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Quantitative metric theory of continued fractions

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Abstract. Quantitative versions of the central results of the metric theory of continued fractions were given primarily by C. De Vroedt. In this paper we give improvements of the bounds involved . For a real number x, let

$$x = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{c_4 + \ddots}}}}.$$

A sample result we prove is that given $\epsilon > 0$,

$$(c_1(x)\cdots c_n(x))^{\frac{1}{n}} = \prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\frac{\log k}{\log 2}} + o\left(n^{-\frac{1}{2}}(\log n)^{\frac{3}{2}}(\log\log n)^{\frac{1}{2}+\epsilon}\right)$$

almost everywhere with respect to the Lebesgue measure.

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1. Introduction

In this paper, we use a quantitative L^2-ergodic theorem to study the metrical theory of the regular continued fraction expansion of real number. Here and throughout the rest of the paper, by a dynamical system (X, β, μ, T) we mean a set X, together with a σ-algebra β of subsets of X, a probability measure μ on the measurable space (X, β) and a measurable self-map T of X that is also measure-preserving. By this we mean that if given an element A of β and if we set $T^{-1}A = \{x \in X : Tx \in A\}$, then $\mu(A) = \mu(T^{-1}A)$. We say a dynamical system is ergodic if $T^{-1}A = A$ for some A in β means that $\mu(A)$ is either zero or one in value. For β-measurable f, let $\|f\|$ denote the L^2 norm $((\int_X |f|^2 \mathrm{d}\mu)^{\frac{1}{2}})$. As a standard, for two functions f and g, we say $f(x) = O(g(x))$ if there exists $C > 0$ such that $|f(x)| \leq C|g(x)|$. We say $f(x) = o(g(x))$, as $x \to \infty$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. The central tool in this note is the following theorem.

Theorem 1. *Let (X, β, μ, T) denote an ergodic dynamical system. Suppose that*

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) - \int_X f \mathrm{d}\mu \right\| \leq \frac{B}{N^{\frac{1}{2}}},$$

for $B > 0$ and $N = 1, 2, \ldots$. Then given $\epsilon > 0$, we have

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int_X f \mathrm{d}\mu + o\left(\frac{(\log N)^{\frac{3}{2}} (\log \log N)^{\frac{1}{2}+\epsilon}}{N^{\frac{1}{2}}} \right),$$

for μ almost all x.

For a real number x, let

$$x = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{c_4 + \ddots}}}}$$

denote its regular continued fraction expansion, which is also written more compactly as $[c_0; c_1, c_2, \ldots]$. The terms $c_0, c_1, \ldots$ are called the partial quotients of the continued fraction expansion and the sequence of rational truncates

$$[c_0; c_1, \ldots, c_n] = \frac{p_n}{q_n}, \quad (n = 1, 2, \ldots)$$

are called the convergents of the continued fraction expansion. In this paper, we use Theorem 1 to study the metric theory of continued fractions. In particular, we refine results of DeVroedt [2, 3]. For more historic background, see [4, 15] and for basic background on continued fractions, see [6].

For a real number y, let $\{y\}$ denote its fractional part. We now consider the particular ergodic properties of the Gauss map, defined on $[0, 1]$ by

$$Tx = \begin{cases} \left\{\frac{1}{x}\right\}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Notice that $c_n(x) = c_{n-1}(Tx)$ $(n = 1, 2, \ldots)$. Let (X, β, μ, T) denote the dynamical system where X denotes $[0, 1]$, β is the σ-algebra of Borel sets on X, $\mu = \gamma$ is the measure on (X, β) defined for any A in β by

$$\gamma(A) = \frac{1}{\log 2} \int_A \frac{\mathrm{d}x}{x+1},$$

and T is the Gauss map. Note that (X, β, μ, T) is ergodic (see [1] for details). The ergodic properties of the Gauss dynamical system (X, β, μ, T) are not quite enough to carry out this investigation. We also need ergodic theoretic information about its natural extension which we now describe. Let $\Omega = ([0, 1) \setminus \mathbf{Q}) \times [0, 1]$. Now let β^* be the σ-algebra of Borel subsets of Ω and let γ^* be the probability measure on the measurable space (Ω, β^*) defined by

$$\gamma^*(A) = \frac{1}{(\log 2)} \int_A \frac{\mathrm{d}x\mathrm{d}y}{(1+xy)^2}.$$

