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MATHEMATICS

Bethe ansatz for the deformed Gaudin model

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Abstract. A deformation of the $\mathfrak{sl}(2)$ Gaudin model by a Jordanian r-matrix depending on the spectral parameter is constructed. The energy spectrum is preserved and recurrent creation operators are proposed.

Key words: Gaudin models, algebraic Bethe ansatz.

The Gaudin model of the interacting spins on a chain [1] can be considered classically with the variables S_n^α satisfying Poisson brackets or as a quantum system with the generators of the Lie algebra $\mathfrak{sl}(2)$ and commutation relations

$$[S_m^\alpha, S_n^\beta] = \imath \varepsilon^{\alpha\beta\gamma} S_m^\gamma \delta_{mn}.$$

It is useful as a particularly simple model to develop different approaches within the framework of the quantum inverse scattering method [2] and for some physical application as well.

The treatment of the model using the classical r-matrix [3] permits us to extend the Gaudin model to any semi-simple Lie algebra and to all cases when the antisymmetric r-matrix $r(\lambda,\mu)$ satisfies the classical Yang–Baxter equation. The existence of a set of integrals of motion in involution (an Abelian subalgebra) for the Gaudin model is an easy consequence of the classical Yang–Baxter equation. However, the difficult problem is to find the spectrum and corresponding eigenvectors through the Bethe ansatz. This problem has been solved on the case by case basis. A particular situation appears when the underlying quantum group or classical r-matrix is deformed by a twist transformation. It was shown in [4] that the integrable system is deformed as well. Even in the simple case of the $\mathfrak{sl}(2)$ Gaudin model deformed by the Jordanian twist [5] the algebraic Bethe ansatz changes, and the eigenvectors are constructed by recurrence creation operators [6]. Here we consider a more complicated deformation given by the classical r-matrix $r(\lambda,\mu)$ with polynomial dependence on the spectral parameter [7,8].

The $\mathfrak{sl}(2)$-invariant r-matrix with an extra Jordanian term

$$r_J(\lambda-\mu) = \frac{c_2}{\lambda-\mu} + \xi(h\otimes X^+ - X^+\otimes h) \tag{1}$$

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was studied in [5,6]. To write the r-matrix (1), the generators of $\mathfrak{sl}(2)$ are used

$$[h,X^+]=2X^+,\quad [X^+,X^-]=h,\quad [h,X^-]=-2X^-,$$

as well as the Casimir operator c_2 in the tensor square of the universal enveloping algebra $U(\mathfrak{sl}(2))$

$$c_2=h\otimes h+2(X^+\otimes X^-+X^-\otimes X^+).$$

In this paper we consider the r-matrix with an extra term (deformation) depending on the spectral parameters λ and μ

$$r_\xi(\lambda,\mu)=\frac{c_2}{\lambda-\mu}+\xi(h\otimes\mu X^+-\lambda X^+\otimes h). \tag{2}$$

Using the fundamental representation of $\mathfrak{sl}(2)$, one gets the following 4×4 matrix:

$$r_\xi(\lambda,\mu)=\begin{pmatrix} \frac{1}{\lambda-\mu} & \xi\mu & -\xi\lambda & 0 \\ 0 & -\frac{1}{\lambda-\mu} & \frac{2}{\lambda-\mu} & \xi\lambda \\ 0 & \frac{2}{\lambda-\mu} & -\frac{1}{\lambda-\mu} & -\xi\mu \\ 0 & 0 & 0 & \frac{1}{\lambda-\mu} \end{pmatrix}. \tag{3}$$

Following the quantum inverse scattering method developed in [9], it is useful to introduce the current algebra generated by

$$L(\lambda)=\begin{pmatrix} h(\lambda) & 2X^-(\lambda) \\ 2X^+(\lambda) & -h(\lambda) \end{pmatrix}, \tag{4}$$

subject to the following classical analogue of the RTT-relations of [9]:

$$[L(\lambda)\otimes\mathbb{1},\mathbb{1}\otimes L(\mu)]=-[r_\xi(\lambda,\mu),L(\lambda)\otimes\mathbb{1}+\mathbb{1}\otimes L(\mu)],$$

