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Characterizations of the power distribution by record values

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables which has absolutely continuous distribution function $F(x)$ with probability density function $f(x)$ and $F(0) = 0$. Assume that X_n belongs to the class C_1^* or C_2. Then X_k has the power distribution if and only if X_k and $\frac{X_{L(n+1)}}{X_{L(n)}}$ or $\frac{X_{L(n+1)}}{X_{L(n)}}$ and $\frac{X_{L(n)}}{X_{L(n-1)}}$ are identically distributed, respectively. Suppose that X_n belongs to the class C_3. Also, X_k has the power distribution if and only if $X_{L(n+1)}$ and $X_{L(n)} \cdot V$ are identically distributed, where V is independent of $X_{L(n)}$ and $X_{L(n+1)}$ and is distributed as X_n's.

Keywords. Characterizations; power distribution; independent and identically distributed; hazard rate; lower record values; theory of functional equations.

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1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Let $Y_n = \min(\max)\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say X_j is a lower(upper) record value of this sequence, if $Y_j < (>) Y_{j-1}$ for $j \geq 2$. By definition, X_1 is a lower record value as well as an upper record value. The indices at which the lower record values occur are given by the record times $\{L(n), n \geq 1\}$, where $L(n) = \min\{j \mid j > L(n-1), X_j > X_{L(n-1)}, n \geq 2\}$ with $L(1) = 1$. Similarly, let $U(n) = \min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$ with $U(1) = 1$ denoting the times of upper record values. We assume that all lower record values $X_{L(i)}$ for $i \geq 1$ occur at a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables.

If F is the distribution function of a nonnegative random variable, we call F as 'new better than used'(NBU), if $\bar{F}(xy) \leq \bar{F}(x)\bar{F}(y)$, $x, y > 1$, and F as 'new worse than used' (NWU), if $\bar{F}(xy) \geq \bar{F}(x)\bar{F}(y)$, $x, y > 1$.

We say the random value X_n belongs to the class C_1 if $F(x)$ is either NBU and NWU. Similarly to the definition of C_1 in the literature (see [2]), we define the following three classes to effectively induce the functional relations of the distribution function $F(x)$ as well as the hazard rate $h(x)$ from the integrands in (3.7), (3.13) and (3.18) to be shown later. X_n is said to belong to the class C_1^* if $F(x)$ is either $F(xy) \geq F(x)F(y)$ or $F(xy) \leq F(x)F(y)$ for all $x > 0$, $y > 0$. For $F(x) > 0$, let $h(x) = \frac{f(x)}{F(x)}$ be the hazard rate in the lower record values with $f(x)$ as the density function of $F(x)$. X_n is said to belong to the class C_2 if either $h(xy)y \geq h(x)$ or $h(xy)y \leq h(x)$ for all $x > 0$, $y > 0$. Finally, X_n is said to belong to the class C_3 if either $F(\frac{y}{x}) \geq \frac{F(y)}{F(x)}$ or $F(\frac{y}{x}) \leq \frac{F(y)}{F(x)}$ for all $x > 0$, $y > 0$.

We call the random variable $X \in POW(a, \alpha)$ if the corresponding probability cdf $F(x)$ of X is of the form

$$F(x) = \begin{cases} \left(\frac{x}{a}\right)^{\alpha}, & 0 \leq x \leq a,\ \alpha > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Ahsanullah [3] characterized that if X_k belongs to the class C_1 and for the upper record values $X_{U(m)}$ and $X_{U(n)}$ with $1 \leq m < n$, $X_{U(n)} - X_{U(m)}$ and $X_{U(m)}$ are identically distributed, then X_k, $k \geq 1$, has the exponential distribution. Also, one can find more details on characterizations under assumption of identical distribution in Ahsanullah and Raqab [4].

In this paper, we obtain characterizations of the power distribution based on lower record values by the assumption of identical distributions.

