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PHYSICS

Deformed surfaces in holographic interferometry. Similar aspects in general gravitational fields

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Abstract. In the introductory part (Section 1) some aspects of the large deformation analysis in holographic interferometry are briefly outlined. The calculus there may also serve as an introduction for a link to the principal part afterwards. Modifications of the set-up at the reconstruction should recover the previously invisible fringes. Their spacing and the contrast are characterized by the fringe and visibility vectors. The relevant derivative of the path difference involves the polar decomposition of the deformation gradient into strain and rotation and the image aberration implies further changes of the geodesic curvature and of surface curvatures. In the principal part (sections 2, 3, 4) these considerations lead then to similar aspects for hypersurfaces, above all to an interpretation of gravitation by two virtual deformations for the Schwarzschild solution. This is further useful for non-spherical gravitational fields, for the invariants there, and for the TOV relation between pressure and density. The null-geodesics or light rays can also be interpreted by these virtual deformations. An approach towards the Kerr solution for rotating stars is added. As to linearization, a connection is outlined, which confirms the non-existence of gravitational waves if they are described by pure geometrical considerations of the field equations. Detailed equations for calculations are presented in Section 4.

Key words: optics holographic interferometry, large deformations, curved (hyper-) surfaces, general gravitational fields.

1. DERIVATIVES OF THE OPTICAL PATH DIFFERENCE, STRAIN, ROTATION, CHANGES OF CURVATURE, FRINGE AND VISIBILITY VECTORS

The basic expression in holographic interferometry is the optical path difference $D=\boldsymbol{u}\cdot(\boldsymbol{k}-\boldsymbol{h})=\lambda\nu$. Here $\boldsymbol{u}$ is the displacement, $\boldsymbol{h}$ and $\boldsymbol{k}$ are unit vectors, λ is the wave length, and ν the fringe order. In the case of a large deformation, when two modified holograms are used [1], the exact expression becomes $D=(\lambda/2\pi)(\tilde{\varphi}-\tilde{\varphi}')-(\tilde{L}-\tilde{L}')$, where $\tilde{L},\tilde{L}'$ denote the distances from the image points $\tilde{\mathrm{P}},\tilde{\mathrm{P}}'$ to a point $\tilde{\mathrm{K}}$ of fringe localization (see Fig. 1). The phases at $\tilde{\mathrm{P}},\tilde{\mathrm{P}}'$ are $\tilde{\varphi}=(2\pi/\lambda)(L_{\mathrm{T}}+L_{\mathrm{S}}+p-q-q_{\mathrm{T}}-\tilde{p}+\tilde{q}+\tilde{q}_{\mathrm{T}})+\pi+\tilde{\psi},\ \tilde{\varphi}'=\ldots+\Delta\tilde{\psi},$ so we obtain in general

$$D=L_{\mathrm{S}}-L_{\mathrm{S}}'-(\tilde{L}+\tilde{p})+(\tilde{L}'+\tilde{p}')+(p-q)-(p'-q')+\tilde{q}-\tilde{q}'-\lambda\Delta\tilde{\psi}/2\pi. \tag{1}$$

Many authors [e.g. 2] have studied the recovering of fringes. The contrast depends on the derivative of D, and the spacing leads to the strains. Therefore the differential $\mathrm{d}D=\mathrm{d}L_{\mathrm{S}}-\mathrm{d}L_{\mathrm{S}}'-\mathrm{d}(\tilde{L}+\tilde{p})+\mathrm{d}(\tilde{L}'+\tilde{p}')+\mathrm{d}(p-q)-\mathrm{d}(p'-q')+\mathrm{d}\tilde{q}-\mathrm{d}\tilde{q}'$ is primary.

We insert now some elements with convenient notations for the geometry and deformation of 2D-surfaces in the 3D-space. In the principal part these concepts will be generalized to 4D-hypersurfaces, embedded in an 8D-space or in a 4D-complex space. We write in particular for the distance L_{S} in Fig. 1 left: $\mathrm{d}L_{\mathrm{S}}=\mathrm{d}\boldsymbol{r}\cdot\boldsymbol{N}\nabla L_{\mathrm{S}}=\mathrm{d}\boldsymbol{r}\cdot\boldsymbol{N}\boldsymbol{h}$ with the *normal projector* $\boldsymbol{N}=\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n}$ [note that for any dyadic $\boldsymbol{a}\otimes\boldsymbol{b}$ the rules $\boldsymbol{x}(\boldsymbol{a}\otimes\boldsymbol{b})=(\boldsymbol{x}\cdot\boldsymbol{a})\boldsymbol{b},\ (\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{y}=\boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{y})$ hold]. The operator $\boldsymbol{N}\nabla=\nabla_{\mathrm{n}}=\boldsymbol{a}^{\alpha}\partial/\partial\theta^{\alpha}$ (α summed

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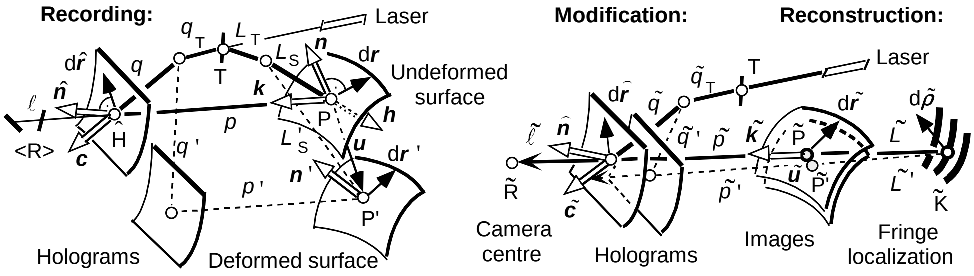

Fig. 1. Recording of a large surface deformation. Modification at the reconstruction to recover fringes.

from 1 to 2) appears as a formal projection of ∇ and the bases on the surface require $\boldsymbol{a}^{\alpha}\cdot\boldsymbol{a}_{\beta}=\delta^{\alpha}_{\beta}$, $\boldsymbol{a}_{\beta}=\partial\boldsymbol{r}/\partial\theta^{\beta}$, $\boldsymbol{a}_{\alpha}\cdot\boldsymbol{a}_{\beta}=a_{\alpha\beta}$, and $\boldsymbol{N}=a_{\alpha\beta}\boldsymbol{a}^{\alpha}\otimes\boldsymbol{a}^{\beta}$. We write also

$$(\nabla_{\mathrm{n}}\otimes\boldsymbol{n})\boldsymbol{N}=-\boldsymbol{B}, \tag{2}$$

$$\nabla_{\mathrm{n}}\otimes\boldsymbol{N}=\boldsymbol{B}\otimes\boldsymbol{n}+\boldsymbol{B}\otimes\boldsymbol{n}]^{T}. \tag{3}$$

The tensor $\boldsymbol{B}=B_{\alpha\beta}\boldsymbol{a}^{\alpha}\otimes\boldsymbol{a}^{\beta}=(1/r_1)\boldsymbol{e}_1\otimes\boldsymbol{e}_1+(1/r_2)\boldsymbol{e}_2\otimes\boldsymbol{e}_2$ describes the *exterior* curvature of a surface with principal values $1/r_1$, $1/r_2$. Equations (2) and (3) correspond to the *Frenet-relations* $\mathrm{d}\boldsymbol{n}/\mathrm{d}s=-\boldsymbol{e}/r$, $\mathrm{d}\boldsymbol{e}/\mathrm{d}s=\boldsymbol{n}/r$ in the case of a plane curve. The bracket $]^{T}$ marks a transposition so that $\boldsymbol{B}\otimes\boldsymbol{n}]^{T}=B_{\alpha\beta}\boldsymbol{a}^{\alpha}\otimes\boldsymbol{n}\otimes\boldsymbol{a}^{\beta}$. Equations (2) and (3) help to decompose the derivative of a vector $\boldsymbol{u}=\boldsymbol{v}+w\boldsymbol{n}$ into an *interior* and a *semi-exterior* part, namely $\nabla_{\mathrm{n}}\otimes\boldsymbol{u}=(\nabla_{\mathrm{n}}\otimes\boldsymbol{v})\boldsymbol{N}-\boldsymbol{B}w+(\boldsymbol{B}\boldsymbol{v}+\nabla_{\mathrm{n}}w)\otimes\boldsymbol{n}.$ The deformations read $\boldsymbol{N}'\mathrm{d}\boldsymbol{r}'=\boldsymbol{F}\boldsymbol{N}\mathrm{d}\boldsymbol{r}$, $\hat{\boldsymbol{N}}\mathrm{d}\hat{\boldsymbol{r}}=\hat{\boldsymbol{F}}\hat{\boldsymbol{N}}\mathrm{d}\hat{\boldsymbol{r}}\ldots$ where only the semi-projection $\boldsymbol{F}\boldsymbol{N}=\boldsymbol{N}+(\nabla_{\mathrm{n}}\otimes\boldsymbol{u})^{T}$ of the 3D-deformation gradient $\boldsymbol{F}=\boldsymbol{I}+(\nabla\otimes\boldsymbol{u})^{T}$ intervenes. The polar decomposition is $\boldsymbol{F}=\boldsymbol{Q}\boldsymbol{U}$ with the (orthogonal) rotation tensor $\boldsymbol{Q}$ $[\boldsymbol{Q}^{T}\boldsymbol{Q}=\boldsymbol{I}]$ and the symmetric dilatation $\boldsymbol{U}$, defined by the Cauchy–Green tensor $\boldsymbol{F}^{T}\boldsymbol{F}=\boldsymbol{U}\boldsymbol{U}$. At the surface the decomposition is with a rotation $\boldsymbol{Q}_{\mathrm{n}}$ $[\boldsymbol{n}'=\boldsymbol{Q}_{\mathrm{n}}\boldsymbol{n}=\boldsymbol{Q}_{\mathrm{i}}\boldsymbol{n}]$ and the in-plane dilatation $\boldsymbol{V}$ $[\boldsymbol{N}\boldsymbol{F}^{T}\boldsymbol{F}\boldsymbol{N}=\boldsymbol{V}\boldsymbol{V}]$

$$\boldsymbol{F}\boldsymbol{N}=\boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}=\boldsymbol{Q}_{\mathrm{i}}\boldsymbol{Q}_{\mathrm{p}}\boldsymbol{V}. \tag{4}$$

For small values, a strain tensor $\boldsymbol{\gamma}$, an *inclination* vector $\boldsymbol{\psi}$, a p-rotation scalar Ω, and the permutation tensor $\boldsymbol{E}=E_{\alpha\beta}\boldsymbol{a}^{\alpha}\otimes\boldsymbol{a}^{\beta}$ $(E_{11}=0,\ E_{12}=-E_{21},\ E_{22}=0)$, the decomposition $\boldsymbol{F}\boldsymbol{N}=\boldsymbol{N}+\boldsymbol{\gamma}-\Omega\boldsymbol{E}+\boldsymbol{n}\otimes\boldsymbol{\psi}$ is additive and $\boldsymbol{Q}_{\mathrm{i}}\boldsymbol{N}\simeq\boldsymbol{N}+\boldsymbol{n}\otimes\boldsymbol{\psi}$, $\boldsymbol{Q}_{\mathrm{p}}\simeq\boldsymbol{N}-\Omega\boldsymbol{E}$, $\boldsymbol{V}\simeq\boldsymbol{N}+\boldsymbol{\gamma}$, $\boldsymbol{E}\boldsymbol{E}=-\boldsymbol{N}$. At an isotropic, elastic surface $\boldsymbol{\gamma}=(\boldsymbol{\tau}+v_0\boldsymbol{E}\boldsymbol{\tau}\boldsymbol{E})/E_0$ holds with factors v_0, E_0, the stress tensor $\boldsymbol{\tau}$, and an *involution* $-\boldsymbol{E}(\ldots)\boldsymbol{E}$. Next, the equation of a geodesic curve, relative to the arc s, can be written as $\boldsymbol{N}\mathrm{d}^2\boldsymbol{r}/\mathrm{d}s^2=0$, because the osculating plane contains the unit normal $\boldsymbol{n}$. More generally, for any curve and its image we obtain with $\boldsymbol{V}^{(-1)}\boldsymbol{V}=\boldsymbol{N}$ the transformation *backwards*

$$\boldsymbol{N}\mathrm{d}^2\boldsymbol{r}=\boldsymbol{V}^{(-1)}[\boldsymbol{Q}_{\mathrm{n}}^{T}\boldsymbol{N}'\mathrm{d}^2\boldsymbol{r}'-(\mathrm{d}\boldsymbol{r}\boldsymbol{D}_{\mathrm{V}}\mathrm{d}\boldsymbol{r})], \tag{5}$$

$$\boldsymbol{D}_{\mathrm{V}}=[(\nabla_{\mathrm{n}}\otimes\boldsymbol{V})\boldsymbol{N}]\,|\,\boldsymbol{N}+[(\nabla_{\mathrm{n}}\otimes\boldsymbol{Q}_{\mathrm{n}})\boldsymbol{V}]\,|\,\boldsymbol{N}'\boldsymbol{Q}_{\mathrm{n}}, \tag{6}$$

where $|\boldsymbol{N}$ marks a projection of the middle factor in a triadic. Finally, $\mathrm{d}\boldsymbol{r}'\cdot\nabla_{\mathrm{n}'}=\mathrm{d}\boldsymbol{r}\cdot\nabla_{\mathrm{n}}=\mathrm{d}\boldsymbol{r}'\cdot\boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)}\nabla_{\mathrm{n}}$ gives the change of surface curvature *forwards*

$$\boldsymbol{B}'=-\boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)}(\nabla_{\mathrm{n}}\otimes\boldsymbol{n}')\boldsymbol{N}'=-\boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)}[\boldsymbol{B}\boldsymbol{Q}_{\mathrm{n}}^{T}-(\nabla_{\mathrm{n}}\otimes\boldsymbol{Q}_{\mathrm{n}})\boldsymbol{n}]\boldsymbol{N}'. \tag{7}$$

Again in holography: The image $<\tilde{\mathrm{P}}>$ is defined by $\mathrm{d}\theta_{\mathrm{P}}=0$ (or Eq. (9)) of $\theta_{\mathrm{P}}=2\pi(\tilde{p}-\tilde{q}-p+q)/\lambda$ for the rays through the aperture. Thus we find with $\boldsymbol{V}'\boldsymbol{Q}_{\mathrm{n}'}'^{T}=\boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)}$

$$\mathrm{d}D=\mathrm{d}\boldsymbol{r}'\cdot\boldsymbol{N}'[(\boldsymbol{k}'-\boldsymbol{h}')-\boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)}(\boldsymbol{k}-\boldsymbol{h})]+\mathrm{d}\tilde{\boldsymbol{\rho}}\cdot\tilde{\boldsymbol{K}}_{\tilde{\rho}}(\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{k}}'), \tag{8}$$

$$\hat{\boldsymbol{N}}[\hat{\boldsymbol{V}}\hat{\boldsymbol{Q}}_{\hat{\mathrm{n}}}^{T}(\tilde{\boldsymbol{k}}-\tilde{c})-(\boldsymbol{k}-\boldsymbol{c})]=0. \tag{9}$$