Also define the map

$$\mathcal{T}(x, y) = \left(Tx, \frac{1}{[\frac{1}{x}] + y} \right).$$

Then the map $\mathcal{T}$ preserves the measure γ^* and the dynamical system $(\Omega, \beta^*, \gamma^*, \mathcal{T})$ is ergodic, which we call the natural extension of the Gauss dynamical system (see [8] for details). Our results are obtained by applying Theorem 1 to the maps T and $\mathcal{T}$. In particular we need the following theorem from [8] (see [1] for a definition of the natural extension). In §2, we prove Theorem 1. In §3 we state and derive our results about the metric theory of continued fractions.

2. Proof of Theorem 1

To prove Theorem 1, we need the following lemma.

Lemma 2.1. *Suppose (X, β, μ, T) is an ergodic dynamical system and suppose $f : X \to \mathbf{R}$ is μ-integrable with $\int_X f \mathrm{d}\mu = 0$. Also suppose, for non-negative integer k, that*

$$\int_X \left| \sum_{n=0}^{2^k-1} f(T^n x) \right|^2 \mathrm{d}\mu \leq B2^k,$$

for $B > 0$. Then also for $k \geq 0$,

$$\int_X \max_{1 \leq j \leq 2^k} \left| \sum_{n=0}^{j-1} f(T^n x) \right|^2 \mathrm{d}\mu = O(k^2 2^k).$$

Proof of Lemma 2.1. Let

$$F(M, N, x) = \sum_{n=M}^{N-1} f(T^n x), \quad (0 \leq M < N)$$

and let

$$m_n(f, x) = \max_{1 \leq l \leq n} |F(0, l, x))|, \quad (n \geq 1).$$

Suppose $1 \leq j < 2^{K+1}$. Note that any natural number j can be written uniquely in the form $j = 2^{a_1} + \cdots + 2^{a_h}$, where a_i depends on j with $a_1 > \cdots > a_h \geq 0$. With our bound on j, it follows that $h = h(j) < K + 1$. With this notation, we have

$$F(0, j, x) = \sum_{i=0}^{h-1} F(l_i, l_{i+1}, x),$$

where $l_0 = 0$, $l_i = l_i(j) = 2^{a_1} + \cdots + 2^{a_i}$ for $1 \leq i \leq h$, and $l_h = j$. Note that $l_i = 0 \pmod{2^{a_i}}$ for all i and all choices of j. By the Cauchy–Schwarz inequality,

$$|F(0, j, x)|^2 \leq (K+1) \sum_{i=0}^{h-1} |F(l_i, l_{i+1}, x)|^2, \quad (j = 1, 2, \ldots, h).$$

Also, regardless of j or of the particular sequence of l_i pertaining to j, we have

$$\sum_{i=1}^{h-1} |F(l_i, l_{i+1}, x)|^2 \leq \sum_{p=1}^{K+1} \sum_{\substack{\nu=0 \\ \nu \text{ even}}}^{2^p - 1} |F(\nu 2^{(K+1)-p}, (\nu+1) 2^{(K+1)-p}, x)|^2.$$

This is because every term on the left occurs as a term on the right.