2. Results

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables which has absolutely continuous distribution function $F(x)$ with pdf $f(x)$ and $F(0) = 0$. Assume that X_n belongs to the class C_1^*. Then $X_k, k \geq 1$ has the power distribution* (1.1) *with $a = 1$ if and only if $W_{n+1,n} = \frac{X_{L(n+1)}}{X_{L(n)}}$ has an identical distribution with X_j for some j between 1 and n.*

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables which has absolutely continuous distribution function $F(x)$ with pdf $f(x)$ and $F(0) = 0$. Assume that X_n belongs to the class C_2. Then $X_k, k \geq 1$ has the power distribution* (1.1) *with $a = 1$ if and only if $W_{n+1,n} = \frac{X_{L(n+1)}}{X_{L(n)}}$ has an identical distribution with $W_{n,n-1} = \frac{X_{L(n)}}{X_{L(n-1)}}$, $n \geq 2$.*

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables which has absolutely continuous distribution function $F(x)$ with pdf $f(x)$ and $F(0) = 0$. Assume that X_n belongs to the class C_3. Then $X_k, k \geq 1$ has the power distribution* (1.1) *with $a = 1$ if and only if $X_{L(n+1)}$ and $X_{L(n)} \cdot V$ are identically distributed, where V is independent of $X_{L(n)}$ and $X_{L(n+1)}$ and is distributed as X_n's.*

3. Proofs

Proof of Theorem 2.1. If $X_k \in POW(1, \alpha)$, then it can easily be seen that $W_{n+1,n}$ has an identical distribution with X_k. We have to prove the reverse.

The joint pdf $f_{n,n+1}(x, y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ can be written as

$$f_{n,n+1}(x, y) = \frac{H(x)^{n-1}}{\Gamma(n)} h(x) f(y), \tag{3.1}$$

where $H(x) = -\ln(F(x))$ and $h(x) = -\frac{\mathrm{d}}{\mathrm{d}x} H(x)$.

Let us use the transformation $U = X_{L(n)}$ and $W_{n+1,n} = \frac{X_{L(n+1)}}{X_{L(n)}}$. The Jacobian of the transformation is $J = u$. Thus we can write the joint pdf $f_{U,W}(u, w)$ of U and $W_{n+1,n}$ as

$$f_{U,W}(u, w) = \frac{H(u)^{n-1}}{\Gamma(n)} h(u) f(uw) u \tag{3.2}$$

for all $0 < u \leq 1$, $0 \leq w < 1$.

The pdf $g_n(w)$ of $W_{n+1,n}$ can be written as

$$g_n(w) = \int_0^1 \frac{H(u)^{n-1}}{\Gamma(n)} h(u) f(uw) u \mathrm{d}u, \qquad 0 \leq w < 1. \tag{3.3}$$

By the assumption of the identical distribution of $W_{n+1,n}$ and X_k, we have

$$\int_0^1 \frac{H(u)^{n-1}}{\Gamma(n)} h(u) f(uw) u \mathrm{d}u = f(w), \qquad 0 \leq w < 1. \tag{3.4}$$

Substituting

$$\Gamma(n) = \int_0^1 H(u)^{n-1} f(u) \mathrm{d}u, \tag{3.5}$$

we get

$$\int_0^1 H(u)^{n-1} f(u) \left\{ f(w) - \frac{f(uw)u}{F(u)} \right\} \mathrm{d}u = 0, \quad 0 \leq w < 1. \tag{3.6}$$

Integration (3.6) with respect to w from 0 to w_1, we get the following equation

$$\int_0^1 H(u)^{n-1} f(u) \left\{ F(w_1) - \frac{F(uw_1)}{F(u)} \right\} \mathrm{d}u = 0, \qquad 0 \leq w_1 < 1. \tag{3.7}$$

If $F(x)$ belongs to the class C_1^*, then (3.7) is true if

$$F(uw_1) = F(u)F(w_1) \tag{3.8}$$

for almost all u, $0 < u \leq 1$ and fixed w_1, $0 \leq w_1 < 1$.

By the theory of functional equations [1], the only continuous solution of (3.8) with boundary conditions $F(0) = 0$ and $F(1) = 1$ is

$$F(x) = x^\alpha, \qquad 0 \leq x \leq 1, \ \alpha > 0.$$

This completes the proof. □

Proof of Theorem 2.2. If $X_k \in POW(1, \alpha)$, then it can easily be seen that $W_{n,n-1}$ and $W_{n+1,n}$ are identically distributed. We have to prove the converse.