Consider now $\theta_{\tilde{\mathrm{R}}} = 2\pi(\ell + q - \tilde{\ell} - \tilde{q})/\lambda$. The condition $\mathrm{d}\theta_{\tilde{\mathrm{R}}} = 0$ leads also to Eq. (9). Therefore we get from neighbouring rays $\mathrm{d}^2(\ell + q - \tilde{\ell} - \tilde{q}) = 0$, $\mathrm{d}^2 q = (\hat{\boldsymbol{N}}\mathrm{d}^2\hat{\boldsymbol{r}})\cdot\boldsymbol{c} + \mathrm{d}\hat{\boldsymbol{r}}\cdot[\hat{\boldsymbol{B}}(\hat{\boldsymbol{n}}\cdot\boldsymbol{c}) + \hat{\boldsymbol{N}}\boldsymbol{C}\hat{\boldsymbol{N}}/q]\mathrm{d}\hat{\boldsymbol{r}}\ldots$. For $\widehat{\boldsymbol{N}}\mathrm{d}^2\widehat{\boldsymbol{r}}$ we apply Eq. (5) so that the total term $\hat{\boldsymbol{N}}\mathrm{d}^2\hat{\boldsymbol{r}}(\ldots)$ is cancelled because of Eq. (9). We use also the affine connection $\mathrm{d}\hat{\boldsymbol{r}} = -\ell\hat{\boldsymbol{M}}^T\mathrm{d}\boldsymbol{k}$ with the oblique projector $\hat{\boldsymbol{M}} = \boldsymbol{I} - \hat{\boldsymbol{n}}\otimes\boldsymbol{k}/\hat{\boldsymbol{n}}\cdot\boldsymbol{k}$. Resolving $\mathrm{d}\hat{\boldsymbol{r}}\hat{\boldsymbol{N}}\cdot\mathrm{d}\bar{\boldsymbol{k}} = -\mathrm{d}^2\ell + \ldots$, we get a linear transformation $\mathrm{d}\bar{\boldsymbol{k}} = \ell\boldsymbol{T}\mathrm{d}\boldsymbol{k}$, where the tensor $\boldsymbol{T} = \hat{\boldsymbol{M}}\{\hat{\boldsymbol{B}}(\hat{\boldsymbol{n}}\cdot(\boldsymbol{k}-\boldsymbol{c})) - \hat{\boldsymbol{N}}\boldsymbol{C}\hat{\boldsymbol{N}}/q - \widehat{\boldsymbol{Q}}_{\hat{\mathrm{n}}}\widehat{\boldsymbol{V}}^{(-1)}\ [\widehat{\boldsymbol{B}}(\widehat{\boldsymbol{n}}\cdot(\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{c}})) - \widehat{\boldsymbol{N}}\tilde{\boldsymbol{C}}\widehat{\boldsymbol{N}}/\tilde{q} - \ \widehat{\boldsymbol{D}}_{\hat{\mathrm{V}}}\,|\widehat{\boldsymbol{V}}^{(-1)}(\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{c}}) - \tilde{\boldsymbol{K}}/\tilde{\ell}]\widehat{\boldsymbol{V}}^{(-1)}\widehat{\boldsymbol{Q}}^T_{\hat{\mathrm{n}}}\}\hat{\boldsymbol{M}}^T$ shows the curvature of the non-spherical wavefront and the astigmatic interval $<\mathrm{R}>$. The ray aberration reads $\boldsymbol{K}\mathrm{d}\boldsymbol{r} = \boldsymbol{K}\mathrm{d}\hat{\boldsymbol{r}} - p\mathrm{d}\bar{\boldsymbol{k}}$; thus the bridge $\ell\mathrm{d}\boldsymbol{k} = \tilde{\ell}\boldsymbol{K}\hat{\boldsymbol{V}}^{(-1)}\hat{\boldsymbol{Q}}^T_{\hat{\mathrm{n}}}\widehat{\boldsymbol{M}}^T\mathrm{d}\tilde{\boldsymbol{k}}$ gives with $\hat{\boldsymbol{V}}^{(-1)}\hat{\boldsymbol{Q}}^T_{\hat{\mathrm{n}}} = \widehat{\boldsymbol{Q}}_{\hat{\mathrm{n}}}\widehat{\boldsymbol{V}}$ the virtual image deformation $\boldsymbol{K}\mathrm{d}\boldsymbol{r} = \tilde{\boldsymbol{G}}_{\tilde{\mathrm{R}}}(\tilde{\boldsymbol{K}}\mathrm{d}\tilde{\boldsymbol{r}})$, where $\tilde{\boldsymbol{G}}_{\tilde{\mathrm{R}}} = \tilde{\ell}(p\boldsymbol{T} + \boldsymbol{K})\widehat{\boldsymbol{Q}}_{\hat{\mathrm{n}}}\widehat{\boldsymbol{V}}\widehat{\boldsymbol{M}}^T/(\widehat{\ell} + \tilde{p})$. If the small areas projected by the aperture overlap sufficiently for the correlation, we have $\tilde{\boldsymbol{k}} - \tilde{\boldsymbol{k}}' \approx -\tilde{\boldsymbol{K}}\tilde{\boldsymbol{u}}/\tilde{L}'$. To apply Eq. (8) we use $\boldsymbol{M}' = \boldsymbol{I} - \boldsymbol{n}'\otimes\boldsymbol{k}'/\boldsymbol{n}'\cdot\boldsymbol{k}'$ as well as $\mathrm{d}\tilde{\boldsymbol{k}}' = -\tilde{\boldsymbol{m}}'\mathrm{d}\tilde{\beta}'$ with a unit vector $\tilde{\boldsymbol{m}}'$ and an angle $\mathrm{d}\tilde{\beta}'$. We write then $\mathrm{d}D_{\tilde{\mathrm{R}}'}/\mathrm{d}\tilde{\beta}' = \tilde{\boldsymbol{m}}'\cdot\tilde{\boldsymbol{f}}'_{\tilde{\mathrm{R}}'}$ and $\mathrm{d}D_{\tilde{\mathrm{K}}}/\mathrm{d}\tilde{\beta}' = -\tilde{\boldsymbol{m}}'\cdot\tilde{\boldsymbol{f}}'_{\tilde{\mathrm{K}}}$. The fringe vector [3] (indicating the spacing) and the visibility vector [4] (marking the distance of the homologous rays) are therefore

$$\tilde{\boldsymbol{f}}'_{\tilde{\mathrm{R}}'} = (\tilde{\ell}' + \tilde{p}')\tilde{\boldsymbol{G}}'^T_{\tilde{\mathrm{R}}'}\boldsymbol{M}'[\boldsymbol{k}' - \boldsymbol{h}' - \boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)}(\boldsymbol{k}-\boldsymbol{h})] - \tilde{\boldsymbol{K}}\tilde{\boldsymbol{u}}(\tilde{\ell}' + \tilde{p}' + \tilde{L}')/\tilde{L}', \tag{10}$$

$$\tilde{\boldsymbol{f}}'_{\tilde{\mathrm{K}}} = \tilde{L}'\tilde{\boldsymbol{G}}'^T_{\tilde{\mathrm{K}}}\boldsymbol{M}'[\boldsymbol{k}' - \boldsymbol{h}' - \boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)}(\boldsymbol{k}-\boldsymbol{h})]. \tag{11}$$

2. ASPECTS OF DEFORMATION AT SPHERICAL AND NONSPHERICAL GRAVITATIONAL FIELDS, GRAVITATIONAL LENS, AND ROTATING BODIES

This section is only indirectly related to the previous subject. An extension should illustrate Eqs (2)–(7). Equation (7) gives the curvature $\boldsymbol{B}'$ of a deformed surface $\mathbb{A}^2 \subset \mathbb{R}^3$. For a hypersurface $\mathbb{A}^k \subset \mathbb{R}^n$, $n > k$ this leads to the Ricci tensor $\boldsymbol{R}$, in components $R_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\lambda,\beta} - \Gamma^{\lambda}_{\alpha\beta,\lambda} + \Gamma^{\mu}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\beta} - \Gamma^{\mu}_{a\beta}\Gamma^{\lambda}_{\mu\lambda}$, where $\Gamma^{\lambda}_{\alpha\beta} = a^{\lambda\mu}(a_{\mu a,\beta} + a_{\mu\beta,a} - a_{\alpha\beta,\mu})/2$ are Christoffel symbols. However, the projector $\boldsymbol{N}' = a_{\alpha\beta}\boldsymbol{a}^{\alpha}\otimes\boldsymbol{a}^{\beta} = \boldsymbol{I} - \boldsymbol{n}'^i\otimes\boldsymbol{n}'_i$ (α, β from 0 to $k-1$; i from 1 to $n-k$) implies both the ‘metric tensor’ $a_{\alpha\beta}$ and the exterior orthonormal vectors $\boldsymbol{n}'^i = \boldsymbol{n}'_i$, which are at the moment supposed to be *real*. For the *complex* version see Section 3, Eqs (40)–(43). If we use these vectors, the Riemann–Christoffel tensor can be written (see also Section 4.1, Eqs (A1)–(A5)) as

$$\begin{aligned}\mathrm{R}^T &= \boldsymbol{N}'|\boldsymbol{N}'[\nabla_{\mathrm{n}'}\otimes\nabla_{\mathrm{n}'}\otimes\boldsymbol{N}' - \nabla_{\mathrm{n}'}\otimes(\nabla_{\mathrm{n}'}\otimes\boldsymbol{N}')^T]\boldsymbol{N}'|\boldsymbol{N}' \\ &= \boldsymbol{B}'_i\otimes\boldsymbol{B}'^{iT} - \boldsymbol{B}'_i\otimes\boldsymbol{B}'^{iT}\rrbracket^T + \boldsymbol{B}'^i\otimes(\boldsymbol{B}'_i - \boldsymbol{B}'^T_i)\rrbracket^T\Big]^T,\end{aligned} \tag{12}$$

according to Eq. (3) and $\boldsymbol{B}'^i = \boldsymbol{B}'_i = -\boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)}(\nabla_{\mathrm{n}}\otimes\boldsymbol{n}'_i)\boldsymbol{N}' = \boldsymbol{B}'^T_i$ (Eq. (7)). The bracket $\rrbracket^T$ indicates a transposition of the factors 2 and 4. The Ricci tensor is the contraction of R, alternatively $\boldsymbol{R} = \boldsymbol{B}'^T_i\boldsymbol{B}'^i - (\boldsymbol{B}'_i\cdot\boldsymbol{N}')\boldsymbol{B}'^{iT}$. For a spherical gravitational field first one uses the Schwarzschild radius $2M = 2G\bar{M}/c^2$, the mass $\bar{M}$, the radius a, the velocity of light c, and polar coordinates r, θ, φ. We define an angle $\psi \to \sin^2\psi = 2\tilde{M}/r$, where $2\tilde{M} = 2M$, for $r > a$ and $2\tilde{M} = \kappa\int_0^r \rho(\hat{r})\hat{r}^2\mathrm{d}\hat{r}$ for $r \le a$, with $\kappa = 8\pi G/c^2$ and the ‘density’ ρ. Writing $r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2 = \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{K}_{\mathrm{n}}\mathrm{d}\boldsymbol{r}$ the fundamental form [5] reads

$$\mathrm{d}\sigma'^2 = -\frac{\cos^2\psi}{\varpi^2}c^2\mathrm{d}t^2 + \frac{1}{\cos^2\psi}\mathrm{d}r^2 + \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{K}_{\mathrm{n}}\mathrm{d}\boldsymbol{r}, \tag{13}$$

where $\varpi = 1$ for $r > a$. The projector $\boldsymbol{K}_{\mathrm{n}} = \boldsymbol{N} - \boldsymbol{k}\otimes\boldsymbol{k}$ refers to the radial unit vector $\boldsymbol{k}(\theta, \varphi)$. The space part $\mathrm{d}s'^2 = \mathrm{d}\boldsymbol{r}\cdot(\boldsymbol{k}\otimes\boldsymbol{k}/\cos^2\psi + \boldsymbol{K}_{\mathrm{n}})\mathrm{d}\boldsymbol{r} = \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{V}\boldsymbol{V}\mathrm{d}\boldsymbol{r}$ gives $\boldsymbol{V}^{(-1)} = \boldsymbol{k}\otimes\boldsymbol{k}\cos\psi + \boldsymbol{K}_{\mathrm{n}}$. We obtain with a vector $\boldsymbol{r}' = r\boldsymbol{k} + w\boldsymbol{n}$ a deformation gradient $\boldsymbol{F}\boldsymbol{N} = (\boldsymbol{k} + w_{,r}\boldsymbol{n})\otimes\boldsymbol{k} + \boldsymbol{K}_{\mathrm{n}}$ and $\mathrm{d}s'^2 = \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{F}^T\boldsymbol{F}\mathrm{d}\boldsymbol{r} = \mathrm{d}\boldsymbol{r}\cdot[(1 + w^2_{,r})\boldsymbol{k}\otimes\boldsymbol{k} + \boldsymbol{K}_{\mathrm{n}}]\mathrm{d}\boldsymbol{r}$, so that $w_{,r} = \mathrm{d}w/\mathrm{d}r = \tan\psi$ implies an inclination and $\boldsymbol{k}' = \boldsymbol{k}\cos\psi +$