where

$$L(\lambda)\otimes\mathbb{1}=\begin{pmatrix} h(\lambda) & 0 & 2X^-(\lambda) & 0 \\ 0 & h(\lambda) & 0 & 2X^-(\lambda) \\ 2X^+(\lambda) & 0 & -h(\lambda) & 0 \\ 0 & 2X^+(\lambda) & 0 & -h(\lambda) \end{pmatrix},$$

and

$$\mathbb{1}\otimes L(\mu)=\mathscr{P}\,(L(\mu)\otimes\mathbb{1})\,\mathscr{P},$$

where $\mathscr{P}$ is the permutation matrix in $\mathbb{C}^2\otimes\mathbb{C}^2$. Using the explicit form of the r-matrix $r_\xi(\lambda,\mu)$ (3), we get the following commutation relations:

$$[h(\lambda),h(\mu)]=2\xi(\lambda X^+(\lambda)-\mu X^+(\mu)),\qquad [h(\lambda),X^+(\mu)]=-2\frac{X^+(\lambda)-X^+(\mu)}{\lambda-\mu},$$

$$[h(\lambda),X^-(\mu)]=2\frac{X^-(\lambda)-X^-(\mu)}{\lambda-\mu}+\xi\mu h(\mu),\qquad [X^-(\lambda),X^-(\mu)]=-\xi(\mu X^-(\lambda)-\lambda X^-(\mu)),$$

$$[X^+(\lambda),X^-(\mu)]=-\frac{h(\lambda)-h(\mu)}{\lambda-\mu}+\xi\mu X^+(\mu),\qquad [X^+(\lambda),X^+(\mu)]=0.$$

One can see that for $\xi = 0$ the polynomial loop algebra $\mathfrak{sl}(2)[\mu]$ is reproduced. The generating function of integrals of motion is given by

$$t(\lambda) = \frac{1}{2}\mathrm{Tr}(L(\lambda)^2) = h^2(\lambda) + 2(X^-(\lambda)X^+(\lambda) + X^+(\lambda)X^-(\lambda))$$
$$= h^2(\lambda) - 2h'(\lambda) + 4X^-(\lambda)X^+(\lambda) + 2\xi\lambda X^+(\lambda).$$

Due to (3) the current algebra $L(\lambda)$ (4) admits the following local representation:

$$L_{0a}(\lambda, z_a) = \begin{pmatrix} \frac{h_a}{\lambda - z_a} + \xi z_a X_a^+ & \frac{2X_a^-}{\lambda - z_a} - \xi\lambda h_a \\ \frac{2X_a^+}{\lambda - z_a} & -\frac{h_a}{\lambda - z_a} - \xi z_a X_a^+ \end{pmatrix}.$$

Here $h, X^{\pm}$ are elements of $\mathfrak{sl}(2)$, index 0 refers to the auxiliary space $\mathbb{C}^2$, and a is the index of the quantum spaces V_a (see [9]). Due to the triangular form of the extra term in (2), $L(\lambda)$ (4) has the highest weight vector $v_0(l_a)$ in V_a of the spin l_a; in other words,

$$h_a v_0(l_a) = l_a v_0(l_a), \qquad X_a^+ v_0(l_a) = 0.$$

Again, due to the triangular form of the extra term in (2), $\otimes_{a=1}^N V_a$ is also the highest weight representation of $L(\lambda)$ (4) with the highest weight vector $\Omega = \otimes_{a=1}^N v_0(l_a)$. In this representation we have

$$h(\lambda) = \sum_{a=1}^{N} \left(\frac{h_a}{\lambda - z_a} + \xi z_a X_a^+ \right),$$
$$X^+(\lambda) = \sum_{a=1}^{N} \frac{X_a^+}{\lambda - z_a},$$
$$X^-(\lambda) = \sum_{a=1}^{N} \left(\frac{X_a^-}{\lambda - z_a} - \frac{1}{2}\xi\lambda h_a \right).$$

The value of $t(\lambda)$ on the highest weight vector Ω is

$$t(\lambda)\Omega = \Lambda_0(\lambda)\Omega = \left(\left(\sum_{a=1}^{N} \frac{l_a}{\lambda - z_a} \right)^2 + 2\sum_{a=1}^{N} \frac{l_a}{(\lambda - z_a)^2} \right) \Omega = \left(\rho^2(\lambda) - 2\rho'(\lambda)\right)\Omega,$$

where

$$h(\lambda)\Omega = \sum_{a=1}^{N} \left(\frac{h_a}{\lambda - z_a} \right) \Omega = \rho(\lambda)\Omega.$$