The pdf $g_n(w)$ of $W_{n+1,n}$ can be written as

$$g_n(w) = \int_0^1 \frac{H(u)^{n-1}}{\Gamma(n)} h(u) f(uw) u \mathrm{d}u, \quad 0 \le w < 1. \tag{3.9}$$

From (3.9), we have

$$P(W_{n+1,n} < w) = \int_0^1 \frac{H(u)^{n-1}}{\Gamma(n)} h(u) F(uw) \mathrm{d}u, \quad 0 \le w < 1. \tag{3.10}$$

Since $W_{n,n-1}$ and $W_{n+1,n}$ are identically distributed, we get

$$\int_0^1 H(u)^{n-1} h(u) F(uw) \mathrm{d}u = (n-1) \int_0^1 H(u)^{n-2} h(u) F(uw) \mathrm{d}u. \tag{3.11}$$

But we know that

$$(n-1) \int_0^1 H(u)^{n-2} h(u) F(uw) \mathrm{d}u = \int_0^1 H(u)^{n-1} f(uw) w \mathrm{d}u. \tag{3.12}$$

Substituting (3.12) in (3.11), we get, on simplification,

$$\int_0^1 H(u)^{n-1} F(uw)[h(u) - h(uw)w] \mathrm{d}u = 0, \qquad 0 \le w < 1. \tag{3.13}$$

Thus, if $F(x)$ belongs to the class C_2, then (3.13) is true if

$$h(uw)w = h(u) \tag{3.14}$$

for almost all $0 < u \le 1$ and $0 \le w < 1$.

Integrating with respect to u and simplifying, we get

$$F(uw) = F(u)F(w) \tag{3.15}$$

for all $0 < u \le 1$ and $0 \le w < 1$.

By the theory of functional equations [1], the only continuous solution of (3.15) with boundary conditions $F(0) = 0$ and $F(1) = 1$ is

$$F(x) = x^{\alpha}, \quad 0 \le x \le 1, \ \alpha > 0.$$

This completes the proof. □

Proof of Theorem 2.3. If $X_k \in POW(1, \alpha)$, then it can easily be seen that $X_{L(n+1)}$ and $X_{L(n)} \cdot U$ are identically distributed. We will prove the sufficient condition.

Suppose that $X_{L(n+1)}$ and $X_{L(n)} \cdot V$ are identically distributed. The pdf $f_1(y)$ of $X_{L(n+1)}$ can be written as

$$\begin{aligned} f_1(y) &= \frac{H(y)^n}{\Gamma(n+1)} f(y) \\ &= \frac{\mathrm{d}}{\mathrm{d}y}\left[F(y)\int_y^1 \frac{H(x)^{n-1}}{\Gamma(n)} h(x)\mathrm{d}x - \int_y^1 \frac{H(x)^{n-1}}{\Gamma(n)} f(x)\mathrm{d}x\right], \qquad 0<y<1. \end{aligned} \tag{3.16}$$

The pdf $f_2(y)$ of $X_{L(n)} \cdot V$ can be written as

$$\begin{aligned} f_2(y) &= \int_y^1 \frac{H(x)^{n-1}}{\Gamma(n)} f(x) f\left(\frac{y}{x}\right)\frac{1}{x}\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}y}\left[\int_y^1 \frac{H(x)^{n-1}}{\Gamma(n)} F\left(\frac{y}{x}\right) f(x)\mathrm{d}x - \int_y^1 \frac{H(x)^{n-1}}{\Gamma(n)} f(x)\mathrm{d}x\right], \quad 0<y<1. \end{aligned} \tag{3.17}$$

Equating (3.16) and (3.17), we get on simplification

$$\int_y^1 \frac{H(x)^{n-1}}{\Gamma(n)} f(x)\left[F\left(\frac{y}{x}\right) - \frac{F(y)}{F(x)}\right]\mathrm{d}x = 0, \quad 0 < y < 1. \tag{3.18}$$

Since $F(x)$ belongs to the class C_3, we must have

$$F\left(\frac{y}{x}\right) = \frac{F(y)}{F(x)} \tag{3.19}$$

for almost all x, y, $1 < y < x < \infty$.

By the theory of functional equations [1], the only continuous solution of (3.19) with the boundary condition $F(0) = 0$, $F(1) = 1$ is $F(x) = x^{\alpha}$ for $0 \le x \le 1$, $\alpha > 0$.

This completes the proof. □

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