$\boldsymbol{n}\sin\psi$, $\boldsymbol{n}' = -\boldsymbol{k}\sin\psi + \boldsymbol{n}\cos\psi$. If we use the key relation $(\sin\psi)_{,r} = -\zeta\sin\psi/2r$ with $\zeta = 1 - \kappa\rho r^3/2\tilde{M}$, we find $\nabla_{\mathrm{n}} \otimes \boldsymbol{n}' = (\boldsymbol{k}\otimes\boldsymbol{k}')\ \ \zeta\tan\psi/2r - \boldsymbol{K}_{\mathrm{n}}\sin\psi/r$ and curvatures

$$\boldsymbol{B}' = -\boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)}(\nabla_{\mathrm{n}} \otimes \boldsymbol{n}')\boldsymbol{N}' = (\sin\psi/r)[-(\zeta/2)\boldsymbol{k}'\otimes\boldsymbol{k}' + \boldsymbol{K}_{\mathrm{n}}] = (1/r_1)\boldsymbol{k}'\otimes\boldsymbol{k}' + (1/r_2)\boldsymbol{K}_{\mathrm{n}}, \tag{14}$$

$$\begin{aligned}\boldsymbol{R}_{3\mathrm{D}} &= \boldsymbol{B}'\boldsymbol{B}' - \boldsymbol{B}'(\boldsymbol{B}'\cdot\boldsymbol{N}') = (\sin^2\psi/r^2)[\zeta(\boldsymbol{k}'\otimes\boldsymbol{k}') + (\zeta/2-1)\boldsymbol{K}_{\mathrm{n}}] \\ &= -(2/r_1r_2)\boldsymbol{k}'\otimes\boldsymbol{k}' - (1/r_1r_2 + 1/r_2r_2)\boldsymbol{K}_{\mathrm{n}},\end{aligned} \tag{15}$$

as well as the known vase-like surface [6]. Second, as for the time-radial terms in Eq. (13), we introduce a vector $\breve{\boldsymbol{r}}' = 2M\boldsymbol{k}\cos\psi/\varpi + \breve{w}\breve{\boldsymbol{n}}$, which gives with $\mathrm{i}c\mathrm{d}t \Rightarrow (2M/r)\mathrm{d}\breve{\boldsymbol{r}}\cdot\boldsymbol{h}$ a real $\breve{\boldsymbol{V}}^{(-1)} = \boldsymbol{h}\otimes\boldsymbol{h}(\varpi r/2M\cos\psi) + \boldsymbol{k}\otimes\boldsymbol{k}\cos\psi$. Defining an angle $\breve{\chi} \to \sin\breve{\chi} = 2M(\cos\psi/\varpi)_{,r}\cos\psi$, $\breve{w}_{,r} = \cos\breve{\chi}/\cos\psi$ we get $\breve{\boldsymbol{Q}}_{\breve{\mathrm{n}}}\breve{\boldsymbol{N}} = \breve{\boldsymbol{F}}\breve{\boldsymbol{V}}^{(-1)} = \boldsymbol{h}\otimes\boldsymbol{h} + \breve{\boldsymbol{k}}'\otimes\boldsymbol{k}$ and with $\breve{\boldsymbol{k}}' = \boldsymbol{k}\sin\breve{\chi} + \breve{\boldsymbol{n}}\cos\breve{\chi}$, $\breve{\boldsymbol{n}}' = -\boldsymbol{k}\cos\breve{\chi} + \breve{\boldsymbol{n}}\sin\breve{\chi}$ another inclination by $\pi/2 - \breve{\chi}$. The 2D-curvatures are therefore

$$\breve{\boldsymbol{B}}' = -\vec{\boldsymbol{Q}}_{\breve{\mathrm{n}}}\breve{\boldsymbol{V}}^{(-1)}(\nabla_{\breve{\mathrm{n}}} \otimes \breve{\boldsymbol{n}}')\breve{\boldsymbol{N}}' = \breve{\eta}(\boldsymbol{h}\otimes\boldsymbol{h})/r - \varpi(\sin\breve{\chi})_{,r}\, r\,(\breve{\boldsymbol{k}}'\otimes\breve{\boldsymbol{k}}')/2M\breve{\eta} = (1/\breve{r}_0)\boldsymbol{h}\otimes\boldsymbol{h} + (1/\breve{r}_1)\breve{\boldsymbol{k}}'\otimes\breve{\boldsymbol{k}}', \tag{16}$$

$$\breve{\boldsymbol{R}}_{2\mathrm{D}} = \breve{\boldsymbol{B}}'\breve{\boldsymbol{B}}' - \breve{\boldsymbol{B}}'(\breve{\boldsymbol{B}}'\cdot\breve{\boldsymbol{N}}') = \varpi(\sin\breve{\chi})_{,r}\,(\boldsymbol{h}\otimes\boldsymbol{h} + \breve{\boldsymbol{k}}'\otimes\breve{\boldsymbol{k}}')/2M = -(1/\breve{r}_0\breve{r}_1)(\boldsymbol{h}\otimes\boldsymbol{h} + \breve{\boldsymbol{k}}'\otimes\breve{\boldsymbol{k}}'). \tag{17}$$

The factor $\breve{\eta} = \varpi r\cos\breve{\chi}/2M\cos\psi$ is not relevant in Eq. (17). Note also that we have $\boldsymbol{N}' = \boldsymbol{N}'^T$ at the moment.

The field equations and the theorem of energy-impulse are with $\boldsymbol{N}'^T \cdot \boldsymbol{N}'^T = 4$ (see also Section 4.3, Eqs (A19)–(A25)

$$\boldsymbol{R} - (\boldsymbol{R}\cdot\boldsymbol{N}'^T)\boldsymbol{N}'^T/2 = -\kappa\boldsymbol{T}, \tag{18}$$

$$\boldsymbol{R} = -\kappa[\boldsymbol{T} - (\boldsymbol{T}\cdot\boldsymbol{N}'^T)\boldsymbol{N}'^T/2], \tag{19}$$

$$(\nabla_{\mathrm{n}'}\boldsymbol{T})\boldsymbol{N}' = 0. \tag{20}$$

In this static case the principal components of the energy-impulse tensor $\boldsymbol{T}$ are $T_0^0 = \rho$, $T_1^1 = T_2^2 = T_3^3 = -p$, where $p(r)$ denotes the 'pressure'. We introduce now the curvature $\boldsymbol{B}'^1 = \boldsymbol{B}_1' = (1/r_0)\boldsymbol{h}\otimes\boldsymbol{h} + (1/r_1)\boldsymbol{k}'\otimes\boldsymbol{k}' + (1/r_2)\boldsymbol{K}_{\mathrm{n}}$ on a stripe $\mathbb{S}^4 \subset \mathbb{R}^6$ [7], where $1/r_0 = -\omega\sin\psi/2r$, $1/r_1 = -\zeta\sin\psi/2r$, and $\boldsymbol{B}'^2 = \boldsymbol{B}_2' = (1/\hat{r}_0)\boldsymbol{h}\otimes\boldsymbol{h} + (1/\hat{r}_1)\boldsymbol{k}'\otimes\boldsymbol{k}'$ where $\cos\beta/\hat{r}_0 = \sin\beta/r_0$, $\cos\beta/\hat{r}_1 = \sin\beta/r_1$ must hold. We define also $\omega \to \kappa p = (\sin^2\psi/r^2)(\omega-1)$. The Ricci tensor $\boldsymbol{R}_{4\mathrm{D}} = \boldsymbol{B}'^i\boldsymbol{B}_i' - \boldsymbol{B}'^i(\boldsymbol{B}_i'\cdot\boldsymbol{N}')$ (i sum 1 to 2) becomes then with $\tan^2\beta = [(2+\zeta)/\omega - 1]2/\zeta - 1$ and Eq. (19)

$$\begin{aligned}\boldsymbol{R}_{4\mathrm{D}} &= (\kappa/2)[(3p+\rho)\boldsymbol{h}\otimes\boldsymbol{h} + (p-\rho)(\boldsymbol{k}'\otimes\boldsymbol{k}' + \boldsymbol{K}_{\mathrm{n}})] \\ &= (\sin^2\psi/2r^2)[(3\omega - \zeta - 2)\boldsymbol{h}\otimes\boldsymbol{h} + (\omega + \zeta - 2)(\boldsymbol{k}'\otimes\boldsymbol{k}' + \boldsymbol{K}_{\mathrm{n}})] \\ &= -(1/\hat{r}_0\hat{r}_1 + 1/r_0r_1 + 2/r_0r_2)\boldsymbol{h}\otimes\boldsymbol{h} - (1/\hat{r}_0\hat{r}_1 + 1/r_0r_1 + 2/r_1r_2)\boldsymbol{k}'\otimes\boldsymbol{k}' - (1/r_0r_2 + 1/r_1r_2 + 1/r_2r_2)\boldsymbol{K}_{\mathrm{n}}.\end{aligned} \tag{21}$$

However, as $2/r_0r_2 = a^{00}\Gamma_{00}^1(\Gamma_{12}^2 + \Gamma_{13}^3) = -a^{00}a^{11}a_{00,1}/r = -2\cos\psi(\cos\psi/\varpi)_{,r}/r$, we have a connection between ω and ϖ (Eq. (22)), another connection between the curvatures (Eq. (23)) and, for a given $\rho(r)$, a linear differential equation for $1/\varpi$ (Eq. (24)):

$$\omega = \varpi r\sin\breve{\chi}/M\sin^2\psi, \tag{22}$$

$$1/\hat{r}_0\hat{r}_1 + 1/r_0r_1 = 1/\breve{r}_0\breve{r}_1, \tag{23}$$

$$[(\cos\psi/\varpi)_{,r}\cos\psi/r]_{,r} = (\sin^2\psi/2r^2)_{,r}/\varpi. \tag{24}$$

Two special cases are: (a) $r > a$, $\rho = 0$, $\zeta = 1$, $\beta = \pi/3$, $\omega = \varpi = 1$, and (b) $r \le a$, $\rho = \rho_0$(const.), $\zeta = -2$, $\beta = 0$, $\omega = \varpi = 2\cos\psi/(3\cos\psi_a - \cos\psi)$, $p = \rho_0(\cos\psi - \cos\psi_a)/(3\cos\psi_a - \cos\psi)$ (TOV equation [8]). In a *nonspherical field* we have $U_{,ss} + 2U_{,s}/\bar{r} = \kappa p/2$ with the potential U, the mean curvature $1/\bar{r} = 1/2\bar{r}_2 + 1/2\bar{r}_3$, and $\boldsymbol{k}$ along the arc s. We define $\psi \to \sin^2\psi = 2(\tilde{M}\mu U_{,s})^{1/2}$, with $(\sin\psi)_{,s} = -\bar{\rho}\zeta\sin\psi/2\bar{r}$, $\bar{\rho}\zeta = 1 - \bar{r}(\tilde{M}\mu)_{,s}/2\tilde{M}\mu - \kappa p/4U_{,s}$. The vector $\boldsymbol{k}$ differs from $\boldsymbol{k}^*$ by an angle α, which must be determined by the conditions of vanishing mixed terms. The inclination ψ appears then between $\boldsymbol{k}^*$ and $\boldsymbol{k}'$. On $\mathbb{S}^4 \subset \mathbb{R}^8$ we use the curvature $1/\hat{r}_0 = \tan\beta/r_0$, $1/\hat{r}_1 = \tan\beta/r_1$, $1/r_0 = -\rho^*\omega\sin\psi/2\bar{r}$, $1/r_1 = -\rho^*\zeta\sin\psi/2\bar{r}$, the tensors $\boldsymbol{B}'^1 = \boldsymbol{B}'_1 = [(1/r_0)\boldsymbol{h}\otimes\boldsymbol{h} + (1/r_1)\boldsymbol{k}'\otimes\boldsymbol{k}' + \boldsymbol{B}^*_{\mathrm{k}}\sin\psi]/\sqrt{2}$, $\boldsymbol{B}'^3 = [(1/r_0)\boldsymbol{h}\otimes\boldsymbol{h} + (1/r_1)\boldsymbol{k}'\otimes\boldsymbol{k}' - \boldsymbol{E}^*\boldsymbol{B}^*_{\mathrm{k}}\boldsymbol{E}^*\sin\psi]/\sqrt{2}$, and $\boldsymbol{B}'^2 = [(1/\hat{r}_0)\boldsymbol{h}\otimes\boldsymbol{h} + (1/\hat{r}_1)\boldsymbol{k}'\otimes\boldsymbol{k}']/\sqrt{2} = \boldsymbol{B}'^4$ and we write

$$\boldsymbol{B}'^1\boldsymbol{B}'_1 - \boldsymbol{B}'^1(\boldsymbol{B}'_1\cdot\boldsymbol{N}') = -(1/2r_0r_1 + 1/r_0\tilde{r})\boldsymbol{h}\otimes\boldsymbol{h} - (\ldots + 1/r_1\tilde{r})\boldsymbol{k}'\otimes\boldsymbol{k}'$$
$$+ \sin^2\psi[\rho^*(\omega+\zeta)\boldsymbol{B}^*_{\mathrm{k}}/4\bar{r} - \boldsymbol{K}^*_{\mathrm{n}}/2r^*_2r^*_3], \tag{25}$$

$$\boldsymbol{B}'^3\boldsymbol{B}'_3 - \boldsymbol{B}'^3(\boldsymbol{B}'_3\cdot\boldsymbol{N}') = -(\ldots + 1/r_0\tilde{r})\boldsymbol{h}\otimes\boldsymbol{h} - (1/2r_0r_1 + 1/r_1\tilde{r})\boldsymbol{k}'\otimes\boldsymbol{k}'$$
$$-\sin^2\psi[\rho^*(\omega+\zeta)\boldsymbol{E}^*\boldsymbol{B}^*_{\mathrm{k}}\boldsymbol{E}^*/4\bar{r} + \boldsymbol{K}^*_{\mathrm{n}}/2r^*_2r^*_3], \tag{26}$$

$$\boldsymbol{B}'^2\boldsymbol{B}'_2 - \boldsymbol{B}'^2(\boldsymbol{B}'_2\cdot\boldsymbol{N}') = -(1/2\hat{r}_0\hat{r}_1)(\boldsymbol{h}\otimes\boldsymbol{h} + \boldsymbol{k}'\otimes\boldsymbol{k}') = \boldsymbol{B}'^4\boldsymbol{B}'_4 - \boldsymbol{B}'^4(\boldsymbol{B}'_4\cdot\boldsymbol{N}'), \tag{27, 28}$$

where $\boldsymbol{E}^*$ is the 2D-permutation tensor normal to $\boldsymbol{k}^*$. With the involution we have here the Gauss curvature $1/r^*_2r^*_3 = -(\boldsymbol{E}^*\boldsymbol{B}^*_{\mathrm{k}}\boldsymbol{E}^*)\cdot\boldsymbol{B}^*_{\mathrm{k}}/2 = K^*$, the mean curvature $1/r^* = (\boldsymbol{B}^*_{\mathrm{k}} - \boldsymbol{E}^*\boldsymbol{B}^*_{\mathrm{k}}\boldsymbol{E}^*)\cdot\boldsymbol{K}^*_{\mathrm{n}}/2$ and $1/\tilde{r} = \sin\psi/r^*$. We obtain then with $1/r_2 = \sin\psi/r^*_2$, $1/r_3 = \sin\psi/r^*_3$, $\tan^2\beta = [(2+\zeta)/\omega - 1]2\bar{r}/r^*\rho^*\zeta - 1$, $\rho^* = \bar{r}r^*K^*$ the Ricci tensor

$$\begin{aligned}\boldsymbol{R}_{4\mathrm{D}} &= (\kappa/2)[(3p+\rho)\boldsymbol{h}\otimes\boldsymbol{h} + (p-\rho)(\boldsymbol{k}'\otimes\boldsymbol{k}' + \boldsymbol{K}^*_{\mathrm{n}})]\\ &= (K^*\sin^2\psi/2)[(3\omega-\zeta-2)\boldsymbol{h}\otimes\boldsymbol{h} + (\omega+\zeta-2)(\boldsymbol{k}'\otimes\boldsymbol{k}' + \boldsymbol{K}^*_{\mathrm{n}})]\\ &= -(1/\hat{r}_0\hat{r}_1 + 1/r_0r_1 + 2/r_0\tilde{r})\boldsymbol{h}\otimes\boldsymbol{h} - (1/\hat{r}_0\hat{r}_1 + 1/r_0r_1 + 2/r_1\tilde{r})\boldsymbol{k}'\otimes\boldsymbol{k}' - (1/r_0\tilde{r} + 1/r_1\tilde{r} + 1/r_2r_3)\boldsymbol{K}^*_{\mathrm{n}}.\end{aligned} \tag{29}$$

