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Exponential function method for solving nonlinear ordinary differential equations with constant coefficients on a semi-infinite domain

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Abstract. A new approach, named the exponential function method (EFM) is used to obtain solutions to nonlinear ordinary differential equations with constant coefficients in a semi-infinite domain. The form of the solutions of these problems is considered to be an expansion of exponential functions with unknown coefficients. The derivative and product operational matrices arising from substituting in the proposed functions convert the solutions of these problems into an iterative method for finding the unknown coefficients. The method is applied to two problems: viscous flow due to a stretching sheet with surface slip and suction; and mageto hydrodynamic (MHD) flow of an incompressible viscous fluid over a stretching sheet. The two resulting solutions are compared against some standard methods which demonstrates the validity and applicability of the new approach.

Keywords. Exponential function method; nonlinear ordinary differential equations; viscous flow; mageto hydrodynamic flow; Navier–Stokes.

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1. Introduction

Many science and engineering models have semi-infinite domains, and a quick and effective approach to finding solutions to such problems is valuable. During the last few decades, several methods have been introduced to solve nonlinear ordinary differential equations on a semi-infinite domain. Guo [12, 13] proposed a method that maps the original problem in an unbounded domain onto a problem in a bounded domain, and used suitable Jacobi polynomials to approximate the resulting functions. Boyd [6] used a domain truncation method thereby replacing the semi-infinite interval by a finite interval.

There are also effective direct approaches to solving these problems based on rational approximations. Christov [9] and Boyd [4, 5] developed spectral methods on unbounded intervals by using mutually orthogonal systems of rational functions. Boyd [4] defined a new spectral basis named rational Chebyshev functions on the semi-infinite interval by mapping to Chebyshev polynomials. Guo *et al.* [14] introduced a new system of rational

Legendre functions which are mutually orthogonal in $L^2[0, +\infty)$. They applied to a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half-line. Boyd *et al.* [7] applied pseudospectral methods on a semi-infinite interval and compared this with the rational Chebyshev, Laguerre and mapped Fourier sine methods.

A number of spectral methods for treating various problems such as semi-infinite domains problems have been proposed by different researchers [20, 21, 23].

A particularly efficient method is called the homotopy analysis method (HAM), and has been presented in [1, 16], and other related methods are given in [3, 11, 17, 18, 22, 25]. However, spectral methods often produce systems of non-linear equations which increase the complexity, and also HAM can produce extra chaotic terms that are computationally intensive.

In the present paper a linearization method is applied using exponential functions on the semi-infinite interval for solving nonlinear ordinary differential equations of the form

$$\sum_{j=0}^{k} d_j f^{(j)}(\eta) + \sum_{i,j=0}^{k} a_{ij} f^{(i)}(\eta) f^{(j)}(\eta) = 0, \tag{1}$$

where $f^{(j)}(\eta)$ is the j's derivative of f with respect to η. From this, the problems of viscous flow due to a stretching sheet with surface slip and suction, and also of MHD flow of an incompressible viscous fluid over a stretching sheet (see, for example, Wang and Chaim [8, 26]) are solved numerically.

The remainder of this paper is organized as follows: in §2 the problem is defined and the theory supporting the exponential function method; in §3 the constructions of the exponential functions operational matrices for the derivative and product are presented; in §4 the application of EFM to the semi-infinite interval is presented; and finally in §5 the application of EFM for the two fluid problems is presented.

2. Exponential functions

The exponential functions can be defined on a semi-infinite plate by

$$E_n(\eta) = \mathrm{e}^{-nL\eta}, \qquad n = 0, 1, 2, \ldots$$

where the parameter L is a positive constant parameter and it sets the length scale of the mapping. $E_n(\eta)$ satisfy the following recurrence relation:

$$E_0'(\eta) = 0, \qquad E_n'(\eta) = -nL\, E_n(\eta), \tag{2}$$

$$E_n(\eta)\, E_m(\eta) = E_{n+m}(\eta). \tag{3}$$

We are going to show that the exponential functions method is based on strong theoretical background. In fact, we want to prove why we have chosen this special basis. Denote

$$\hat{C}[0, \infty) = \{f : [0, \infty) \to \mathbb{C} \,\big|\, f \text{ is continuous and } \lim_{\eta\to\infty} f(\eta) \text{ exists}\}.$$