Integrating over X, and using the premise that T is measure-preserving, we obtain that

$$\begin{aligned}\int_X m_{2^K}^2(f, x) \mathrm{d}\mu &\leq \int_X \left((K+1) \sum_{p=1}^{K+1} \sum_{\substack{\nu=0 \\ \nu \text{ even}}}^{2^p - 1} |F(0, 2^{K+1-p}, x)|^2 \right) \mathrm{d}\mu \\ &= (K+1) \sum_{p=1}^{K+1} 2^{p-1} \int_X |F(0, 2^{K+1-p}, x)|^2 \mathrm{d}\mu.\end{aligned}$$

By the hypothesis of this lemma, the last quantity is no more than

$$\begin{aligned}(K+1) \sum_{p=1}^{K+1} 2^{p-1} B(2^{K+1-p}) &= B(K+1) \sum_{p=1}^{K+1} 2^{p-1} 2^{K+1-p} \\ &\leq BC'(K+1)^2 2^K \leq C'' K^2 . 2^K\end{aligned}$$

for some positive constants C', C''. This completes the proof of Lemma 2.1.

We now complete the proof of Theorem 1. Given $\epsilon > 0$, we set

$$g(N) = (BN)^{\frac{1}{2}} (\log N)^{\frac{3}{2}} (\log \log N)^{\frac{1}{2} + \epsilon}$$

and set

$$E_\epsilon = \left\{ x \in X : \limsup_{N \to \infty} \frac{|\sum_{n=0}^{N-1} f(T^n x) - N \int_X f \mathrm{d}\mu|}{g(N)} > 0 \right\}.$$

To prove Theorem 1, we need to show $\mu(E_\epsilon) = 0$. Set

$$A_k = \left\{ x : \max_{4^{k-1} \le l < 4^k} \left| \sum_{n=0}^{l-1} f(T^n x) - l \int_X f \, \mathrm{d}\mu \right| > (\log k)^{-\epsilon/2} g(4^k) \right\}.$$

One checks easily that

$$E_\epsilon \subseteq \bigcap_{r=1}^{\infty} \bigcup_{k=r}^{\infty} A_k.$$

By the Borel–Cantelli lemma, we need to show $\sum_{k=1}^{\infty} \mu(A_k) < \infty$. Note that

$$\mu(A_k)(\log k)^{-\epsilon} g(4^k)^2 \le \int_{A_k} \max_{4^{k-1} \le l < 4^k} \left| \sum_{n=0}^{l-1} f(T^n x) - l \int_X f \, \mathrm{d}\mu \right|^2 \mathrm{d}\mu.$$

By the premise of Theorem 1, for all k,

$$\int_X \left| \sum_{n=0}^{2^k - 1} f(T^n x) - 2^k \int_X f \, \mathrm{d}\mu \right|^2 \mathrm{d}\mu \le B 2^k.$$

Thus, by Lemma 2.1, there exists $C_1 > 0$ such that

$$\int_X \max_{0 \le l \le 2^{2k}} \left| \sum_{n=0}^{l-1} f(T^n x) - l \int_X f \, \mathrm{d}\mu \right|^2 \mathrm{d}\mu \le C'' k^2 2^{2k}.$$

Hence, there exists $C_0 > 0$ such that

$$\mu(A_k)(\log k)^{-\epsilon} 4^k (k \log 4)^3 (\log(k \log 4))^{1+2\epsilon} \le C_0 k^2 \cdot 4^k$$

and so

$$\mu(A_k) = O\left(\frac{1}{k (\log k)^{1+\epsilon}} \right).$$

It follows that $\sum_{k=1}^{\infty} \mu(A_k) < \infty$ and so Theorem 1 is proved.