For the *general gravitational lens* with $\zeta = 1$, $\omega = \varpi = 1$, we use the equation of a geodesic curve $\boldsymbol{N}'\mathrm{d}^2\boldsymbol{r}'/\mathrm{d}s'^2 = 0$. A type of Eqs (5) and (6) gives then the backwards deformation into the flat space. We have similar to Eqs (25)–(28) four parts

$$\boldsymbol{N}_1\mathrm{d}^2\boldsymbol{r} = \boldsymbol{k}^*\sin^2\psi[\rho^*(\mathrm{d}\hat{\boldsymbol{r}}\cdot\boldsymbol{h})^2 + (\rho^*/\cos^2\psi)\mathrm{d}\hat{r}^2 - 2\bar{r}\mathrm{d}\boldsymbol{r}\cdot\boldsymbol{B}^*_{\mathrm{k}}\mathrm{d}\boldsymbol{r}]/4\bar{r}, \tag{30}$$

$$\boldsymbol{N}_3\mathrm{d}^2\boldsymbol{r} = \boldsymbol{k}^*\sin^2\psi[\ldots + 2\bar{r}\mathrm{d}\boldsymbol{r}\cdot\boldsymbol{E}^*\boldsymbol{B}^*_{\mathrm{k}}\boldsymbol{E}^*\mathrm{d}\boldsymbol{r}]/4\bar{r}, \tag{31}$$

$$\boldsymbol{N}_2\mathrm{d}^2\boldsymbol{r} = \boldsymbol{k}^*\sin^2\psi\tan\beta[\rho^*(\mathrm{d}\hat{\boldsymbol{r}}\cdot\boldsymbol{h})^2 + (\rho^*/\cos^2\psi)\mathrm{d}\hat{r}^2]/4\bar{r} = \boldsymbol{N}_4\mathrm{d}^2\boldsymbol{r}. \tag{32, 33}$$

The image equation is $\boldsymbol{N}\mathrm{d}^2\boldsymbol{r}_{4\mathrm{D}}(\sigma') = (\boldsymbol{N}_1\mathrm{d}^2\boldsymbol{r} + \boldsymbol{N}_3\mathrm{d}^2\boldsymbol{r})\cos\beta + (\boldsymbol{N}_2\mathrm{d}^2\boldsymbol{r} + \boldsymbol{N}_4\mathrm{d}^2\boldsymbol{r})\sin\beta$. A 4D-nullgeodesic (light ray) requires $[\mathrm{d}\hat{r} = \mathrm{d}r/\sqrt{2},\ \mathrm{d}\hat{\boldsymbol{r}}\cdot\boldsymbol{h} = \mathrm{i}c\mathrm{d}t\cos\psi/\sqrt{2}]$ $\mathrm{d}\sigma'^2 = -k\mathrm{d}\vartheta^2$, $k \to 0$ so that $\boldsymbol{N}\mathrm{d}^2\boldsymbol{r}_{4\mathrm{D}}(\vartheta) = \boldsymbol{k}^*\sqrt{K^*}\sin^2\psi(-3\mathrm{d}\boldsymbol{r}\cdot\boldsymbol{K}^*_{\mathrm{n}}\mathrm{d}\boldsymbol{r}/2)$. The surrounding field of a *rotating star* for instance is nonspherical. In the rotating system there, we may write for the scalar of the inertial force $V = -(\Omega^2/2c^2)\boldsymbol{r}\cdot\boldsymbol{K}_0\boldsymbol{r}$, where $\boldsymbol{K}_0 = \boldsymbol{N} - \boldsymbol{k}_0\otimes\boldsymbol{k}_0$ and Ω denotes the angular velocity. The gravitational potential reads $U = -G\bar{M}/rc^2 = -M/r$ and the gradient of the sum is $\nabla_{\mathrm{n}}(U+V) = M(\boldsymbol{r} - \chi\boldsymbol{K}_0\boldsymbol{r})/r^3$, where $\chi = \Omega^2r^3/Mc^2$. Without writing here all the details this gives finally for the equatorial plane

$$(\ln\mu)_{,r} = (4-\chi)\chi/(2-\chi)(1-\chi)r, \tag{34}$$

$$\mu = (1-\chi/2)^{2/3}(1-\chi)^{-1}, \tag{35}$$

$$\cos^2\psi = 1-(2M/r)(1-\chi/2)^{1/3}. \tag{36}$$

For small Ω we have $\cos^2\psi \simeq 1-2M/r+\Omega^2r^2/3c^2$. A Lorentz transformation leads with $1-\Omega^2r^2/c^2 = 1/\upsilon$ to Eq. (37). In comparison Eq. (38) shows the Kerr solution [9] or [10], Eq. (10.58), where $\Delta/r^2 = 1-2M/r+a^2/r^2$,

$$\mathrm{d}\sigma'^2 = -(c\mathrm{d}\bar{t}-\Omega r^2\mathrm{d}\bar{\varphi}/c)^2\,\upsilon\cos^2\psi + (r\mathrm{d}\bar{\varphi}-\Omega r\mathrm{d}\bar{t})^2\upsilon + \mathrm{d}r^2/\cos^2\psi + r^2\mathrm{d}\theta^2, \tag{37}$$

$$\mathrm{d}\bar{\sigma}'^2 = -(c\mathrm{d}\bar{t}-a\mathrm{d}\bar{\varphi})^2\,\Delta/r^2 + (r^2\mathrm{d}\bar{\varphi}+a^2\mathrm{d}\bar{\varphi}-ac\mathrm{d}\bar{t})^2/r^2 + \mathrm{d}r^2\,(r^2/\Delta) + r^2\mathrm{d}\theta^2. \tag{38}$$

3. THE PROBLEM OF LINEARIZATION. CONFIRMATION OF THE GEOMETRICAL NON-EXISTENCE OF GRAVITATIONAL WAVES

The usual linearization $a_{\alpha\beta} \approx \eta_{\alpha\beta} + 2\psi_{\alpha\beta}$ of the metric components concerns the small quantities $\psi_{\alpha\beta}$. Here $\eta_{\alpha\beta}$ refer to a constant tensor in the Minkowski space $\mathbb{M}^4$. The Christoffel symbols $\Gamma^{\lambda}_{\alpha\beta} \approx \eta^{\lambda\mu}(\psi_{\alpha\mu,\beta}+\psi_{\beta\mu,\alpha}-\psi_{\alpha\beta,\mu})$ would apparently give, together with a certain condition $2\psi^{\lambda}_{\alpha\lambda,} = \psi^{\lambda}_{\lambda,\alpha}$, a *wave equation* (?) for the Ricci tensor in vacuum:

$$R_{\alpha\beta} \approx \Gamma^{\lambda}_{\alpha\lambda,\beta} - \Gamma^{\lambda}_{\alpha\beta,\lambda} = \psi^{\lambda}_{\ \lambda,\alpha\beta} - \psi^{\lambda}_{\ \alpha\lambda,\beta} - \psi^{\lambda}_{\ \beta,\alpha\lambda} + \psi^{\lambda}_{\ \alpha\beta,\lambda} = \psi^{\lambda}_{\ \alpha\beta,\lambda} = \Box\psi_{\alpha\beta} = 0. \tag{39}$$

Let us now look at the dynamic case from another standpoint. We mention that the general Lorentz transformation requires second order spinors $\boldsymbol{n}^i$, first in the complementary space $\widehat{\mathbb{M}}^4$ instead of the previous unit vectors $\boldsymbol{n}^i$. This implies a triadic connection $\widehat{\boldsymbol{S}}$ to real unit 'vectors' $\boldsymbol{m}^i$. We refer to the Pauli matrices in components [10], p. 145, [11]:

$$\boldsymbol{n}^i = \boldsymbol{m}^i\widehat{\boldsymbol{S}}, \tag{40}$$

$$\widehat{\boldsymbol{S}} \mathrel{\hat{=}} [\boldsymbol{S}(\widehat{\mathrm{M}}^4)] = \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix}\ \frac{1}{\sqrt{2}}\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}\ \frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\ \frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]. \tag{41}$$

The covariant spinors read $\boldsymbol{n}_i = \widehat{\boldsymbol{S}}\boldsymbol{m}_i$; thus we find $\boldsymbol{n}^i\cdot\boldsymbol{n}_i = \boldsymbol{m}^i\cdot\widehat{\boldsymbol{S}}\widehat{\boldsymbol{S}}\boldsymbol{m}_i = 1$ (i not summed) and $\boldsymbol{n}^i\cdot\boldsymbol{n}_j = \delta^i_j$. Further, if we use once more an *involution* $-\widehat{\boldsymbol{E}}(\ldots)^T\widehat{\boldsymbol{E}}$ with the symplectic matrix $\widehat{\boldsymbol{E}}$, we get the conjugates and i.e. a product:

$$\bar{\boldsymbol{n}}^k = -\widehat{\boldsymbol{E}}\widehat{\boldsymbol{S}}^T\widehat{\boldsymbol{E}}\boldsymbol{m}^k, \tag{42}$$

$$-\widehat{\boldsymbol{E}}\widehat{\boldsymbol{S}}^T\widehat{\boldsymbol{E}} \mathrel{\hat{=}} \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix}\ \frac{1}{\sqrt{2}}\begin{pmatrix}0&\mathrm{i}\\-\mathrm{i}&0\end{pmatrix}\ \frac{1}{\sqrt{2}}\begin{pmatrix}-1&0\\0&1\end{pmatrix}\ \frac{1}{\sqrt{2}}\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\right], \tag{43}$$

$$\widehat{\boldsymbol{E}} \mathrel{\hat{=}} \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \tag{44}$$

$$-\widehat{\boldsymbol{E}}\widehat{\boldsymbol{S}}^T\widehat{\boldsymbol{E}}\widehat{\boldsymbol{S}} \mathrel{\hat{=}} \left[\frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\ \frac{1}{2}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\ \frac{1}{2}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\ \frac{1}{2}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right]. \tag{45}$$

Applying $\bar{\boldsymbol{n}}_k = -\boldsymbol{m}_k\widehat{\boldsymbol{E}}\widehat{\boldsymbol{S}}^T\widehat{\boldsymbol{E}}$ we obtain therefore $\mathrm{d}\hat{\sigma}^2 = -\mathrm{d}\hat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{E}}\widehat{\boldsymbol{S}}^T\widehat{\boldsymbol{E}}\widehat{\boldsymbol{S}}\mathrm{d}\hat{\boldsymbol{r}} = +(\mathrm{d}\hat{x}_1)^2-(\mathrm{d}\hat{x}_2)^2-(\mathrm{d}\hat{x}_3)^2-(\mathrm{d}\hat{x}_4)^2$. But the fundamental form Eq. (13) has incidentally *reversed* signs, implying $[\boldsymbol{S}(\widehat{\mathbb{M}}^4)] = -\mathrm{i}[\boldsymbol{S}(\mathbb{M}^4)]$ or shortly $\widehat{\boldsymbol{S}} = -\mathrm{i}\boldsymbol{S}$, $\boldsymbol{S} = \mathrm{i}\widehat{\boldsymbol{S}}$. With a small parameter ψ of inclination we write tentatively $\boldsymbol{S}' = e^{-\mathrm{i}\psi}\boldsymbol{S} \approx (1-\mathrm{i}\psi-\psi^2/2)\boldsymbol{S} = (1-\psi^2/2)\boldsymbol{S}+\psi\widehat{\boldsymbol{S}}$, in $2\times4=8$ matrices: $\boldsymbol{S}' \Rightarrow [\boldsymbol{S}(1-\psi^2/2), \psi\widehat{\boldsymbol{S}}]$ and

$\widehat{\boldsymbol{S}}' \Rightarrow [\widehat{\boldsymbol{S}}(1-\psi^2/2), -\psi\boldsymbol{S}]$. Each of the two involutions from $\boldsymbol{E} \Rightarrow \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \text{ or } \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\right]$ alters signs: in three matrices from $\mathbb{M}^4$ but in one only of the last two from $\widehat{\mathbb{M}}^4$. Accordingly, $-\boldsymbol{E}'\boldsymbol{S}'^T\boldsymbol{E}'\boldsymbol{S}' \Rightarrow \left[-(1-\psi^2)\boldsymbol{E}\boldsymbol{S}^T\boldsymbol{E}\boldsymbol{S}, \ldots + \widehat{\boldsymbol{S}}\widehat{\boldsymbol{S}} - \left[0, 0, \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\right]$ is in fact related to Eq. (13); for $\mathrm{d}\boldsymbol{r}' \approx (1+\psi^2/2)\mathrm{d}\boldsymbol{r} + \psi\mathrm{d}\widehat{\boldsymbol{r}}'$, $\mathrm{d}\boldsymbol{r} \mathrel{\widehat{=}} (c\mathrm{d}t, \mathrm{d}r, r\mathrm{d}\theta, r\sin\theta\mathrm{d}\varphi)$, $\mathrm{d}\widehat{\boldsymbol{r}} \mathrel{\widehat{=}} (c\mathrm{d}t, \mathrm{d}r, 0, 0)$, we find $\{\varpi = 1, \cos^2\psi \approx 1-\psi^2\}$

$$\mathrm{d}\sigma'^2 = -\mathrm{d}\boldsymbol{r}' \cdot \boldsymbol{E}'\boldsymbol{S}'^T\boldsymbol{E}'\boldsymbol{S}'\mathrm{d}\boldsymbol{r}' \approx -\mathrm{d}\boldsymbol{r} \cdot \boldsymbol{E}\boldsymbol{S}^T\boldsymbol{E}\boldsymbol{S}\mathrm{d}\boldsymbol{r} + \psi^2\mathrm{d}\widehat{\boldsymbol{r}} \cdot \widehat{\boldsymbol{S}}\widehat{\boldsymbol{S}}\mathrm{d}\widehat{\boldsymbol{r}}$$

$$= -(1-\psi^2)c^2\mathrm{d}t^2 + (1+\psi^2)\mathrm{d}r^2 + \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{K}_\mathrm{n}\mathrm{d}\boldsymbol{r}. \tag{13a}$$