Also, define

$$\psi : C[0, 1] \to \hat{C}[0, \infty) \text{ by } f \mapsto f \circ g, \tag{4}$$

where $g(\eta) = \mathrm{e}^{-L\eta}$ for some positive L. So, $\psi(f)(\eta) = f(\mathrm{e}^{-L\eta}), 0 \le \eta < \infty$. Obviously $\psi(0) = 0$ and for any $f_1, f_2 \in C[0,1]$ and any complex number c, there holds $\psi(cf_1 + f_2)(\eta) = (cf_1 + f_2)(\mathrm{e}^{-L\eta}) = c\psi(f_1) + \psi(f_2)$, hence ψ is linear. Let ω be a positive measure on $[0,\infty)$ such that $\mathrm{d}\omega = \mathrm{e}^{-L\eta}\mathrm{d}\eta$, then

$$\begin{aligned} \langle \psi(f_1), \psi(f_2)\rangle_\omega &= \int_0^\infty f_1 \circ g(\eta)\, \overline{f_2 \circ g(\eta)}\, \mathrm{e}^{-L\eta}\mathrm{d}\eta \\ &= \frac{1}{L}\int_0^1 f_1(t)\, \overline{f_2(t)}\mathrm{d}t = \frac{1}{L}\langle f_1, f_2\rangle_{C[0,1]}, \end{aligned}$$

shows that ψ is a preserving inner product as well. The kernel of ψ is 0 and for any $\phi \in \hat{C}[0,\infty)$, $\psi(\alpha) = \phi$, where

$$\alpha(t) = \begin{cases} \phi(\frac{-\ln(t)}{L}), & 0 < t \le 1, \\ \lim\limits_{\eta\to\infty} \phi(\eta), & t = 0. \end{cases}$$

Therefore ψ is an isometry.

According to Stone–Weierstrass theorem [24] for any complex continuous function f defined on $[0,1]$, there exists a sequence of polynomials, converging to f, subject to the sup norm. So, span $\{f_n, n = 0, 1, 2, \ldots\}$ where $f_n(x) = x^n$ is dense in $C[0,1]$. We proved that ψ in (4) is an isometry, so $C[0,1]$ is isometric with $\hat{C}[0,\infty)$, therefore, span $\{\psi(f_n), n = 0, 1, 2, \ldots\}$ would also be dense in $\hat{C}[0,\infty)$. Since

$$\psi(f_n)(\eta) = f_n(\mathrm{e}^{-L\eta}) = (\mathrm{e}^{-L\eta})^n = \mathrm{e}^{-nL\eta},$$

therefore, span $\{\mathrm{e}^{-nL\eta}, n = 0, 1, 2, \ldots\}$ is dense in $\hat{C}[0,\infty)$, and now we ensure that any continuous function that has a limit at infinity, can be considered as limit of a sequence in span $\{\mathrm{e}^{-nL\eta},\ n = 0, 1, 2, \ldots\}$. On the other hand, any $f \in \hat{C}[0,\infty)$ can be expressed in exponential expansion

$$f(\eta) = \sum_{n=0}^{+\infty} a_n \mathrm{e}^{-Ln\eta}, \quad a_n \in \mathbb{C}, \tag{5}$$

which guarantees that if f is the solution of an equation in $\hat{C}[0,\infty)$, then $\sum_{n=0}^{N} a_n \mathrm{e}^{-Ln\eta}$ practically converges to $f(\eta)$ as N grows up.

Now we proceed the arithmetic computations.

2.1 *Function approximation*

Let $f \in \hat{C}[0,\infty)$, consider its expansion as shown in (5), denote

$$A = [a_0, a_1, a_2, \ldots]^T, \qquad E(\eta) = [1, \mathrm{e}^{-L\eta}, \mathrm{e}^{-2L\eta}, \ldots]^T, \tag{6}$$

and then we can reform (5) in matrix form as follows

$$f(\eta) = \sum_{n=0}^{+\infty} a_n \mathrm{e}^{-Ln\eta} = A^T E(\eta) = E^T(\eta) A. \tag{7}$$