3. Mixing properties of continued fraction maps

Let $\mathcal{B}_1^k$ denote the σ-algebra of the subsets of X generated by the sequence of functions $a_1(\alpha), \ldots, a_k(\alpha)$ and let $\mathcal{B}_k^\infty$ denote the σ-algebra generated by the sequence of functions $a_k(\alpha), a_{k+1}(\alpha), \ldots$. For a Borel probability measure μ on $[0, 1)$ and for each natural number n, set

$$\psi_\mu(n) = \sup \left| \frac{\mu(A \cap B)}{\mu(A)\mu(B)} - 1 \right|,$$

where the supremum is taken over all sets $A \in \mathcal{B}_1^k$ and $B \in \mathcal{B}_{k+n}^\infty$, with $\mu(A)\mu(B) \neq 0$ for all natural numbers k. We call the sequence $(a_n)_{n \in \mathbf{N}}$ ψ-mixing if $\psi(n) = o(1)$. In [14], it is shown that $\psi_\gamma(n) \le \rho^n$ where $\rho \in (0, 0.8)$. It then follows readily that $\psi_{\gamma^*}(n) \le \rho^n$.

From Chapter 1 of [5], we have

Lemma 3.1. *Let $(a(T^n x))_{n\in\mathbf{N}}$, be a ψ-mixing for dynamical system (X, β, μ, T) and suppose $\sum_{n\in\mathbf{N}} \psi(n) < \infty$. Also suppose for a function f with domain containing the range of a, that $\int_X f((a(x))^2 \mathrm{d}\mu < \infty$. Then the series*

$$\sigma^2 = \int_X f^2(a(x))\mathrm{d}\mu - \left(\int_X f(a(x))\mathrm{d}\mu\right)^2 + 2\sum_{n\in\mathbf{N}} \int_X \left\{\left(f(a(x)) - \int_X f(a(x))\mathrm{d}\mu\right)\left(f(a(T^n x)) - \int_X f(a(x))\mathrm{d}\mu\right)\right\}\mathrm{d}\mu,$$

is absolutely convergent and non-negative and we have

$$\left\|\sum_{n=1}^{N} f(a(T^n x)) - N\int_X f\mathrm{d}\mu\right\|^2 = \sigma(N + o(1)).$$

Evidently $\sum_{n=1}^{\infty} \rho^n < \infty$. We have shown that $\psi_\gamma(n), \psi_{\gamma^*}(n) = O(n)$ as n tends to infinity.

4. Statistical properties of continued fractions

In this section, we use Theorem 1 to refine classical results of metric theory of continued fractions [2, 3, 7, 9–11, 13]. See also [12] for related matter on subsequence averages. This is possible because we use Theorem 1 instead of the classical and well known method of I. S. Gál and J. F. Koksma, which for any $\epsilon > 0$ gives an error term of order $o(n^{-\frac{1}{2}}(\log n)^{\frac{3}{2}+\epsilon})$. The improvement seems small but is the first in nearly 50 years.

For $i^{(m)} = (i_1, \ldots, i_m) \in \mathbf{N}^m$, suppose $\frac{p_n(\alpha)}{q_n(\alpha)} = [i_1, \ldots, i_n]$ $(n = 1, 2, \ldots)$ are the convergents of the continued fraction expansion of the real number α, with g.c.d.$(p_{n-1}, q_{n-1}) =$ g.c.d.$(p_n, q_n) = 1$ and $p_0 = q_0 = 1$. Let

$$K(i^{(m)}) = \begin{cases} \frac{p_m+p_{m-1}}{q_m+q_{m-1}}, & \text{if } m \text{ is odd,} \\ \frac{p_m}{q_m}, & \text{if } m \text{ is even} \end{cases}$$

and

$$L(i^{(m)}) = \begin{cases} \frac{p_m+p_{m-1}}{q_m+q_{m-1}}, & \text{if } m \text{ is even,} \\ \frac{p_m}{q_m}, & \text{if } m \text{ is odd.} \end{cases}$$

We have the following application of Theorem 1.

