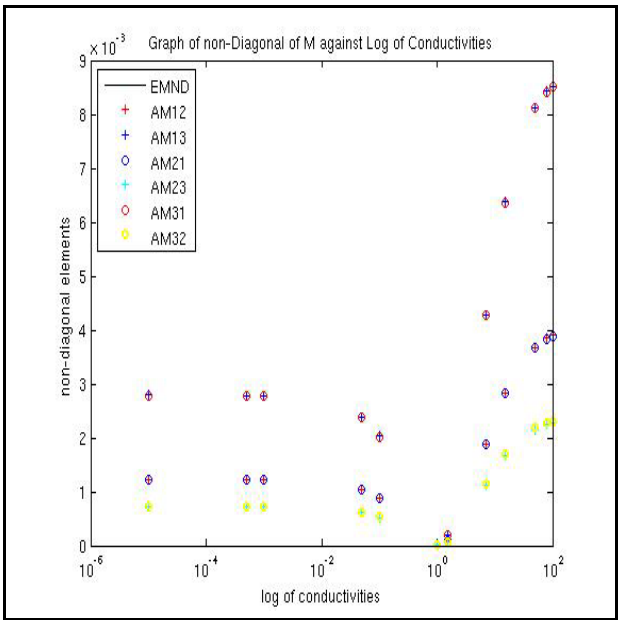
Table 2: The Approximated First Order PT for Rotated B

Ellipsoid	N	$\hat{A}$	$APT1$
$(\frac{x}{2})^2+(\frac{y}{3})^2+z^2=1$	8342	48.8291	$\begin{bmatrix}11.0249 & -0.0004 & 0.0001\\ -0.0002 & 11.5820 & 0\\ -0.0003 & -0.0002 & 9.7223\end{bmatrix}$
$x^2+(\frac{y}{2})^2+(\frac{z}{3})^2=1$	8882	48.8348	$\begin{bmatrix}9.7240 & 0.0001 & 0\\ 0 & 11.0271 & 0.0003\\ 0 & 0.0002 & 11.5869\end{bmatrix}$
$(\frac{x}{3})^2+y^2+(\frac{z}{2})^2=1$	7870	48.8275	$\begin{bmatrix}11.5811 & -0.0001 & 0.0003\\ -0.0011 & 9.7168 & 0.0084\\ 0.0004 & 0.0020 & 10.9279\end{bmatrix}$

Finally, in order to demonstrate the transformation of the first order PT and investigate the approximation for each rotated B, instead of modifying formula (7), Theorem 2 is applied to compute the exact first order PT for the rotated B. The relation $M(1.5,B')=RM(1.5,B)R^T$ in this theorem with appropriate rotation matrix R as defined in Definition

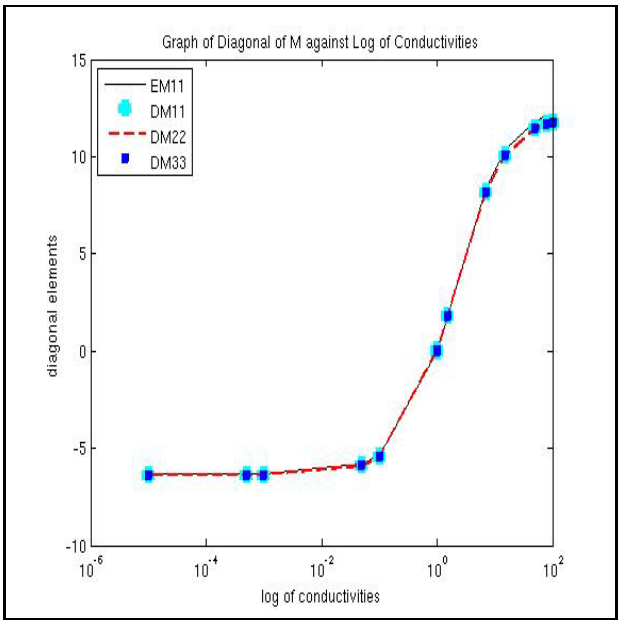


(a)