In order of obtain the one-magnon eigenstate, it is of interest to notice that

$$t(\lambda)X^-(\mu) = X^-(\mu)\left(t(\lambda) - \frac{4h(\lambda)}{\lambda - \mu} + 4\xi\lambda X^+(\lambda) \right)$$
$$+ 4\frac{X^-(\lambda)h(\mu)}{\lambda - \mu} + 2\xi\left(\mu h(\lambda) + 1\right)h(\mu) + 2\xi^2\mu^2 X^+(\mu).$$

Thus the one-magnon eigenstate is given by $X^-(\mu_1)\Omega$:

$$t(\lambda)X^-(\mu_1)\Omega = \Lambda_1(\lambda;\mu_1)X^-(\mu_1)\Omega = \left(\Lambda_0(\lambda) - \frac{4\rho(\lambda)}{\lambda - \mu_1} \right) X^-(\mu_1)\Omega,$$

provided the Bethe equation $\rho(\mu_1) = 0$ is valid.

In order to obtain the M-magnon eigenstate, it is necessary to introduce the following operators symmetric with respect to the permutation of the quasi-momenta $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_M)$:

$$B_M(\vec{\mu}) = X^-(\mu_1)(X^-(\mu_2) + \xi\mu_2) \cdots (X^-(\mu_M) + (M-1)\xi\mu_M),$$

$$\begin{aligned} B_M^{(k)}(\vec{\mu}) &= (X^-(\mu_1) + k\xi\mu_1)(X^-(\mu_2) + (k+1)\xi\mu_2) \cdots (X^-(\mu_M) + (k+M-1)\xi\mu_M) \\ &= B_M^{(k-1)}(\vec{\mu}) + \xi \textstyle\sum_1^M \mu_j B_{M-1}^{(k)}(\vec{\mu}^{(j)}), \end{aligned}$$

where $\vec{\mu}^{(j)} = (\mu_1, \mu_2, \ldots, \mu_{j-1}, \mu_{j+1}, \ldots, \mu_M)$ means that the variable μ_j is absent. We introduce the operator

$$\hat{\beta}_M(\lambda; \vec{\mu}) = h(\lambda) + \sum_{\mu \in \vec{\mu}} \frac{2}{\mu - \lambda},$$

in particular, $\hat{\beta}(\mu; \emptyset) = h(\mu)$ and we notice that

$$\hat{\beta}_M(\lambda; \vec{\mu})\Omega = \beta_M(\lambda; \vec{\mu})\Omega = \left(\rho(\lambda) + \sum_{\mu \in \vec{\mu}} \frac{2}{\mu - \lambda}\right)\Omega.$$

Consider the two-magnon eigenstate

$$\begin{aligned} B_2(\mu_1, \mu_2)\Omega &= X^-(\mu_1)(X^-(\mu_2) + \xi\mu_2)\Omega \\ &= \frac{1}{2}\left(X^-(\mu_1)X^-(\mu_2) + X^-(\mu_2)X^-(\mu_1) + \xi\left(\mu_2 X^-(\mu_1) + \mu_1 X^-(\mu_2)\right)\right)\Omega. \end{aligned}$$

In order to obtain the action of $t(\lambda)$ on this state, we calculate the commutator $[t(\lambda), B_2(\mu_1, \mu_2)]$. Calculation is done, having in mind the formulas with the generators $X^{\pm}(\lambda), h(\lambda)$ given below which are obtained by induction. So, for any natural number M we have

$$\begin{aligned} X^+(\lambda)B_M(\vec{\mu}) = &B_M^{(1)}(\vec{\mu})X^+(\lambda) - \sum_1^M B_{M-1}^{(1)}(\vec{\mu}^{(j)}) \left(\frac{\hat{\beta}(\lambda; \vec{\mu}^{(j)}) - \hat{\beta}(\mu_j; \vec{\mu}^{(j)})}{\lambda - \mu_j}\right) \\ &- \sum_{1 \leqslant i < j \leqslant M} \frac{2B_{(M-1)}^{(1)}(\lambda \cup \vec{\mu}^{(i,j)})}{(\lambda - \mu_i)(\lambda - \mu_j)}, \end{aligned}$$