PROPOSITION 4.1

Suppose $m \in \mathbf{N}$ and let $F : \mathbf{N}^m \to \mathbf{R}$ be such that

$$\int_X F^2(c_1(x), \ldots, c_m(x))\mathrm{d}\gamma(x) < \infty$$

or equivalently that

$$\sum_{i^{(m)}\in\mathbf{N}^m} |F(i^{(m)})|^2 (L(i^{(m)}) - K(i^{(m)})) < \infty.$$

Let

$$\delta_m = \frac{1}{\log 2} \sum_{i^{(m)}\in\mathbf{N}^m} F(i^{(m)}) \log \frac{1+K(i^{(m)})}{1+L(i^{(m)})}.$$

Then

$$\frac{1}{n}\sum_{k=1}^{n-1} F(c_k(x), \dots, c_{k+m-1}(x)) = \delta_m + o(n^{-\frac{1}{2}} (\log n)^{\frac{3}{2}} (\log\log n)^{1+\epsilon})$$

almost everywhere with respect to Lebesgue measure.

Proof. Take $f(x) = F(c_1(x), \dots, c_m(x))$ in Lemma 3.1 . □

Proposition 4.1 has a number of consequences, that follow from self-evident choices of m and F, which we collect together in the following proposition. We do, however, need the following lemma from [13].

Lemma 4.2. *We have* $\sum_{p=0}^{\infty} \log \frac{(l+pm+1)^2}{(l+pm)(l+pm+2)} = \log \left(\frac{\Gamma(\frac{l}{m})\Gamma(\frac{l+2}{m})}{\Gamma(\frac{l+1}{m})^2} \right)$.

For brevity set $\Delta(n,\epsilon) = n^{-\frac{1}{2}} (\log n)^{\frac{3}{2}} (\log\log n)^{\frac{1}{2}+\epsilon}$ $(n = 1, 2, \dots)$.

PROPOSITION 4.2

Given $\epsilon > 0$, *we have*

(*a*) $\frac{1}{n}\#\{k \in [1,n] : c_k(x) = i\} = \frac{1}{2}\log\left(1 + \frac{1}{i(i+1)}\right) + o(\Delta(n,\epsilon))$;
(*b*) $\frac{1}{n}\#\{k \in [1,n] : c_k(x) \geq i\} = \frac{1}{2}\log\left(\frac{i+1}{i}\right) + o(\Delta(n,\epsilon))$;
(*c*) $\frac{1}{n}\#\{k \in [1,n] : i \leq c_k(x) \leq j\} = \frac{1}{2}\log\left(\frac{(i+1)(j+1)}{i(j+2)}\right) + o(\Delta(n,\epsilon))$;
(*d*) $\frac{1}{n}\#\{k \in [1,n] : (c_k(x), \dots, c_{k+m-1}(x)) = i^{(m)}\} = \frac{1}{2}\log\left(\frac{1+K(i^{(m)})}{1+L(i^{(m)})}\right) + o(\Delta(n,\epsilon))$;
(*e*) $\frac{1}{n}\#\{k \in [1,n] : c_k(x) \equiv l \bmod m\} = \frac{1}{2}\log\left(\frac{\Gamma(\frac{l}{m})\Gamma(\frac{l+2}{m})}{\Gamma(\frac{l+1}{m})}\right) + o(\Delta(n,\epsilon))$;
and
(*f*) *for* $p \leq 1, p \neq 0$ *we have* $\left(\frac{c_1(x)^p + \cdots + c_n(x)^p}{N}\right)^{\frac{1}{p}} = K_p + o(\Delta(n,\epsilon))$, *where* $K_1 = \prod_{k=1}^{\infty}\left(1 + \frac{1}{k(k+2)}\right)^{\frac{\log i}{\log 2}}$ *and* $K_p = \left(\frac{1}{2}\int_0^1 \frac{(\lfloor\frac{1}{t}\rfloor)^p}{t+1}\right)^{\frac{1}{p}}$, *almost everywhere with respect to the Lebesgue measure.*