After this preparation, again in general, $\boldsymbol{N} = \tilde{\boldsymbol{I}} - \boldsymbol{n}^i \otimes \boldsymbol{n}_i = \tilde{\boldsymbol{I}} - \bar{\boldsymbol{n}}_k \otimes \bar{\boldsymbol{n}}^k$ (i, k summed from 1 to 4) denotes here a *modified projector* for which the transpose is equal to the conjugate: $\boldsymbol{N}^T = \bar{\boldsymbol{N}}$. We have also $\tilde{\boldsymbol{I}}\boldsymbol{n}^i = \boldsymbol{n}^i$, $\boldsymbol{n}_i\tilde{\boldsymbol{I}} = \boldsymbol{n}_i, \ldots$ where $\tilde{\boldsymbol{I}}$ is the 'identity' in a complex space $\widehat{\mathbb{M}}^4 \oplus \mathbb{M}^4 = \tilde{\mathbb{C}}^4$ (see Section 4.5 and Eqs (A44)). We write then the development $\boldsymbol{n}'^i \approx \boldsymbol{n}^i - \psi^i - (\psi^i \cdot \psi_j)\boldsymbol{n}^j/2$, $\boldsymbol{n}'_i \approx \boldsymbol{n}_i - \psi_i - (\psi_i \cdot \psi^j)\boldsymbol{n}_j/2$ with small 'inclination vectors' or spinors in the flat space $\mathbb{M}^4$: $\psi^i = \boldsymbol{N}\psi^i = \phi^i\boldsymbol{S}$, $\psi_i = \psi_i\boldsymbol{N} = \boldsymbol{S}\phi_i$. Therefore the development of the projector on the curved space becomes $\boldsymbol{N}' \approx \boldsymbol{N} + \psi^i \otimes \boldsymbol{n}_i + \boldsymbol{n}^i \otimes \psi_i - \psi^i \otimes \psi_i + (\psi^i \cdot \psi_j)(\boldsymbol{n}^j \otimes \boldsymbol{n}_i)$. The derivative of this projector reads according to Eq. (3), $\nabla_{\mathrm{n}'} \otimes \boldsymbol{N}' = \boldsymbol{B}'^i \otimes \boldsymbol{n}'_i + \boldsymbol{B}'_i \otimes \boldsymbol{n}'^i]^T$. This leads to two exterior curvature tensors, here we have $\boldsymbol{B}'^i \neq \boldsymbol{B}'_i \neq \boldsymbol{B}'^T_i$, see also in particular Section 4.4, Eqs (A38) and (A39)

$$\boldsymbol{B}'^i = (\nabla_{\mathrm{n}'} \otimes \boldsymbol{N}')\boldsymbol{n}'^i = -(\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'^i)\boldsymbol{N}'^T \approx (\nabla_\mathrm{n} \otimes \psi^i)\boldsymbol{N}^T, \tag{46}$$

$$\boldsymbol{B}'_i = -(\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'_i)\boldsymbol{N}' \approx (\nabla_\mathrm{n} \otimes \psi_i)\boldsymbol{N}. \tag{47}$$

We can also use a base $\boldsymbol{e}^\alpha$, $\bar{\boldsymbol{e}}^\lambda$ in the flat space instead of a base $\boldsymbol{a}^\alpha \approx \boldsymbol{e}^\alpha + \psi_i^\alpha\boldsymbol{n}'^i$, $\bar{\boldsymbol{a}}^\lambda \approx \bar{\boldsymbol{e}}^\lambda + \bar{\psi}_k^\lambda\bar{\boldsymbol{n}}'^k$ on the curved space. The derivative of the spinor $\psi^i = \bar{\psi}^i_\beta\bar{\boldsymbol{e}}^\beta$ has the interior part $(\nabla_\mathrm{n} \otimes \psi^i)\boldsymbol{N}^T = \bar{\psi}^i_{\beta;\mu}(\boldsymbol{e}^\mu \otimes \bar{\boldsymbol{e}}^\beta)$ with the (covariant) derivative $\bar{\psi}^i_{\beta;\mu} = \bar{\psi}^i_{\beta,\mu} - \Gamma^\lambda_{\mu\beta}\bar{\psi}^i_\lambda = \bar{\psi}^i_{\beta,\mu}$. The Ricci tensor $\boldsymbol{R} = \boldsymbol{B}'^T_i\boldsymbol{B}'^i - (\boldsymbol{B}'_i \cdot \boldsymbol{N}')\boldsymbol{B}'^{iT}$ is in vacuum (see also sections 4.1, Eq. (A6), 4.2, Eq. (A16), and 4.6, Eqs (A58)–(A60))

$$\boldsymbol{R} \approx [(\nabla_\mathrm{n} \otimes \psi_i)\boldsymbol{N}]^T(\nabla_\mathrm{n} \otimes \psi^i)\boldsymbol{N}^T - [(\nabla_\mathrm{n} \otimes \psi_i) \cdot \boldsymbol{N}][(\nabla_\mathrm{n} \otimes \psi^i)\boldsymbol{N}^T]^T$$

$$\approx (\bar{\psi}^i_{\lambda,\alpha}\psi^\lambda_{i\beta,} - \bar{\psi}^i_{\alpha,\beta}\psi^\lambda_{i\lambda,})(\boldsymbol{e}^a \otimes \bar{\boldsymbol{e}}^\beta) = 0. \tag{48}$$

This is **not** a wave equation. Therefore gravitational waves, based on the incorrect Eqs (39), cannot exist (see also [12]). The contradiction between Eqs (48) and (39) originates from the semi-exterior part $\psi^i \otimes \boldsymbol{n}_i + \boldsymbol{n}^i \otimes \psi_i$, which is of first order small whereas $\psi^i \otimes \psi_i$ and $2\psi_{\alpha\beta}$ are of second order. We get in fact $2\psi^\lambda_{\alpha\beta,\lambda} = (\psi^i_a\bar{\psi}_{i\beta})^\lambda_{,\lambda} \neq 2(\bar{\psi}^i_{\lambda,\alpha}\psi^\lambda_{i\beta,} - \bar{\psi}^i_{\alpha,\beta}\psi^\lambda_{i\lambda,})$. We could also develop the complementary projector $\widehat{\boldsymbol{N}}' = \tilde{\boldsymbol{I}} - \boldsymbol{N}' \approx (\widehat{\boldsymbol{N}} - \boldsymbol{\Psi})(\widehat{\boldsymbol{N}} - \boldsymbol{\Psi}^*) = \widehat{\boldsymbol{N}} - \boldsymbol{\Psi}\widehat{\boldsymbol{N}} - \widehat{\boldsymbol{N}}\boldsymbol{\Psi}^* + \boldsymbol{\Psi}\boldsymbol{\Psi}^*$ where $\boldsymbol{\Psi} = \psi^i \otimes \boldsymbol{n}_i + \ldots$, $\boldsymbol{\Psi}^* = \boldsymbol{n}^i \otimes \psi_i + \ldots$ have small semi-exterior parts. The Ricci tensor $\boldsymbol{R} = \boldsymbol{B}'^{\cdot} \cdot \boldsymbol{B}'^T \cdot - \boldsymbol{B}'^{\cdot}\boldsymbol{B}'^T \cdot \boldsymbol{N}'$ is then obtained from the 'one-third-exterior triadic' curvatures $\boldsymbol{B}' = \boldsymbol{B}'^i \otimes \boldsymbol{n}'_i = (\nabla_{\mathrm{n}'} \otimes \boldsymbol{N}')\widehat{\boldsymbol{N}}' \mid \boldsymbol{N}' \approx (\nabla_\mathrm{n} \otimes \boldsymbol{\Psi})\widehat{\boldsymbol{N}} \mid \boldsymbol{N}, \ldots$.

4. APPENDIX, DETAILED CALCULATIONS

4.1. Riemann–Christoffel tensor

Some remarks concern Eq. (12) in the general dynamic case where spinors $\boldsymbol{n}'^i \neq \boldsymbol{n}'_i$ and a modified projector $\boldsymbol{N}' = \tilde{\boldsymbol{I}} - \boldsymbol{n}'^i \otimes \boldsymbol{n}'_i = \tilde{\boldsymbol{I}} - \bar{\boldsymbol{n}}'_k \otimes \bar{\boldsymbol{n}}'^k = a_{\alpha\beta}\boldsymbol{a}^\alpha \otimes \bar{\boldsymbol{a}}^\beta$ are used. The first derivative $(\nabla_{\mathrm{n}'} = \boldsymbol{N}'\nabla)$ of $\boldsymbol{N}'$

reads (pay attention to the non-commutative products and the transposition of the factors 2 and 3 by the sign $]^T$)

$$\begin{aligned}\nabla_{\mathrm{n}'} \otimes \boldsymbol{N}' &= -(\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'^i) \otimes \boldsymbol{n}'_i - (\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'_i) \otimes \boldsymbol{n}'^i\,]^T \\ &= -(\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'^i)\boldsymbol{N}'^T \otimes \boldsymbol{n}'_i + (\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'^i)\boldsymbol{n}'_j \otimes \boldsymbol{n}'^j \otimes \boldsymbol{n}'_i - (\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'_i)\boldsymbol{N}' \otimes \boldsymbol{n}'^i\,]^T \\ &\quad + (\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'_i)\boldsymbol{n}'^j \otimes \boldsymbol{n}'_j \otimes \boldsymbol{n}'^i\,]^T .\end{aligned}$$

As $(\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'_i)\boldsymbol{n}'^j + (\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'^j)\boldsymbol{n}'_i = \nabla_{\mathrm{n}'}(\boldsymbol{n}'^j \cdot \boldsymbol{n}'_i) = 0$ we have $(\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'_i)\boldsymbol{n}'^j \otimes \boldsymbol{n}'_j \otimes \boldsymbol{n}'^i\,]^T = -(\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'^j)\boldsymbol{n}'_i \otimes \boldsymbol{n}'^i \otimes \boldsymbol{n}'_j$, so that we get in accordance with Eqs (46) and (47)

$$\nabla_{\mathrm{n}'} \otimes \boldsymbol{N}' = -(\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'^i) \otimes \boldsymbol{n}'_i - (\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'_i) \otimes \boldsymbol{n}'^i\,]^T = \boldsymbol{B}'^i \otimes \boldsymbol{n}'_i + \boldsymbol{B}'_i \otimes \boldsymbol{n}'^i\,]^T , \tag{A1}$$

$$(\nabla_{\mathrm{n}'} \otimes \boldsymbol{N}')^T = -\boldsymbol{n}'_i \otimes (\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'^i)^T - (\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'_i)^T \otimes \boldsymbol{n}'^i\,]^T = \boldsymbol{n}'_i \otimes \boldsymbol{B}'^{iT} + \boldsymbol{B}'^T_i \otimes \boldsymbol{n}'^i\,]^T . \tag{A2}$$

The second derivatives of the projector become afterwards

$$\nabla_{\mathrm{n}'} \otimes (\nabla_{\mathrm{n}'} \otimes \boldsymbol{N}') = \nabla_{\mathrm{n}'} \otimes \boldsymbol{B}'^i \otimes \boldsymbol{n}'_i - \boldsymbol{B}'_i \otimes \boldsymbol{B}'^{iT} \rrbracket^T + \nabla_{\mathrm{n}'} \otimes \boldsymbol{B}'_i \otimes \boldsymbol{n}'^i \Big]^T - \boldsymbol{B}'^i \otimes \boldsymbol{B}'^T_i \rrbracket^T \Big]^T , \tag{A3}$$

$$\nabla_{\mathrm{n}'} \otimes (\nabla_{\mathrm{n}'} \otimes \boldsymbol{N}')^T = -\boldsymbol{B}'_i \otimes \boldsymbol{B}'^{iT} + \nabla_{\mathrm{n}'} \otimes \boldsymbol{B}'^{iT} \otimes \boldsymbol{n}'_i \rrbracket^T + \nabla_{\mathrm{n}'} \otimes \boldsymbol{B}'^T_i \otimes \boldsymbol{n}'^i \Big]^T - \boldsymbol{B}'^i \otimes \boldsymbol{B}'_i \rrbracket^T \Big]^T . \tag{A4}$$

The bracket $]^T$ indicates here a transposition of the factors 3 and 4, the double bracket $\rrbracket^T$ indicates, as previously, a transposition of the factors 2 and 4. Referring to the $2 \times 4 = 8$ terms in Eqs (A3) and (A4), we see on one hand that the 1st, 3rd, 6th, and 7th quarter-exterior terms vanish all together if the full projection $\boldsymbol{N}' | \{\boldsymbol{N}'[.....]\boldsymbol{N}'\} | \boldsymbol{N}'$ is applied. One term disappears by the first $\boldsymbol{N}' | = \boldsymbol{N}'^T(\ldots)$ left (applied to the 2nd factor), two by the fourth $| \boldsymbol{N}' = (\ldots)\boldsymbol{N}'^T$ right (applied to the 3rd factor), and one by the third $\boldsymbol{N}'$ right (applied to the 4th factor). The second $\boldsymbol{N}'$ left is actually not necessary, but it serves to accentuate the *interior character* of these 4th order tensors. In the difference appearing in the Riemann–Christoffel tensor only four terms remain therefore:

$$\begin{aligned}&\boldsymbol{N}' | \boldsymbol{N}'[\nabla_{\mathrm{n}'} \otimes \nabla_{\mathrm{n}'} \otimes \boldsymbol{N}' - \nabla_{\mathrm{n}'} \otimes (\nabla_{\mathrm{n}'} \otimes \boldsymbol{N}')^T\,]\boldsymbol{N}' | \boldsymbol{N}' \\ &\qquad = \boldsymbol{B}'_i \otimes \boldsymbol{B}'^{iT} - \boldsymbol{B}'_i \otimes \boldsymbol{B}'^{iT} \rrbracket^T + \boldsymbol{B}'^i \otimes (\boldsymbol{B}'_i - \boldsymbol{B}'^T_i) \rrbracket^T \Big]^T = \mathbf{R}^T .\end{aligned} \tag{A5}$$

The Ricci tensor $\boldsymbol{R}$ is the 'outside' contraction of $\mathbf{R}$, in components $R_{\alpha\beta} = \mathrm{R}^{\lambda}_{\alpha\beta\lambda}$, the last skew-symmetric factor in Eq. (A5) vanishes after this contraction. Then $\boldsymbol{R}$ can be written in two simple conjugate expressions (the real components $R_{\alpha\beta} = R_{\beta\alpha}$ are symmetric)

$$\boldsymbol{R} = \boldsymbol{B}'^T_i \boldsymbol{B}'^i - (\boldsymbol{B}'_i \cdot \boldsymbol{N}')\boldsymbol{B}'^{iT} \tag{A6}$$

or

$$\boldsymbol{R} = \bar{\boldsymbol{B}}'^{iT} \bar{\boldsymbol{B}}'_i - (\bar{\boldsymbol{B}}'^i \cdot \bar{\boldsymbol{N}}')\bar{\boldsymbol{B}}'^T_i = \bar{\boldsymbol{R}}^T . \tag{A7}$$

4.2. Components of the exterior curvatures and the Ricci tensor

The derivatives of the spinors $\boldsymbol{n}'^i \neq \boldsymbol{n}'_i$ can be written

$$\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'^i = \boldsymbol{a}^\alpha \otimes \boldsymbol{n}'^i_{,\alpha} , \tag{A8}$$

$$\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'_i = \boldsymbol{a}^\alpha \otimes \boldsymbol{n}'_{i,\alpha} , \tag{A9}$$

$$\boldsymbol{n}'^{i}_{,\alpha} = -\bar{\Gamma}^{i}_{\alpha\beta}\bar{\boldsymbol{a}}^{\beta} - \Gamma^{i}_{\alpha j}\boldsymbol{n}'^{j}, \tag{A10}$$

$$\boldsymbol{n}'_{i\alpha} = \Gamma^{\mu}_{i\alpha}\boldsymbol{a}_{\mu} + \Gamma^{j}_{i\alpha}\boldsymbol{n}'_{j}. \tag{A11}$$