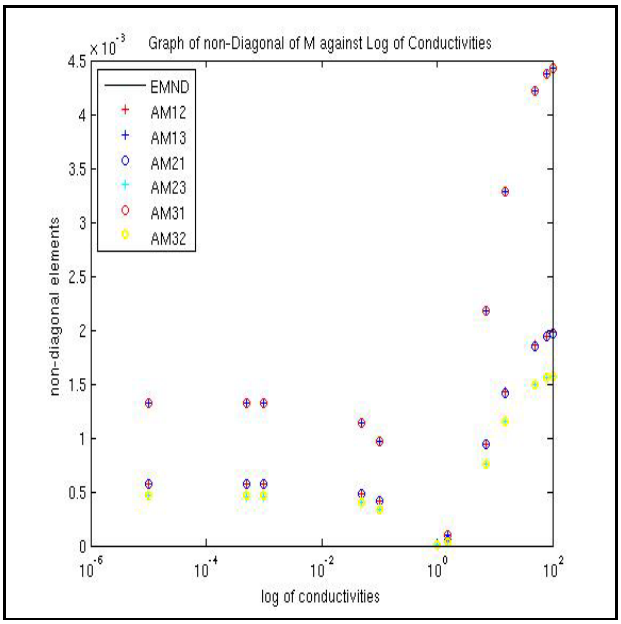


(b)

Figure 2: A Comparison between the Approximated First Order PT for a Sphere Triangularized with 620 Elements and the Exact First Order PT for the Same Sphere in the Same Graph

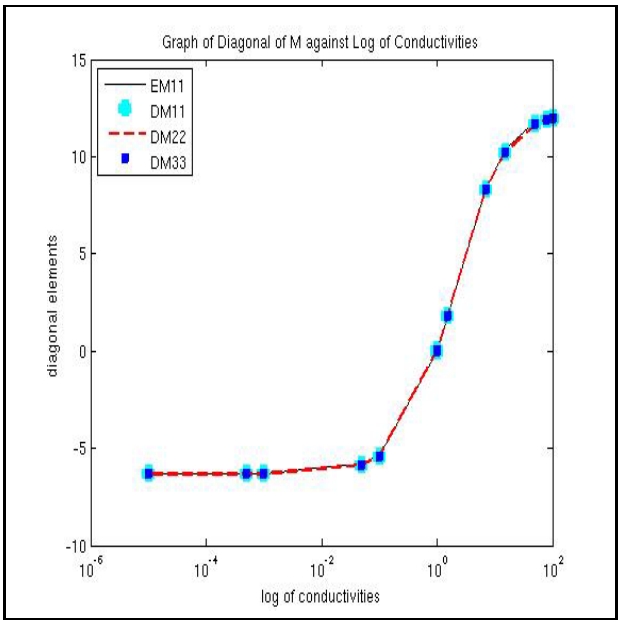


(a)

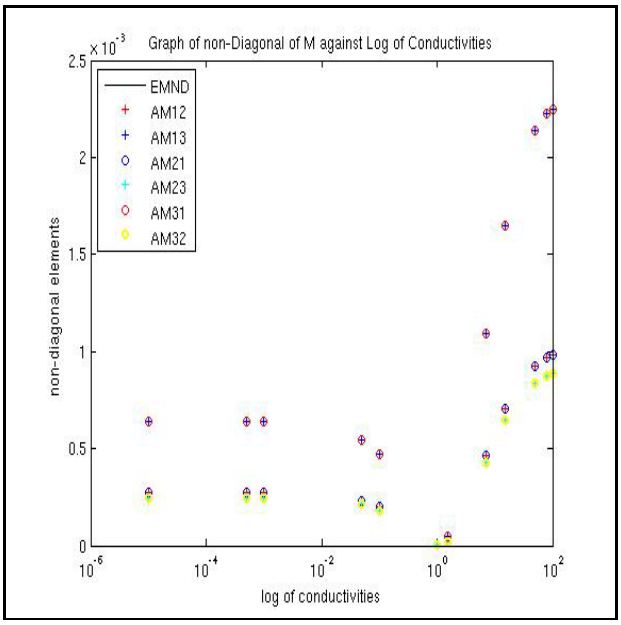


(b)

Figure 3: A Comparison between the Approximated First Order PT for a Sphere Triangularized with 2480 Elements and the Exact First Order PT for the Same Sphere in the Same Graph



(a)



(b)

Figure 4: A Comparison between the Approximated First Order PT for a Sphere Triangularized with 9920 Elements and the Exact First Order PT for the Same Sphere in the Same Graph

2 is used to achieve this and then presented in Table 3. The approximation first order PT for the original B is also obtained in Table 4 for further comparison.