To the sequence of strictly stationary random variables $(a_n(x))_{n\in\mathbf{N}}$, defined on the measurable space $([0,1), \mathcal{B}([0,1))$ via the natural extension construction we can construct the doubly infinite sequence of random variables $(a_n^*)_{n\in\mathbf{N}}$ on $([0,1)^2, \mathcal{B}([0,1)^2))$ defined μ almost everywhere on $[0,1)^2$ for any probability measure μ assigning zero to the measure of $[0,1)^2 \backslash \Omega^2$ (e.g. $\mu = \gamma^*$), where $\Omega = [0,1) \backslash \mathbf{Q}$. We define $s_l^* = [a_l^*, a_{l-1}^*, \ldots]$ and $y_l^* = \frac{1}{s_l^*}$. Evidently $s_l^* = s_0^* \circ \mathcal{T}^l$ and $y_l^* = y_0^* \circ \mathcal{T}^l$. Also for $a \in [0,1)$, put $s_0^a = a$ and $s_l^a = \frac{1}{s_{l-1}^a + a_n}$.

Samur [16, 17] showed that

$$\sigma^2 = \lim_{n\to\infty} \int_{[0,1)\times([0,1)\backslash\mathbf{Q})} \left(\sum_{i=0}^{n-1} (\log y_i^*(x,y) - \frac{\pi^2}{12\log 2} \right) \frac{\mathrm{d}x\mathrm{d}y}{((1+xy)}$$

exists and is positive. Theorem 1 applied to $\mathcal{T}$, for $\epsilon > 0$, gives

$$\frac{1}{n}\sum_{i=0}^{n-1} f(\mathcal{T}^i(x,y)) = \int_{[0,1)\times([0,1)\backslash\mathbf{Q})} f(x,y)\frac{\mathrm{d}x\mathrm{d}y}{1+xy} + o(\Delta(n,\epsilon))$$

for $f \in L^2(\gamma^*)$. Let $f = \chi_B$ denote the characteristic function of the set $B \in \mathcal{B}^2([0,1))$. Now noting that from the definition of $(s^k *)_{k=1}^{\infty}$, the fact that

$$\mathcal{T}(x,y) = \left(Tx, \frac{1}{c_1(y)+x} \right), \quad (x,y) \in ([0,1)\backslash\mathbf{Q} \times [0,1)),$$

that for $(i \geq 2)$,

$$\mathcal{T}^i(x,y) = (T^i(x), [c_i(x), \ldots, c_2(x), c_1(x)+y]), \quad (x,y) \in ([0,1)\backslash\mathbf{Q}\times[0,1)),$$

and that for $(i \geq 0)$,

$$\mathcal{T}^i(x,y) = (T^i(x), s_i^*), \quad (x,y) \in ([0,1)\backslash\mathbf{Q}) \times [0,1).$$

Theorem 1 gives the following proposition.

PROPOSITION 4.3

Given $\epsilon > 0$, *we have*

$$\frac{1}{n}\sum_{i=0}^{n-1} \log y_n^* = -\frac{1}{n}\sum_{i=0}^{n-1} \log s_i^* = \frac{\pi^2}{12\log 2} + o(\Delta(n,\epsilon))$$

almost everywhere with respect to the Lebesgue measure on $([0,1)\backslash\mathbf{Q}) \times [0,1)$.

We also have the following result.

PROPOSITION 4.4

Given $\epsilon > 0$, *for any* $a \in [0,1)$,

$$\frac{1}{n}\sum_{i=0}^{n-1} \log y_i^* = -\frac{1}{n}\sum_{i=0}^{n-1} \log s_i^a = \frac{\pi^2}{12\log 2} + o(\Delta(n,\epsilon))$$

almost everywhere with respect to the Lebesgue measure on $[0,1)\backslash\mathbf{Q}$.