similarly,

$$h(\lambda)B_M(\vec{\mu}) = B_M(\vec{\mu})h(\lambda) + 2\sum_{j=1}^M \frac{B_M(\lambda \cup \vec{\mu}^{(j)}) - B_M(\vec{\mu})}{\lambda - \mu_j} + \xi \sum_{j=1}^M \mu_j B_{M-1}^{(1)}(\vec{\mu}^{(j)})\hat{\beta}(\mu_j; \vec{\mu}^{(j)}),$$

and

$$\begin{aligned} X^-(\lambda)B_M^{(k)}(\vec{\mu}) &= B_M^{(k)}(\vec{\mu})(X^-(\lambda) + M\xi\lambda) - \xi \sum_{j=1}^M \mu_j B_M^{(k)}(\lambda \cup \vec{\mu}^{(j)}) \\ &= B_{M+1}^{(k)}(\lambda \cup \vec{\mu}) - k\xi\lambda B_M^{(k)}(\vec{\mu}) - \xi \sum_{j=1}^M \mu_j B_M^{(k)}(\lambda \cup \vec{\mu}^{(j)}). \end{aligned}$$

Also, the following formula

$$[h(\lambda), B_2(\lambda,\mu_2)] = 2B_2'(\lambda,\mu_2) + 2\frac{B_2(\lambda,\lambda)-B_2(\lambda,\mu_2)}{\lambda-\mu_2} + \xi\mu_2 B_1^{(1)}(\lambda)\hat{\beta}(\mu_2;\lambda) + \xi\lambda B_1^{(1)}(\mu_2)\hat{\beta}(\lambda;\mu_2),$$

is used to obtain

$$\begin{aligned}
&[h(\lambda),[h(\lambda),B_2(\mu_1,\mu_2)]] - 2[h'(\lambda),B_2(\mu_1,\mu_2)] \\
&= \frac{8B_2(\mu_1,\mu_2)}{(\lambda-\mu_1)(\lambda-\mu_2)} - \frac{8B_2(\lambda,\mu_2)}{(\lambda-\mu_1)(\lambda-\mu_2)} - \frac{8B_2(\lambda,\mu_1)}{(\lambda-\mu_1)(\lambda-\mu_2)} + \frac{8B_2(\lambda,\lambda)}{(\lambda-\mu_1)(\lambda-\mu_2)} \\
&+ 4\xi\lambda\frac{B_1^{(1)}(\lambda)}{(\lambda-\mu_1)(\lambda-\mu_2)} + \xi\mu_1\left(4\frac{B_1^{(1)}(\lambda)-B_1^{(1)}(\mu_2)}{\lambda-\mu_2} - 2\xi + \xi\mu_2\hat{\beta}(\mu_2;\emptyset)\right)\hat{\beta}(\mu_1;\mu_2) \\
&+ \xi\mu_2\left(4\frac{B_1^{(1)}(\lambda)-B_1^{(1)}(\mu_1)}{\lambda-\mu_1} - 2\xi + \xi\mu_1\hat{\beta}(\mu_1;\emptyset)\right)\hat{\beta}(\mu_2;\mu_1) \\
&+ 2\xi^2\mu_2 B_1^{(1)}(\mu_1)(\lambda X^+(\lambda) - \mu_2 X^+(\mu_2)) + 2\xi^2\mu_1 B_1^{(1)}(\mu_2)(\lambda X^+(\lambda) - \mu_1 X^+(\mu_1)) \\
&+ 2\xi\left(B_1^{(1)}(\mu_1)\left(\frac{\lambda\hat{\beta}(\lambda;\mu_1) - \mu_2\hat{\beta}(\mu_2;\mu_1)}{\lambda-\mu_2}\right) + B_1^{(1)}(\mu_2)\left(\frac{\lambda\hat{\beta}(\lambda;\mu_2) - \mu_1\hat{\beta}(\mu_1;\mu_2)}{\lambda-\mu_1}\right)\right).
\end{aligned}$$