Table 3: The Exact First Order PT for Rotated B

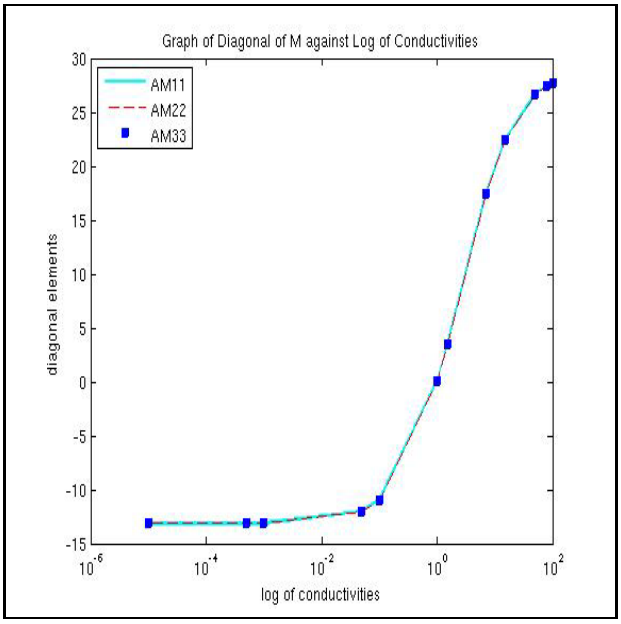
Rotated Surface, B'	Relation	$M(1.5, B')$
$(\frac{x}{2})^2 + (\frac{y}{3})^2 + z^2 = 1$	$R_z(90°)M(1.5, B)R_z^T(90°)$	$\begin{bmatrix} 11.0856 & 0 & 0 \\ 0 & 11.6555 & 0 \\ 0 & 0 & 9.7544 \end{bmatrix}$
$x^2 + (\frac{y}{2})^2 + (\frac{z}{3})^2 = 1$	$R_y(90°)M(1.5, B)R_y^T(90°)$	$\begin{bmatrix} 9.7544 & 0 & 0 \\ 0 & 11.0856 & 0 \\ 0 & 0 & 11.6555 \end{bmatrix}$
$(\frac{x}{3})^2 + y^2 + (\frac{z}{2})^2 = 1$	$R_x(90°)M(1.5, B)R_x^T(90°)$	$\begin{bmatrix} 11.6555 & 0 & 0 \\ 0 & 9.7544 & 0 \\ 0 & 0 & 11.0856 \end{bmatrix}$

4.3 First Order PT for other Shapes

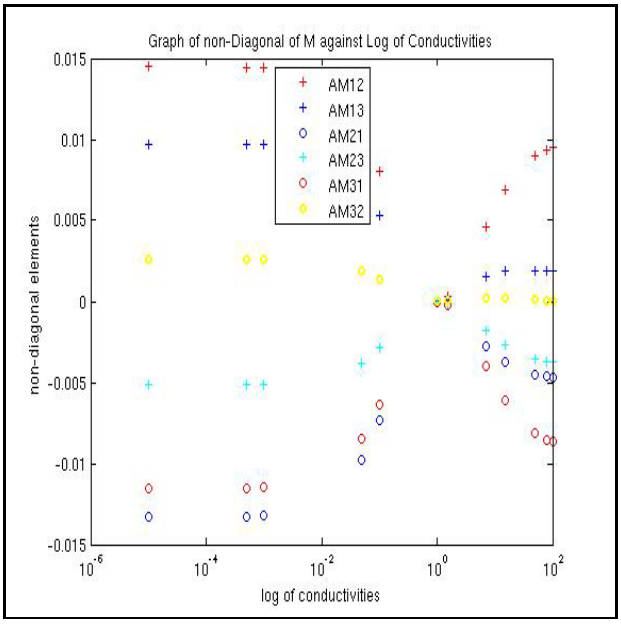
In order to further check the positivity and negativity of the first order PT, instead of depending on PT for the sphere and ellipsoid, the first order PT for few common shapes that are frequently appear in engineering applications are also calculated for some conductivities $0 < k < 1$ and $k > 1$ by the same numerical procedures where each of them are firstly triangularized by around 10,000 elements. They are cube with dimension $2 \times 2 \times 2$, cylinder with both diameter and height equal to 2 and hemisphere with radius 1.5. Similarly as the sphere, the elements of approximated first order PT for these three shapes at different values of conductivity are shown in both parts of Figure 5, Figure 6 and Figure 7 respectively.