Proof. By the mean value theorem we can see that $|\frac{\log x - \log y}{x-y}| \le \frac{1}{\min(x,y)}$ for any $0 < x, y \le 1$ with $x \ne y$. Now note that if $(F_m)_{m=1}^{\infty}$ denotes the Fibonacci sequence, then

$$0 < \frac{L(i^{(m)})}{K(i^{(m)})} - 1 \le \left(\frac{1}{F_{m-1}F_m}, \frac{1}{F_m^2}\right)$$

for any interval $I(i^{(m)}) = [0,1)\backslash \mathbf{Q}(K(i^{(m)}), L(i^{(m)})(i^{(m)} \in \mathbf{N}^m$ and $k \in \mathbf{N})$, as follows from basic properties of continued fractions. For $a \in [0,1)$, we have

$$\left|\log s_m^* - \log s_m^a\right| \le \max\left(\frac{1}{F_{m-1}F_m}, \frac{1}{F_m^2}\right) = o(g^{2k}),$$

where g denotes the golden ratio, as m tends to ∞ for all $(x,y) \in ([0,1)\backslash \mathbf{Q}) \times [0,1)$. The proof of Proposition 4.3 is complete. □

In the case $a = 0$, we have $s_k^0 = \frac{q_k}{q_{k-1}}$ $(k \ge 1)$ so we get the following Proposition.

PROPOSITION 4.5

Given $\epsilon > 0$, *we have*

$$q_n^{\frac{1}{n}}(x) = \mathrm{e}^{\frac{\pi^2}{12\log 2}} + o(\Delta(n,\epsilon))$$

almost everywhere with respect to the Lebesgue measure on $[0,1)\backslash \mathbf{Q}$.

PROPOSITION 4.6

Given $\epsilon > 0$, *we have*

$$\frac{1}{n}\log\left|x - \frac{p_n}{q_n}\right| = -\frac{\pi^2}{6\log 2} + o(\Delta(n,\epsilon))$$

almost everywhere with respect to the Lebesgue measure on $[0,1)\backslash \mathbf{Q}$.

Proof. Note the classical inequality $\frac{1}{2q_{n+1}^2} < \left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$. The proposition now follows from Proposition 4.4. □

PROPOSITION 4.7

Let $I(a_1,\ldots,a_n) = \{x \in [0,1) : c_1(x) = a_1, \ldots, c_n(x) = a_n\}$. *Given* $\epsilon > 0$, *we have*

$$\frac{1}{n}\log\lambda(I(a_1,a_2,\ldots,a_n)) = -\frac{\pi^2}{6\log 2} + o(\Delta(n,\epsilon))$$

almost everywhere with respect to the Lebesgue measure on $[0,1)\backslash \mathbf{Q}$.

Proof. Let λ denote the Lebesgue measure, and let $I(c_1,\ldots,c_n)$ be a cylinder in $[0,1)$. Recall that $s_n = \frac{q_{n-1}}{q_n}$ and that $\lambda(I(c_1,\ldots,c_n)) = \frac{1}{q_n(q_n+q_{n-1})}$. This means that

$$\log\lambda(I(c_1,\ldots,c_n)) = -2\log q_n - \log(s_n+1)$$

for $(n \in \mathbf{N})$. Since $s_n \in [0,1)$, the proposition follows from Proposition 4.5. □

We also have the following proposition.

PROPOSITION 4.8

Given $\epsilon > 0$, *we have*

$$p_n^{\frac{1}{n}}(x) = x^{\frac{1}{n}} \mathrm{e}^{\frac{\pi^2}{12 \log 2}} + o(\Delta(n, \epsilon))$$

almost everywhere with respect to the Lebesgue measure on $[0, 1)\backslash\mathbf{Q}$.

Proof. This proposition follows from the inequality

$$\left| p_n(x)^{\frac{1}{n}} - (x q_n(x))^{\frac{1}{n}} \right| \leq \frac{1}{F_{n+1} F_n^{\frac{(n-1)}{n}}}$$

for all $x \in [0, 1)\backslash\mathbf{Q}$ and $n \in \mathbf{N}$, which is easily checked. □

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