Using the auxiliary relations $\nabla_{\mathrm{n}'} = \boldsymbol{a}^{\alpha}\partial/\partial\theta^{\alpha}$, $\Gamma^{\mu}_{i\alpha} = a^{\mu\beta}\Gamma_{i\alpha\beta}$, $a^{\mu\beta} = -\boldsymbol{a}^{\mu}\cdot\bar{\boldsymbol{a}}^{\beta}$, where $\bar{\boldsymbol{a}}^{\beta}$ is the conjugate of $\boldsymbol{a}^{\beta}$, we obtain therefore

$$\boldsymbol{B}'^{i} = -(\nabla_{\mathrm{n}'}\otimes\boldsymbol{n}'^{i})\boldsymbol{N}'^{T} = \bar{\Gamma}^{i}_{\alpha\beta}\boldsymbol{a}^{\alpha}\otimes\bar{\boldsymbol{a}}^{\beta}, \tag{A12}$$

$$\boldsymbol{B}'_{i} = -(\nabla_{\mathrm{n}'}\otimes\boldsymbol{n}'_{i})\boldsymbol{N}' = -\Gamma^{\mu}_{i\alpha}\boldsymbol{a}^{\alpha}\otimes\boldsymbol{a}_{\mu} = -\Gamma_{i\alpha\beta}\boldsymbol{a}^{\alpha}\otimes\bar{\boldsymbol{a}}^{\beta}. \tag{A13}$$

Now as for the development in case of a small curvature, we get with $\psi^{i} = \bar{\psi}^{i}{}_{\lambda}\bar{\boldsymbol{e}}^{\lambda}$ and $\psi_{i} = \bar{\psi}_{i\mu}\bar{\boldsymbol{e}}^{\mu}$ successively

$$\boldsymbol{B}'^{i} \approx (\nabla_{\mathrm{n}}\otimes\psi^{i})\boldsymbol{N}^{T} = \bar{\psi}^{i}{}_{\lambda;\alpha}\boldsymbol{e}^{\alpha}\otimes\bar{\boldsymbol{e}}^{\lambda} \approx \bar{\psi}^{i}{}_{\lambda,\alpha}\boldsymbol{e}^{\alpha}\otimes\bar{\boldsymbol{e}}^{\lambda},\ \ \boldsymbol{B}'^{iT} \approx \bar{\psi}^{i}{}_{\lambda,\alpha}\bar{\boldsymbol{e}}^{\lambda}\otimes\boldsymbol{e}^{\alpha} = \psi^{i}{}_{\alpha,\beta}\boldsymbol{e}^{\alpha}\otimes\bar{\boldsymbol{e}}^{\beta}, \tag{A14}$$

$$\mathbf{B}'_{i} \approx (\nabla_{\mathrm{n}}\otimes\mathbf{n}_{i})\mathbf{N} \approx \bar{\psi}_{i\alpha,\lambda}\mathbf{e}^{\lambda}\otimes\bar{\mathbf{e}}^{\alpha}, \qquad \mathbf{B}'^{T}_{i} \approx \bar{\psi}_{i\alpha,\lambda}\bar{\mathbf{e}}^{\alpha}\otimes\mathbf{e}^{\lambda} = \psi_{i\alpha,\lambda}\mathbf{e}^{\alpha}\otimes\bar{\mathbf{e}}^{\lambda},$$

$$\boldsymbol{B}'^{T}_{i}\boldsymbol{B}'^{i} \approx \psi_{i\alpha,\lambda}e^{\lambda\mu}\bar{\psi}^{i}{}_{\beta;\mu}\boldsymbol{e}^{\alpha}\otimes\bar{\boldsymbol{e}}^{\beta} = \bar{\psi}_{i\alpha,\lambda}\psi^{i\lambda}_{\ \ \beta;}\boldsymbol{e}^{\alpha}\otimes\bar{\boldsymbol{e}}^{\beta},$$

$$\boldsymbol{B}'_{i}\cdot\boldsymbol{N} \approx \bar{\psi}_{i\lambda,\beta}(\boldsymbol{e}^{\beta}\otimes\bar{\boldsymbol{e}}^{\lambda})\cdot(\boldsymbol{e}^{\mu}\otimes\boldsymbol{e}_{\mu}) = \psi_{i\lambda,\beta}e^{\lambda\beta} = \psi^{\lambda}_{i\lambda,}, \qquad (\boldsymbol{B}'_{i}\cdot\boldsymbol{N}')\boldsymbol{B}'^{iT} \approx \psi^{\lambda}_{i\lambda,}\bar{\psi}^{i}{}_{\beta,\alpha}\boldsymbol{e}^{\alpha}\otimes\bar{\boldsymbol{e}}^{\beta}, \tag{A15}$$

so that the Ricci tensor of Eq. (48) becomes in fact

$$\boldsymbol{R} = \boldsymbol{B}'^{T}_{i}\boldsymbol{B}'^{i} - (\boldsymbol{B}'_{i}\cdot\boldsymbol{N}')\boldsymbol{B}'^{iT} \approx (\bar{\psi}^{i}{}_{\lambda,\alpha}\psi^{\lambda}_{i\beta,} - \bar{\psi}^{i}{}_{\alpha,\beta}\psi^{\lambda}_{i\lambda,})(\boldsymbol{e}^{a}\otimes\bar{\boldsymbol{e}}^{\beta}). \tag{A16}$$

In general we have also

$$\boldsymbol{R} = \boldsymbol{B}'^{T}_{i}\boldsymbol{B}'^{i} - (\boldsymbol{B}'_{i}\cdot\boldsymbol{N}')\boldsymbol{B}'^{iT} = -(\bar{\Gamma}^{i}_{\alpha\lambda}\Gamma^{\lambda}_{i\beta} - \bar{\Gamma}^{i}_{\alpha\beta}\Gamma^{\lambda}_{i\lambda})(\boldsymbol{a}^{a}\otimes\bar{\boldsymbol{a}}^{\beta}). \tag{A17}$$

The components of Eq. (A17) correspond to the usual representation $R_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\lambda,\beta} - \Gamma^{\lambda}_{\alpha\beta,\lambda} + \Gamma^{\mu}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\beta} - \Gamma^{\mu}_{\alpha\beta}\Gamma^{\lambda}_{\mu\lambda}$ because

$$(\Gamma^{\lambda}_{\alpha\lambda,\beta} - \Gamma^{\lambda}_{\alpha\beta,\lambda} + \Gamma^{\mu}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\beta} - \Gamma^{\mu}_{\alpha\beta}\Gamma^{\lambda}_{\mu\lambda}) + (\bar{\Gamma}^{i}_{\alpha\lambda}\Gamma^{\lambda}_{i\beta} - \bar{\Gamma}^{i}_{\alpha\beta}\Gamma^{\lambda}_{i\lambda}) = 0 \tag{A18}$$

(α, β from 0 to 3, i from 1 to 4) expresses the vanishing Ricci tensor in the space $\widehat{\mathbb{M}}'^{4}\oplus\mathbb{M}'^{4} = \widehat{\mathbb{M}}^{4}\oplus\mathbb{M}^{4} = \tilde{\mathbb{C}}^{4}$.

4.3. Theorem of energy impulse

It is well known that Eq. (20) is compatible with Eqs (18) and (19), because of some integrability (Bianchi) equations. Here we give an alternative explanation, which refers to the present notation by exterior spinors. We use the rules $\nabla(\boldsymbol{X}\boldsymbol{Y}) = (\nabla\boldsymbol{X})\boldsymbol{Y} + (\nabla\otimes\boldsymbol{Y})^{T}\cdot\boldsymbol{X}$ and $\nabla(\boldsymbol{X}\cdot\boldsymbol{Y}) = (\nabla\otimes\boldsymbol{X})\cdot\boldsymbol{Y} + (\nabla\otimes\boldsymbol{Y})\cdot\boldsymbol{X}$, valid for any 2nd order tensors $\boldsymbol{X}$, $\boldsymbol{Y}$. First, the derivative of the trace $\boldsymbol{R}\cdot\boldsymbol{N}'^{T} = \boldsymbol{B}'^{T}_{i}\cdot\boldsymbol{B}'^{i}\cdot - (\boldsymbol{B}'_{i}\cdot\boldsymbol{N}')(\boldsymbol{B}'^{iT}\cdot\boldsymbol{N}'^{T})$ is (here we have $\boldsymbol{N}'\neq\boldsymbol{N}'^{T}$)

$$\begin{aligned}\nabla_{\mathrm{n}'}(\boldsymbol{R}\cdot\boldsymbol{N}'^{T}) &= (\nabla_{\mathrm{n}'}\otimes\boldsymbol{B}'^{T}_{i})\cdot\boldsymbol{B}'^{i} + (\nabla_{\mathrm{n}'}\otimes\boldsymbol{B}'^{i})\cdot\boldsymbol{B}'^{T}_{i} - [\nabla_{\mathrm{n}'}(\boldsymbol{B}'_{i}\cdot\boldsymbol{N}')](\boldsymbol{B}'^{iT}\cdot\boldsymbol{N}'^{T}) - [\nabla_{\mathrm{n}'}(\boldsymbol{B}'^{iT}\cdot\boldsymbol{N}'^{T})](\boldsymbol{B}'_{i}\cdot\boldsymbol{N}')\\ &= 2(\nabla_{\mathrm{n}'}\otimes\boldsymbol{B}'^{T}_{i})\cdot\boldsymbol{B}'^{i} - 2[\nabla_{\mathrm{n}'}(\boldsymbol{B}'_{i}\cdot\boldsymbol{N}')](\boldsymbol{B}'^{iT}\cdot\boldsymbol{N}'^{T}),\end{aligned} \tag{A19}$$

because $(\nabla_{\mathrm{n}'}\otimes\boldsymbol{B}'^{i})\cdot\boldsymbol{B}'^{T}_{i} = (\nabla_{\mathrm{n}'}\otimes\boldsymbol{B}'^{T}_{i})\cdot\boldsymbol{B}'^{i}$, $[\nabla_{\mathrm{n}'}(\boldsymbol{B}'^{iT}\cdot\boldsymbol{N}'^{T})](\boldsymbol{B}'_{i}\cdot\boldsymbol{N}') = [\nabla_{\mathrm{n}'}(\boldsymbol{B}'_{i}\cdot\boldsymbol{N}')](\boldsymbol{B}'^{iT}\cdot\boldsymbol{N}'^{T})$. The integrability equation can be written with the 4D-permutation tensor $\mathbf{E}$ in the form

$$\nabla_{n'}\mathbf{E}\nabla_{n'} \otimes \boldsymbol{n}'_i) = 0, \tag{A20}$$

meaning that the two operators $\nabla_{n'}$ can be permutated. Thus we have $\nabla_{n'}(\boldsymbol{B}'_i \cdot \boldsymbol{N}') = (\nabla_{n'}\boldsymbol{B}'_i)\boldsymbol{N}'$ so that we obtain

$$\nabla_{n'}(\boldsymbol{R} \cdot \boldsymbol{N}'^T)/2 = (\nabla_{n'} \otimes \boldsymbol{B}'^T_i) \cdot \boldsymbol{B}'^i - (\nabla_{n'}\boldsymbol{B}'_i)\boldsymbol{N}'(\boldsymbol{B}'^{iT} \cdot \boldsymbol{N}'^T). \tag{A21}$$

Second, the derivative of the Ricci tensor $\boldsymbol{R}$ is

$$\begin{aligned}(\nabla_{n'}\boldsymbol{R})\boldsymbol{N}' &= (\nabla_{n'}\boldsymbol{B}'^T_i)\boldsymbol{B}'^i + (\nabla_{n'} \otimes \boldsymbol{B}'^i)^T \cdot \boldsymbol{B}'^T_i - [\nabla_{n'}(\boldsymbol{B}'_i \cdot \boldsymbol{N}')]\boldsymbol{B}'^{iT} - (\nabla_{n'}\boldsymbol{B}'^{iT})\boldsymbol{N}'^T(\boldsymbol{B}'_i \cdot \boldsymbol{N}') \\ &= (\nabla_{n'}\boldsymbol{B}'^T_i)\boldsymbol{B}'^i + (\nabla_{n'} \otimes \boldsymbol{B}'^T_i)^T \cdot \boldsymbol{B}'^i - (\nabla_{n'}\boldsymbol{B}'_i)\boldsymbol{B}'^{iT} - (\nabla_{n'}\boldsymbol{B}'^{iT})\boldsymbol{N}'^T(\boldsymbol{B}'_i \cdot \boldsymbol{N}').\end{aligned} \tag{A22}$$

From the integrability Eq. (A20) we have also $(\nabla_{n'} \otimes \boldsymbol{B}'^T_i)^T = (\nabla_{n'} \otimes \boldsymbol{B}'^T_i)$ and further $(\nabla_{n'}\boldsymbol{B}'_i)\boldsymbol{B}'^{iT} = (\nabla_{n'}\boldsymbol{B}'^T_i)\boldsymbol{B}'^i$; therefore we get

$$(\nabla_{n'}\boldsymbol{R})\boldsymbol{N}' = (\nabla_{n'} \otimes \boldsymbol{B}'^T_i) \cdot \boldsymbol{B}'^i - (\nabla_{n'}\boldsymbol{B}'_i)\boldsymbol{N}'(\boldsymbol{B}'^{Ti} \cdot \boldsymbol{N}'^T). \tag{A23}$$

The elimination of two terms from Eqs (A21) and (A23) leads finally to the equations sought (see also [8], p. 53)

$$(\nabla_{n'}\boldsymbol{R})\boldsymbol{N}' = \nabla_{n'}(\boldsymbol{R} \cdot \boldsymbol{N}'^T)/2, \tag{A24}$$

$$-\kappa(\nabla_{n'}\boldsymbol{T})\boldsymbol{N}' = \{\nabla_{n'}[\boldsymbol{R} - (\boldsymbol{R} \cdot \boldsymbol{N}'^T)\boldsymbol{N}'^T/2]\}\boldsymbol{N}' = 0. \tag{A25}$$

4.4. Three illustrative steps

In order to relate the concepts of Section 2 with those of Section 3, we write:

(a) In Eq. (14) the projector $\boldsymbol{K}_n$ onto the plane normal to $\boldsymbol{k}$ appearing in the 3D-space. Here we have the derivative of a distance r to a *fixed point* (the centre) and then the derivative of the resulting unit vector $\boldsymbol{k}$:

$$\nabla_n r = \nabla_n \sqrt{\boldsymbol{r} \cdot \boldsymbol{r}} = \frac{1}{r}(\nabla_n \otimes \boldsymbol{r})\boldsymbol{r} = \boldsymbol{k}, \tag{A26}$$

$$\nabla_n \otimes \boldsymbol{k} = \frac{1}{r}\nabla_n \otimes \boldsymbol{r} - \frac{1}{r^2}\nabla_n r \otimes \boldsymbol{r} = \frac{1}{r}(\boldsymbol{N} - \boldsymbol{k} \otimes \boldsymbol{k}) = \frac{1}{r}\boldsymbol{K}_n. \tag{A27}$$