5 Discussion

According to the results obtained, the diagonal elements of the approximated first order PT for the sphere $x^2 + y^2 + z^2 = 1$ are close to the exact value except for larger conductivity. In addition, the non-diagonal elements also converge to 0 especially when the conductivity is near to 1. Based on Figure 2, Figure 3 and Figure 4, it can be seen that as the number of element due to triangularization increase, the convergence of diagonal elements becomes better including for high level of conductivity while the non-diagonal elements also become smaller for all conductivities. In order to obtain the approximated first order PT for the sphere to be almost diagonal matrix, the proposed numerical method must be applied with finer meshes such that with more number of element during triangularization. However,

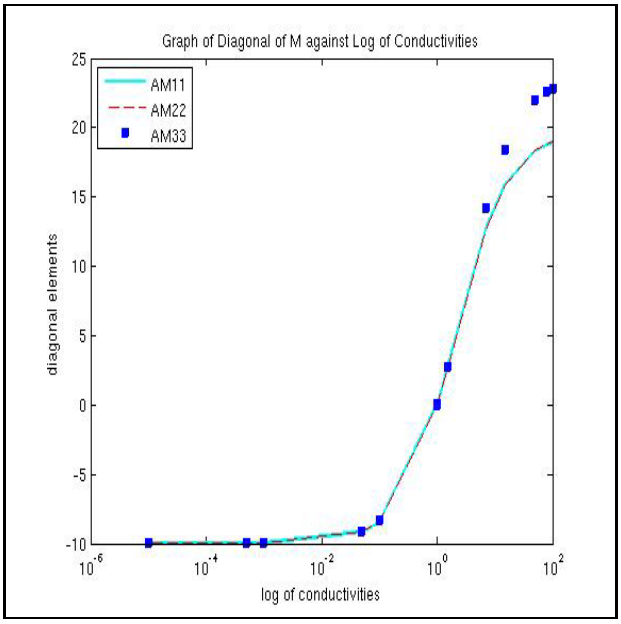


(a)

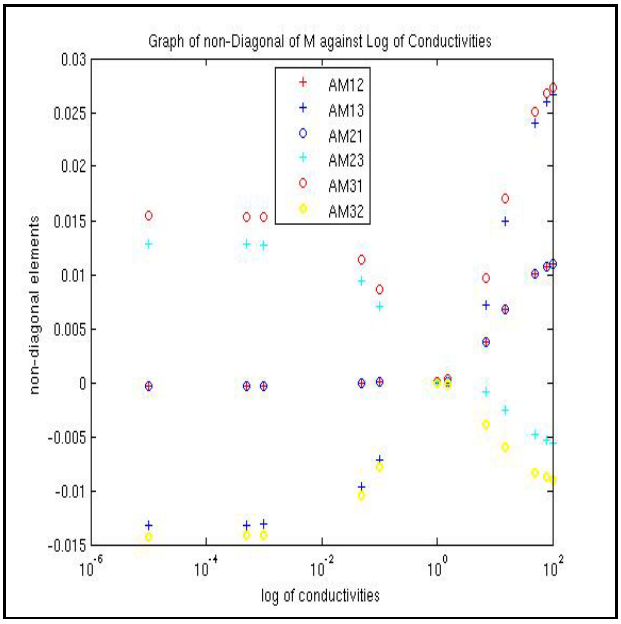


(b)

Figure 5: The Approximated First Order PT for a Cube

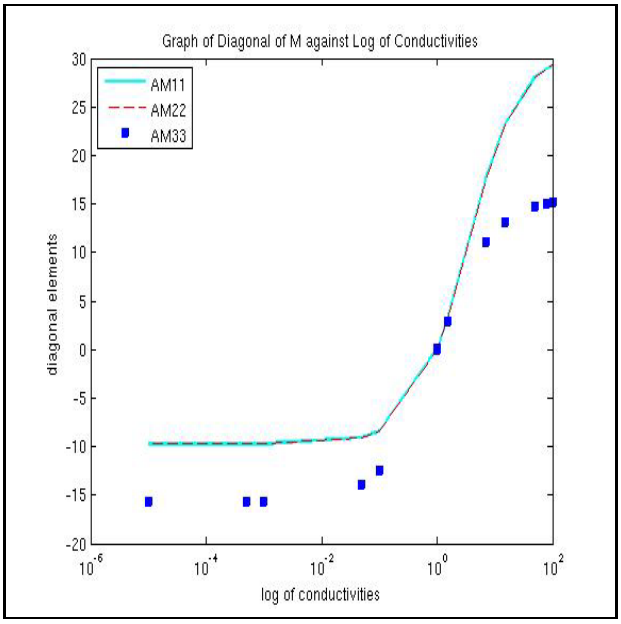


(a)