Finally, we obtain the action of the transfer matrix $t(\lambda)$ on the two-magnon creation operator

$$\begin{aligned}
t(\lambda)B_2(\mu_1,\mu_2) &= B_2(\mu_1,\mu_2)\left(t(\lambda) - \sum_1^2 \frac{4h(\lambda)}{\lambda-\mu_j} + \frac{8}{(\lambda-\mu_1)(\lambda-\mu_2)} + 8\xi\lambda X^+(\lambda)\right) \\
&+ \frac{4B_2(\lambda,\mu_1)}{\lambda-\mu_2}\hat{\beta}(\mu_2;\mu_1) + \frac{4B_2(\lambda,\mu_2)}{\lambda-\mu_1}\hat{\beta}(\mu_1;\mu_2) \\
&+ 2\xi\mu_2 B_1^{(1)}(\mu_1)h(\lambda)\hat{\beta}(\mu_2;\mu_1) + 2\xi\mu_1 B_1^{(1)}(\mu_2)h(\lambda)\hat{\beta}(\mu_1;\mu_2) \\
&+ 4\xi\mu_2\frac{B_1^{(1)}(\lambda)-B_1^{(1)}(\mu_1)}{\lambda-\mu_1}\hat{\beta}(\mu_2;\mu_1) + 4\xi\mu_1\frac{B_1^{(1)}(\lambda)-B_1^{(1)}(\mu_2)}{\lambda-\mu_2}\hat{\beta}(\mu_1;\mu_2) \\
&+ \xi^2\mu_1\mu_2(\hat{\beta}(\mu_2;\emptyset)\hat{\beta}(\mu_1;\mu_2) + \hat{\beta}(\mu_1;\emptyset)\hat{\beta}(\mu_2;\mu_1)) \\
&+ 2\xi^2(\mu_2^2 B_1^{(1)}(\mu_1)X^+(\mu_2) + \mu_1^2 B_1^{(1)}(\mu_2)X^+(\mu_1)) \\
&- 2\xi^2(\mu_1\hat{\beta}(\mu_1;\mu_2) + \mu_2\hat{\beta}(\mu_2;\mu_1)) + 2\xi(B_1^{(1)}(\mu_1)\hat{\beta}(\mu_2;\mu_1) + B_1^{(1)}(\mu_2)\hat{\beta}(\mu_1;\mu_2)).
\end{aligned}$$

Thus, the eigenvalue of the two-magnon state is

$$t(\lambda)B_2(\vec{\mu})\Omega = \Lambda_2(\lambda;\vec{\mu})B_2(\vec{\mu})\Omega,$$

$$\Lambda_2(\lambda;\vec{\mu}) = \Lambda_0(\lambda) + \frac{8}{(\lambda-\mu_1)(\lambda-\mu_2)} - \frac{4\rho(\lambda)}{\lambda-\mu_1} - \frac{4\rho(\lambda)}{\lambda-\mu_2},$$

provided the Bethe equations $\beta(\mu_j;\vec{\mu}^{(j)}) = 0$ are satisfied.

The comparison with the previous calculations for the $\mathfrak{sl}(2)$ Gaudin model deformed by the Jordanian r-matrix [6] support our conjecture that the M-magnon eigenstate is given by

$$B_M(\vec{\mu})\Omega = X^-(\mu_1)(X^-(\mu_2)+\xi\mu_2)\cdots(X^-(\mu_M)+(M-1)\xi\mu_M)\Omega.$$

The proof by induction is under construction.

Finally, for each rational $\mathfrak{sl}(2)$-valued r-matrix we have a corresponding Gaudin model. For future development it is interesting to consider the algebraic Bethe ansatz for deformed Gaudin models related to higher rank Lie algebras. For example, in the case of the Lie algebra $\mathfrak{sl}(3)$, it may be of interest to consider Gaudin models based on the classification of the rational $\mathfrak{sl}(3)$-valued r-matrices given in [7,8]. In particular, in this case there are families of rational r-matrices of the following form:

$$r(\lambda,\mu) = \frac{c_2}{\lambda-\mu} + r,$$

where r is a constant triangular r-matrix or a polynomial of degree one with values in the Lie algebra $\mathfrak{sl}(3)$. Also, the papers [10,11] are related to this subject.

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Bethe eeldus deformeeritud Gaudini mudeli jaoks

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On konstrueeritud $\mathfrak{sl}(2)$ Gaudini mudeli deformatsioon Jordani r-maatriksi abil, mis sõltub spektraalsest parameetrist. Energiaspekter ei muutu ja on pakutud rekurrentsed tekkeoperaatorid.