In the 4D-space we form the derivative of a distance p_1 to a *fixed straight line* (representing in some way the ‘time’) from a point with ‘position vector’ $\boldsymbol{q}_1$. We obtain thus with $\boldsymbol{p}_1 = \boldsymbol{q}_1 - \xi\boldsymbol{k}_0$, $\nabla_n\bar{\xi} = \bar{\boldsymbol{k}}^0$, $\boldsymbol{k}^0 \cdot \boldsymbol{k}_1 = 0$

$$\nabla_n p_1 = \nabla_n \sqrt{\bar{\boldsymbol{p}}_1 \cdot \boldsymbol{p}_1} = \frac{1}{p_1}(\nabla_n \otimes \bar{\boldsymbol{p}}_1)\boldsymbol{p}_1 = \frac{1}{p_1}(\nabla_n \otimes \bar{\boldsymbol{q}}_1 - \nabla_n\bar{\xi} \otimes \bar{\boldsymbol{k}}_0)\boldsymbol{p}_1 = \frac{1}{p_1}(\boldsymbol{N}^T - \boldsymbol{k}_0 \otimes \boldsymbol{k}^0)\boldsymbol{p}_1 = \boldsymbol{k}_1. \tag{A28}$$

The derivative of this (unit) spinor leads to a modified projector $\boldsymbol{K}_n$ and its transpose $\boldsymbol{K}^T_n$. But with reference to the position vector $\boldsymbol{r} \neq \boldsymbol{q}_1$, we obtain

$$(\nabla_n \otimes \boldsymbol{k}_1)_{\boldsymbol{q}_1} = \frac{1}{p_1}(\nabla_n \otimes \boldsymbol{p}_1) - \frac{1}{p_1^2}(\nabla_n p_1 \otimes \boldsymbol{p}_1) = \frac{1}{p_1}(\nabla_n \otimes \boldsymbol{q}_1 - \nabla_n\xi \otimes \boldsymbol{k}_0) - \frac{1}{p_1^2}(\nabla_n p_1 \otimes \boldsymbol{p}_1), \tag{A29}$$

$$(\nabla_n \otimes \boldsymbol{k}_1)_{\boldsymbol{r}} = \frac{1}{r}(\boldsymbol{N} - \boldsymbol{k}^0 \otimes \boldsymbol{k}_0 - \boldsymbol{k}^1 \otimes \boldsymbol{k}_1) = \frac{1}{r}\boldsymbol{K}_n, \quad (\nabla_n \otimes \boldsymbol{k}^1)_{\boldsymbol{r}} = \frac{1}{r}(\boldsymbol{N}^T - \boldsymbol{k}_0 \otimes \boldsymbol{k}^0 - \boldsymbol{k}_1 \otimes \boldsymbol{k}^1) = \frac{1}{r}\boldsymbol{K}^T_n. \tag{A30}$$

(b) The *derivative of the spinor* $\boldsymbol{n}'^1 = \boldsymbol{m}^1\widehat{\boldsymbol{S}}' = \boldsymbol{m}^1\widehat{\boldsymbol{S}}e^{-\mathrm{i}\psi}$ in the 'inclined' complementary space $\widehat{\mathbb{M}}'^4$ reads

$$\nabla_{\mathrm{n}} \otimes \boldsymbol{n}'^1 = -\mathrm{i}\nabla_{\mathrm{n}}\psi \otimes (\boldsymbol{m}^1\widehat{\boldsymbol{S}}e^{-\mathrm{i}\psi}) + (\nabla_{\mathrm{n}} \otimes \boldsymbol{m}^1\widehat{\boldsymbol{S}})\,e^{-\mathrm{i}\psi} = -\nabla_{\mathrm{n}}\psi \otimes \boldsymbol{k}'^1 - \frac{\sin\psi}{r}\boldsymbol{K}_{\mathrm{n}}^T, \tag{A31}$$

with the (unit) spinor $\boldsymbol{m}^1\boldsymbol{S} = \boldsymbol{k}^1$ in the Minkowski space $\mathbb{M}^4$ and because

$$(\nabla_{\mathrm{n}} \otimes \boldsymbol{m}^1\widehat{\boldsymbol{S}})\cos\psi = (\nabla_{\mathrm{n}} \otimes \boldsymbol{n}^1)\cos\psi = 0,$$

$$(\nabla_{\mathrm{n}} \otimes \boldsymbol{m}^1\widehat{\boldsymbol{S}})\mathrm{i}\sin\psi = (\nabla_{\mathrm{n}} \otimes \boldsymbol{m}^1\boldsymbol{S})\sin\psi = (\nabla_{\mathrm{n}} \otimes \boldsymbol{k}^1)\sin\psi = \frac{\sin\psi}{r}\boldsymbol{K}_{\mathrm{n}}^T.$$

In the space $\mathbb{M}'^4$ we have $\boldsymbol{k}'^1 = \boldsymbol{m}^1\boldsymbol{S}' = \boldsymbol{m}^1\boldsymbol{S}e^{-\mathrm{i}\psi}$, $\boldsymbol{k}_1' = \boldsymbol{S}'\boldsymbol{m}_1 = \boldsymbol{S}\boldsymbol{m}_1e^{-\mathrm{i}\psi}$, and $\boldsymbol{k}'^1 \cdot \boldsymbol{k}_1' = -e^{-2\mathrm{i}\psi}$, but $\boldsymbol{k}_1' \cdot \boldsymbol{k}'^1 = -e^{+2\mathrm{i}\psi}$. Thus we can write, now with $\boldsymbol{m}_1 \,\widehat{=}\, [\sin\gamma, \cos\gamma, 0, 0] \,\widehat{=}\, \boldsymbol{m}^1$,

$$\boldsymbol{k}_1' \otimes \boldsymbol{k}'^1 = \boldsymbol{m}_1\boldsymbol{E}\boldsymbol{S}^T\boldsymbol{E}e^{\mathrm{i}\psi} \otimes \boldsymbol{E}\boldsymbol{S}^T\boldsymbol{E}\boldsymbol{m}^1e^{\mathrm{i}\psi} \,\widehat{=}\, \begin{bmatrix} \dfrac{\sin^2\gamma}{2}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} & \dfrac{\cos^2\gamma}{2}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} & 0 & 0\end{bmatrix} e^{2\mathrm{i}\psi}, \tag{A32}$$

$$\begin{aligned} \boldsymbol{k}_1' \otimes \boldsymbol{k}'^1 &= (-\boldsymbol{h}_0 \otimes \boldsymbol{h}^0 \sin^2\gamma - \boldsymbol{h}_1 \otimes \boldsymbol{h}^1 \cos^2\gamma)\,e^{2\mathrm{i}\psi},\\ (\boldsymbol{k}_1' \otimes \boldsymbol{k}'^1)(\boldsymbol{k}'^1 \otimes \boldsymbol{k}_1')^T &= \boldsymbol{h}_0 \otimes \boldsymbol{h}^0 \sin^2\gamma + \boldsymbol{h}_1 \otimes \boldsymbol{h}^1 \cos^2\gamma, \end{aligned} \tag{A33}$$

$$\boldsymbol{h}_0 \otimes \boldsymbol{h}^0 \,\widehat{=}\, \begin{bmatrix} \dfrac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & 0 & 0 & 0\end{bmatrix}, \tag{A34}$$

$$\boldsymbol{h}_1 \otimes \boldsymbol{h}^1 \,\widehat{=}\, \begin{bmatrix} 0 & \dfrac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & 0 & 0\end{bmatrix}. \tag{A35}$$

(c) The definition $\psi \to \sin^2\psi = 2\tilde{M}/r$ of the inclination in the static spherical case, according to that in front of Eq. (14), leads with $\zeta = 1 - \kappa\rho r^3/2\tilde{M}$, $\nabla_{\mathrm{n}} r = \nabla_{\mathrm{n}} p_1/\cos^2\gamma$ to the key relation and in addition to $\nabla_{\mathrm{n}'}\psi$

$$\nabla_{\mathrm{n}}(\sin^2\psi) = \left(\frac{\kappa\rho r^2}{r} - \frac{2\tilde{M}}{r^2}\right)\nabla_{\mathrm{n}} r = -\zeta\frac{\sin^2\psi}{r\cos^2\gamma}\boldsymbol{k}_1, \tag{A36}$$

$$\nabla_{\mathrm{n}}\psi = -\zeta\frac{\tan\psi}{2r\cos^2\gamma}\boldsymbol{k}_1, \quad \nabla_{\mathrm{n}'}\psi = -\zeta\frac{\sin\psi}{2r\cos^2\gamma}\boldsymbol{k}_1'. \tag{A37}$$

The exterior curvature tensor $\boldsymbol{B}'^1 = -(\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'^1)\boldsymbol{N}^T$, according to Eq. (46), is therefore

$$\boldsymbol{B}'^1 = -\zeta\frac{\sin\psi}{2r\cos^2\gamma}(\boldsymbol{k}_1' \otimes \boldsymbol{k}'^1) + \frac{\sin\psi}{r}\boldsymbol{K}_{\mathrm{n}}^T = \zeta\frac{\sin\psi}{2r}(\boldsymbol{h}_0 \otimes \boldsymbol{h}^0 \tan^2\gamma + \boldsymbol{h}_1 \otimes \boldsymbol{h}^1)e^{2\mathrm{i}\psi} + \frac{\sin\psi}{r}\boldsymbol{K}_{\mathrm{n}}^T. \tag{A38}$$

By comparison with Section 2 we find now $\tan^2\gamma = \omega/\zeta$. In the same way, but in addition with $\nabla_{\mathrm{n}} \otimes \boldsymbol{k}^2 = 0$, we have $\nabla_{\mathrm{n}} \otimes \boldsymbol{n}'^2 = -\mathrm{i}\nabla_{\mathrm{n}}\psi \otimes (\boldsymbol{m}^2\widehat{\boldsymbol{S}}e^{-\mathrm{i}\psi}) = -\nabla_{\mathrm{n}}\psi \otimes \boldsymbol{k}'^2$. Therefore we get with $\nabla_{\mathrm{n}}\psi = -\zeta\boldsymbol{k}_2\tan\psi\tan\beta/2r\cos^2\gamma$

$$\boldsymbol{B}'^2 = -(\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}'^2)\boldsymbol{N}^T = \zeta\frac{\sin\psi}{2r}\tan\beta(\boldsymbol{h}_0 \otimes \boldsymbol{h}^0 \tan^2\gamma + \boldsymbol{h}_1 \otimes \boldsymbol{h}^1)\,e^{2\mathrm{i}\psi}. \tag{A39}$$

The expressions (A38) and (A39) lead then with $\boldsymbol{B}_1'$, $\boldsymbol{B}_2'$ and Eqs (A6) and (A33) to a relation for the Ricci tensor $\boldsymbol{R}$ similar to Eq. (21).

4.5. The role of the surrounding complex space

In the 4D-complex space $\widehat{\mathbb{M}}^4 \oplus \mathbb{M}^4 = \tilde{\mathbb{C}}^4$ we have the coordinates $\hat{x}_1 = u,\ \mathrm{i}\hat{x}_2 = \mathrm{i}ct_x,\ \mathrm{i}\hat{x}_3 = \mathrm{i}ct_y,\ \mathrm{i}\hat{x}_4 = \mathrm{i}ct_z$ and $\mathrm{i}x_0 = \mathrm{i}ct,\ \ x_1 = x,\ \ x_2 = y,\ \ x_3 = z.$ First, an 'ordinary' vector $\boldsymbol{u}$ and its conjugate have the components

$$\boldsymbol{u} \mathrel{\hat{=}} (\mathrm{i}u_0 + \hat{u}_1, u_1 + \mathrm{i}\hat{u}_2, u_2 + \mathrm{i}\hat{u}_3, u_3 + \mathrm{i}\hat{u}_4), \quad \bar{\boldsymbol{u}} \mathrel{\hat{=}} (-\mathrm{i}u_0 + \hat{u}_1, u_1 - \mathrm{i}\hat{u}_2, u_2 - \mathrm{i}\hat{u}_3, u_3 - \mathrm{i}\hat{u}_4), \tag{A40}$$

and the norm reads $\boldsymbol{u}\cdot\bar{\boldsymbol{u}} = u_0^2 + u_1^2 + u_2^2 + u_3^2 + \hat{u}_1^2 + \hat{u}_2^2 + \hat{u}_3^2 + \hat{u}_4^2 > 0.$ The scalar product of two vectors $\boldsymbol{u}$, $\bar{\boldsymbol{v}}$ is

$$\begin{aligned}\boldsymbol{u}\cdot\bar{\boldsymbol{v}} &= u_0v_0 + u_1v_1 + u_2v_2 + u_3v_3 + \hat{u}_1\hat{v}_1 + \hat{u}_2\hat{v}_2 + \hat{u}_3\hat{v}_3 + \hat{u}_4\hat{v}_4 \\ &\quad + \mathrm{i}\,(u_0\hat{v}_1 - v_0\hat{u}_1 - u_1\hat{v}_2 + v_1\hat{u}_2 - u_2\hat{v}_3 + v_2\hat{u}_3 - u_3\hat{v}_4 + v_3\hat{u}_4).\end{aligned} \tag{A41}$$

In particular, if $\boldsymbol{v} = \boldsymbol{u}_\perp \mathrel{\hat{=}} (-\mathrm{i}\hat{u}_1 + u_0, \hat{u}_2 - \mathrm{i}u_1, \hat{u}_3 - \mathrm{i}u_2, \hat{u}_4 - \mathrm{i}u_3),$ this scalar product is *purely imaginary*:

$$\boldsymbol{u}\cdot\bar{\boldsymbol{u}}_\perp = \mathrm{i}\,(u_0^2 + \hat{u}_1^2 + u_1^2 + \hat{u}_2^2 + u_2^2 + \hat{u}_3^2 + u_3^2 + \hat{u}_4^2). \tag{A42}$$

We have for instance two unit vectors $\boldsymbol{u} \mathrel{\hat{=}} (0, \cos\alpha + \mathrm{i}\sin\alpha, 0, 0)$ and $\boldsymbol{u}_\perp \mathrel{\hat{=}} (0, \sin\alpha - \mathrm{i}\cos\alpha, 0, 0)$ for which $\boldsymbol{u}\cdot\bar{\boldsymbol{u}}_\perp = \mathrm{i}.$ As this corresponds to a complex product $\mathrm{i}z\bar{z} = \mathrm{i},$ we can say that $\boldsymbol{u}_\perp$ is in some way 'orthonormal' to $\boldsymbol{u}.$ Second, in a similar manner the two unit spinors $\boldsymbol{n}^2 = \boldsymbol{m}^2\widehat{\boldsymbol{S}}$ in the complementary space and $\boldsymbol{k}_2 = \boldsymbol{S}\boldsymbol{m}_2 = \mathrm{i}\widehat{\boldsymbol{S}}\boldsymbol{m}_2$ in the Minkowski space can be said to be 'orthonormal' to each other, because we have

$$\boldsymbol{S} = \mathrm{i}\,\widehat{\boldsymbol{S}}, \tag{A43}$$

$$\boldsymbol{n}^2 \cdot \boldsymbol{k}_2 = \boldsymbol{m}^2\widehat{\boldsymbol{S}}\boldsymbol{S}\boldsymbol{m}_2 = \mathrm{i}. \tag{A44}$$