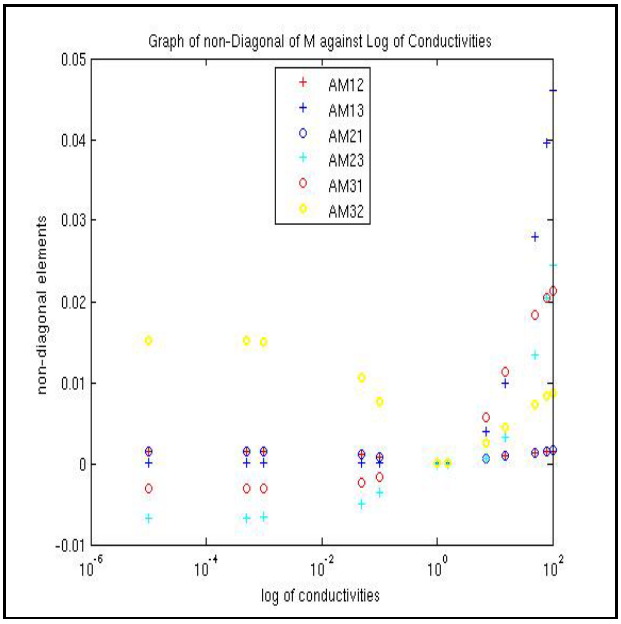


(b)

Figure 6: The Approximated First Order PT for a Cylinder



(a)



(b)

Figure 7: The Approximated First Order PT for a Hemisphere

Table 4: The Approximated First Order PT for B

N	$\hat{A}$	$APT1$
7806	48.8271	$\begin{bmatrix} 11.5814 & 0.0007 & -0.0002 \\ 0.0015 & 10.8046 & 0.0126 \\ -0.0002 & 0.0496 & 9.6919 \end{bmatrix}$
13992	48.8512	$\begin{bmatrix} 11.6029 & 0.0001 & -0.0001 \\ -0.0001 & 10.9231 & 0.0021 \\ -0.0004 & 0.0085 & 9.7178 \end{bmatrix}$

large ammount of memory will be required to implement this method by MATLAB in usual computer.

Next, the first order PT for ellipsoid $(x/3)^2 + (y/2)^2 + z^2 = 1$ is evaluated by the exact formula (7) and then transformed to obtain the first order PT for the same ellipsoid after the surface is rotated three different times (see Table 3). Besides, the proposed numerical method is used to approximate the first order PT for these ellipsoids to further compare the convergence of the results at conductivity 1.5 since previous results indicate that a good approximation of the tensor is obtained when the conductivity is near to 1. This times, each ellipsoid is triangularized such that their approximated surface areas are equal to two decimal places as shown in Table 2. Based on Table 3 and Table 2, the approximated first order PT for each rotated ellipsoid are accurate to the exact at zero decimal place. However, the precision of the computation is still much depend on the total number of elements used during triangularization.

Moreover, the approximated first order PT for ellipsoid $(x/3)^2 + (y/2)^2 + z^2 = 1$ as shown in Table 4 also converge to the exact one as the total number of elementes increase. If this approximated first order PT when N equal to 13992 is transformed to obtain the approximated first order PT for each shape in Table 2, the result now accurate to one decimal place only for $(x/3)^2 + y^2 + (z/2)^2 = 1$. This suggest that the convergence of numerical method implemented also depend on the object.

Since the approximated first order PT for sphere and ellipsoid reasonably converge to the exact, they will converge to positive and negative definite matrix for $k > 1$ and $0 < k < 1$ respectively as stated in Theorem 1. In order to provide further evidence about positivity and negativity of first order PT, the approximated first order PT for cube, cylinder and hemisphere are determined. By inspecting Figure 5, Figure 6 and 7, it is fair to say that all first order PT listed are almost diagonal matrices for all conductivities. For $0 < k < 1$, the diagonal elements of approximated first order PT for each object are negative and very much smaller than the non diagonal elements and thus by Definition 1, the tensor are negative definite matrices. In contrast, the diagonal elements of every PT when $k > 1$ are positive and very much larger than the non diagonal elements which concludes that they are positive definite.

6 Conclusion and Suggestion

The quadrature method is applied to numerically determine the first order PT for several 3-D objects in this report with acceptable accuracy where the results obtained from the computation are used to review transformation and types of the first order PT. By investigating these properties, some factors that generally influence the proposed numerical method are revealed which are the conductivity, the number of elements during triangularization and possibly the triangularized shape itself. All of them must be carefully considered in order to further improve the method use to approximate the first order PT with better convergence.

Furthermore, the establishment of this study might become a useful outline to further and deeper explore not only the first order but also the higher order 3-D PT systematically. Besides, the understanding about first order PT gained here can also help to properly apply this promising terminology in any relevant scientific problems especially in the field of electrical or electromagnectic. These are the areas that will be seriously dealt and highlighted in our future studies of the PT.

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