Finally we recall the well-known Lorentz transformation, for instance in the Minkowski space for the first pair $(\mathrm{i}x_0, x_1),$ which transforms into $(\mathrm{i}x_0', x_1')$ and where $\mathrm{i}x_0 = \mathrm{i}ct = y_0,\ \ x_0' = \mathrm{i}ct' = y_0'.$ We can either write two matrix representations or a complex representation:

$$\begin{Bmatrix} x_1' \\ \mathrm{i}x_0' \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} x_1 \\ \mathrm{i}x_0 \end{Bmatrix}, \tag{A45}$$

$$\begin{Bmatrix} x_1' \\ y_0' \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} x_1 \\ y_0 \end{Bmatrix}, \tag{A46}$$

$$x_1' + \mathrm{i}x_0' = (a - \mathrm{i}b)(x_1 + \mathrm{i}x_0) = x_1(\cos\phi - \sin\phi) + \mathrm{i}x_0(\sin\phi + \cos\phi), \tag{A47, 48}$$

$$a = \cos\phi = 1\Big/\sqrt{1 - {\beta_0}^2}\,, \tag{A49}$$

$$b = -\mathrm{i}\sin\phi = \beta_0\Big/\sqrt{1 - {\beta_0}^2} = \beta_0 a. \tag{A50}$$

For $x_1 = vt = \mathrm{i}x_0/c$ we obtain $0 = x_1' = avx_0/c - a\beta_0x_0,\ \ \beta_0 = v/c,\ \ t' = (t - x_1v/c^2)\Big/\sqrt{1 - v^2/c^2}.$

4.6. The problem of linearization, an alternative development with details

In order to point out explicitly a sort of kinematic meaning of Eq. (48) and also of the statement $2\psi^{\lambda}_{\alpha\beta,\lambda} = (\psi^{i}_{a}\bar{\psi}_{i\beta})^{\lambda}_{,\lambda} \neq 2(\bar{\psi}^{i}_{\lambda,\alpha}\psi^{\lambda}_{i\beta,} - \bar{\psi}^{i}_{\alpha,\beta}\psi^{\lambda}_{i\lambda,})$ there, we introduce a generalized displacement $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ on the 4D-flat space defining the near 4D-hypersurface of small curvature. The spinor $\boldsymbol{v} = \boldsymbol{N}\boldsymbol{v}$ is interior (in the Minkowski space $\mathbb{M}^4$) and $\boldsymbol{w} = w_i\boldsymbol{n}^i = w^i\boldsymbol{n}_i = \widehat{\boldsymbol{N}}\boldsymbol{w}$ is an exterior spinor (in the complementary space $\widehat{\mathbb{M}}^4$). We assume further that $|\boldsymbol{w}|$ is smooth of first order small, whereas $|\boldsymbol{v}|$ is smooth of second order small. Analogously to the nonlinear kinematic equations in the 3D-shell theory, we define here a modified strain tensor $\gamma = \overline{\gamma}^T$ (also in accordance with the full projection $\boldsymbol{NF}^T\boldsymbol{FN} = \boldsymbol{VV}$ in Section 1, before Eq. (4)):

$$\gamma = \frac{1}{2}(\boldsymbol{N}\overline{\boldsymbol{F}}^T\boldsymbol{FN} - \boldsymbol{N}) \approx \frac{1}{2}[(\nabla_{\mathrm{n}} \otimes \overline{\boldsymbol{v}}) + (\nabla_{\mathrm{n}} \otimes \boldsymbol{v})^T + (\nabla_{\mathrm{n}} \otimes \overline{\boldsymbol{w}})(\nabla_{\mathrm{n}} \otimes \boldsymbol{w})^T]. \tag{A51}$$

For the last term we can write $(\nabla_{\mathrm{n}} \otimes \overline{\boldsymbol{w}})(\nabla_{\mathrm{n}} \otimes \boldsymbol{w})^T = \nabla_{\mathrm{n}} w_i \otimes \nabla_{\mathrm{n}} w^i = \psi_i \otimes \psi^i$. We introduce further the *restrictive* kinematic equation

$$2\nabla_{\mathrm{n}}(\gamma + \gamma^T) = \nabla_{\mathrm{n}}(\gamma \cdot \boldsymbol{N}^T + \gamma^T \cdot \boldsymbol{N}), \tag{A52}$$

where we have with $\nabla_{\mathrm{n}} \cdot \nabla_{\mathrm{n}} = \Delta$ the auxiliary relations

$$2\nabla_{\mathrm{n}}\gamma = \Delta\overline{\boldsymbol{v}} + \nabla_{\mathrm{n}}(\nabla_{\mathrm{n}} \cdot v) + (\nabla_{\mathrm{n}} \otimes \nabla_{\mathrm{n}} w^i)^T \nabla_{\mathrm{n}} w_i + \Delta w_i \nabla_{\mathrm{n}} w^i, \tag{A53}$$

$$2\nabla_{\mathrm{n}}\gamma^T = \Delta\boldsymbol{v} + \nabla_{\mathrm{n}}(\nabla_{\mathrm{n}} \cdot \overline{v}) + (\nabla_{\mathrm{n}} \otimes \nabla_{\mathrm{n}} w_i)^T \nabla_{\mathrm{n}} w^i + \Delta w^i \nabla_{\mathrm{n}} w_i, \tag{A54}$$

$$2\nabla_{\mathrm{n}}(\gamma \cdot \boldsymbol{N}^T) = \nabla_{\mathrm{n}}(\nabla_{\mathrm{n}} \cdot \overline{v}) + \nabla_{\mathrm{n}}(\nabla_{\mathrm{n}} \cdot v) + \nabla_{\mathrm{n}}(\nabla_{\mathrm{n}} w_i \cdot \nabla_{\mathrm{n}} w^i), \tag{A55}$$

$$2\nabla_{\mathrm{n}}(\gamma^T \cdot \boldsymbol{N}) = \nabla_{\mathrm{n}}(\nabla_{\mathrm{n}} \cdot v) + \nabla_{\mathrm{n}}(\nabla_{\mathrm{n}} \cdot \overline{v}) + \nabla_{\mathrm{n}}(\nabla_{\mathrm{n}} w^i \cdot \nabla_{\mathrm{n}} w_i). \tag{A56}$$

From the last four equations we get with $\widehat{\boldsymbol{N}} = \boldsymbol{n}^i \otimes \boldsymbol{n}_i$ a differential equation for the displacements $\boldsymbol{v}$ and $\boldsymbol{w}$

$$\Delta\overline{\boldsymbol{v}} = -\Delta w_i \nabla_{\mathrm{n}} w^i = -(\nabla_{\mathrm{n}} \otimes \boldsymbol{w})\widehat{\boldsymbol{N}}\Delta\boldsymbol{w}, \qquad \Delta\boldsymbol{v} = -\Delta w^i \nabla_{\mathrm{n}} w_i = -\Delta\boldsymbol{w}\widehat{\boldsymbol{N}}(\nabla_{\mathrm{n}} \otimes \boldsymbol{w})^T, \tag{A57}$$

which expresses Eq. (A52) in another manner. If we apply now the operator Δ onto Eq. (A51) we obtain with Eq. (A57)

$$\begin{aligned}
\Delta\gamma &= \frac{1}{2}[\nabla_{\mathrm{n}} \otimes \Delta\overline{\boldsymbol{v}} + (\nabla_{\mathrm{n}} \otimes \Delta\boldsymbol{v})^T + \Delta(\nabla_{\mathrm{n}} w_i \otimes \nabla_{\mathrm{n}} w^i)] \\
&= \frac{1}{2}[-\nabla_{\mathrm{n}}\Delta w_i \otimes \nabla_{\mathrm{n}} w^i - \Delta w_i(\nabla_{\mathrm{n}} \otimes \nabla_{\mathrm{n}} w^i)^T - \nabla_{\mathrm{n}} w_i \otimes \nabla_{\mathrm{n}}\Delta w^i - (\nabla_{\mathrm{n}} \otimes \nabla_{\mathrm{n}} w_i)\Delta w^i] \\
&\quad + \frac{1}{2}[\Delta\nabla_{\mathrm{n}} w_i \otimes \nabla_{\mathrm{n}} w^i + 2(\nabla_{\mathrm{n}} \otimes \nabla_{\mathrm{n}} w_i)^T(\nabla_{\mathrm{n}} \otimes \nabla_{\mathrm{n}} w^i) + \nabla_{\mathrm{n}} w_i \otimes \Delta\nabla_{\mathrm{n}} w^i] \\
&= (\nabla_{\mathrm{n}} \otimes \nabla_{\mathrm{n}} w_i)^T(\nabla_{\mathrm{n}} \otimes \nabla_{\mathrm{n}} w^i) - \Delta w_i(\nabla_{\mathrm{n}} \otimes \nabla_{\mathrm{n}} w^i)^T = \boldsymbol{R}^0.
\end{aligned} \tag{A58}$$

The tensor $\boldsymbol{R}^0$ may be called the image in the flat space of the true Ricci tensor on the curved space

$$\boldsymbol{R} \approx \boldsymbol{R}^0 + \boldsymbol{R}^0(\nabla_{\mathrm{n}} \otimes \boldsymbol{w}) + (\nabla_{\mathrm{n}} \otimes \boldsymbol{w})^T \boldsymbol{R}^0. \tag{A59}$$

Because Eq. (48) represents already an approximation after the forgoing development of the projector, $\boldsymbol{R}$ has two semi-exterior parts of higher order. We arrive therefore at the following ***conclusion***:

The field Eq. (19) becomes in vacuum $\boldsymbol{R} = 0$. We would then get from Eqs (A58) and (A59) and from Eq. (A52) *simultaneously* two differential equations:

$$\boldsymbol{R}^0 = \Delta\gamma = 0, \tag{A60}$$

$$2\nabla_{\mathrm{n}}(\gamma + \gamma^T) = \nabla_{\mathrm{n}}(\gamma \cdot \boldsymbol{N}^T + \gamma^T \cdot \boldsymbol{N}). \tag{A52}$$

This is not possible; therefore gravitational waves, described by pure geometrical considerations, *cannot* exist. As for the (incorrect) Eq. (39), note that the operator $\Box = (...)^{\lambda}_{,\lambda}$ is only applied on the components $\psi_{\alpha\beta} = (\psi^{i}_{\alpha}\overline{\psi}_{i\beta})/2$, whereas in Eqs (A58) and (A60) the equivalent operator $\nabla_{\mathrm{n}} \cdot \nabla_{\mathrm{n}} = \Delta$ is applied on the complete tensor $\boldsymbol{\gamma}$ including its base. Moreover, one has $\psi_{\alpha\beta} \neq \gamma_{\alpha\beta}$, because the interior generalized displacement (spinor) $\boldsymbol{v}$ does unfortunately not intervene in Eq. (39). The commonly used condition $2\psi^{\lambda}_{\alpha\lambda,} = \psi^{\lambda}_{\lambda,\alpha}$, serving to obtain this equation, is stated as a choice of simplifying special coordinates. However, *this statement disguises the fact of the additional restrictive kinematic relation* Eq. (A52).

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REFERENCES

1. Champagne, E. B. Holographic interferometry extended. In *Digest of Papers, Int. Opt. Computing Conf., Zurich (IEEE, New York)*. 1974, 73–74.
2. Cuche, D. E. Modification methods in holographic and speckle interferometry. In *Interferometry in Speckle Light* (Jacquot, P. and Fournier, J. M., eds). Springer, Berlin, Heidelberg, New York, 2000, 109–114.
3. Stetson, K. A. Fringe interpretation for hologram interferometry of rigid-body motions and homogenous deformations. *J. Opt. Soc. Am.*, 1974, **64**, 1–10.
4. Walles, S. Visibility and localization of fringes in holographic interferometry of diffusely reflecting surfaces. *Ark. Fys.*, 1970, **40**, 299–403.
5. Schwarzschild, K. Ueber das Gravitationsfeld eines Massenpunktes. *Sitzungsberichte der deutschen Akademie der Wissenschaften, Kl. Math.-Phys. Tech.* Berlin, 1916, 189–196.
6. Misner, C. W., Thorne, K. S. and Wheeler, J. A. *Gravitation*. W. H. Freeman and Company, New York, 1972.
7. Plebanski, J. and Krasinski, A. *An Introduction to General Relativity and Cosmology*. Cambridge University Press, 2007, 196–200.
8. Sexl, R. U. and Urbantke, H. K. *Gravitation und Kosmologie*. 2nd edn. Wissenschaftsverlag, Wien, 1981, 53 and 240–243.
9. Kerr, R. P. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, 1963, **11**(5), 237–238.
10. Goenner, H. *Einführung in die spezielle und allgemeine Relativitätstheorie*. Spektrum, Akademischer Verlag, Heidelberg, Oxford, 1996, 145–149 and 304.
11. Penrose, R. Structure of space-time. In *Battelle Rencontres 1967* (Dewitt, C. M. and Wheeler, J. A., eds). Benjamin, New York, Amsterdam, 1968, 121–235.
12. Poisson, E. *A Relativist's Toolkit*. Cambridge University Press, 2004.

Deformeeritud pinnad holograafilises interferomeetrias. Sarnased aspektid üldistes gravitatsiooniväljades

Walter Schumann

Esimeses osas on antud lühiülevaade pinnadeformatsioonide meetodist holograafilises interferomeetrias. See on sissejuhatuseks järgnevatele osadele, kus sama meetodit on kasutatud üldrelatiivsusteooria võrrandite ja nende lahendite uurimiseks.

Artikli põhiosas (2., 3. ja 4. osa) on vaadeldud Einsteini võrrandite lahendi poolt antud neljamõõtmelist kõverat aegruumi hüperpinnana kõrgemamõõtmelises tasases ruumis. Virtuaalsete deformatsioonide meetodit on rakendatud hüperpindadele, mis vastavad Schwarzschildi ja Kerri lahenditele, aga ka üldise gravitatsiooniläätse isotroopsetele geodeetilistele ehk valguskiirtele. On väidetud, et tühjas (ilma mateeriata) aegruumis on lainevõrrandi tuletamine Einsteini võrrandite lineariseerimisel ebakorrektne ja artiklis toodud meetodil lineariseerimine lainevõrrandit ei tekita. Sellest asjaolust on järeldatud, et puhtgeomeetrilisi gravitatsioonilaineid pole olemas.

Lisas on toodud kasutatud matemaatiliste mõistete ja arvutuste üksikasjad.