The right-hand side of (7) says $f = f^T$ and this is because f is a one-dimensional function. Since,

$$\begin{aligned}\langle \mathrm{e}^{-nL\eta}, \mathrm{e}^{-mL\eta}\rangle_\omega &= \int_0^\infty \mathrm{e}^{-nL\eta}\overline{\mathrm{e}^{-mL\eta}}\mathrm{e}^{-L\eta}\mathrm{d}\eta \\ &= \frac{1}{L(m+n+1)}, \qquad m, n = 0, 1, 2, ...,\end{aligned}$$

then

$$L\langle E(\eta), E^T(\eta)\rangle_\omega = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which conventionally is the Hilbert matrix H and so,

$$L\langle f(\eta), E^T(\eta)\rangle_\omega = A^T H,$$

which yields

$$A^T = L\langle f(\eta), E^T(\eta)\rangle_\omega H^{-1}.$$

However the cost of computing the Hilbert matrix inverse is expensive, but also for a finite approximation its entries can be expressed in closed form using binomial coefficients as follows:

$$\begin{aligned}(H^{-1})_{nm} &= (-1)^{n+m}(n+m+1)\binom{N+n}{N-m+1}\binom{N+m}{N-n+1}\binom{n+m}{n}^2, \\ &\qquad n, m = 0, 1, 2, \ldots,\end{aligned}$$

where N is the order of the matrix. Recent formula shows that all entries of the inverse matrix are integer numbers. The Hilbert matrices are known as canonical examples of ill-conditioned matrices, making them notoriously difficult to use in numerical computation, but in our method we have skillfully avoided this complexity as has been explained in §4.

3. EFM operational matrix

3.1 *Derivative matrix for exponential functions*

The derivative of the vector $E(\eta)$ defined in eq. (6) can be expressed as

$$E'(\eta) = \frac{\mathrm{d}E}{\mathrm{d}\eta} = DE(\eta), \tag{8}$$

where D is the operational matrix for the derivative. Applying eq. (2), it is then deduced that

$$(D)_{nm} = -nL\delta_{nm}, \qquad n, m = 0, 1, 2, \ldots.$$

Remark 1. In general, for the j-order ($j = 0, 1, 2, \ldots$) derivative of the function $f \in \hat{C}[0, \infty)$, we have

$$f^{(j)}(\eta) = A^T D^j E(\eta), \qquad (D^j)_{nm} = (-nL)^j \delta_{nm}, \qquad n, m = 0, 1, 2, \ldots. \tag{9}$$

Thus, all derivative matrices of exponential functions are diagonal which can efficiently decrease the amount of computation for solving differential equations.

3.2 *The product operational matrix*

Again with respect to (6) and (7), we have

$$\begin{aligned} E(\eta)E^T(\eta)A &= \begin{pmatrix} E^T(\eta) \\ \mathrm{e}^{-L\eta}E^T(\eta) \\ \mathrm{e}^{-2L\eta}E^T(\eta) \\ \vdots \end{pmatrix} A = \begin{pmatrix} E^T(\eta)A \\ \mathrm{e}^{-L\eta}E^T(\eta)A \\ \mathrm{e}^{-2L\eta}E^T(\eta)A \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} A^T E(\eta) \\ \mathrm{e}^{-L\eta}A^T E(\eta) \\ \mathrm{e}^{-2L\eta}A^T E(\eta) \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} a_0 + a_1 e^{-L\eta} + a_2 e^{-2L\eta} + a_3 e^{-3L\eta} + \ldots \\ a_0 e^{-L\eta} + a_1 e^{-2L\eta} + a_2 e^{-3L\eta} + \ldots \\ a_0 e^{-2L\eta} + a_1 e^{-3L\eta} + a_2 e^{-4L\eta} + \ldots \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \ldots \\ 0 & a_0 & a_1 & a_2 & \ldots \\ 0 & 0 & a_0 & a_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} E(\eta) \\ &= \tilde{A}E(\eta), \text{ where } (\tilde{A})_{nm} = \begin{cases} a_{m-n}, & m \geq n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{10}$$

If g is also a function that belongs to $\hat{C}[0, \infty)$, according to (7) we can write $g(\eta) = \sum_{n=0}^{\infty} b_n \mathrm{e}^{-Ln\eta} = B^T E(\eta) = E^T(\eta)B$, where $B^T = [b_0, b_1, b_2, \ldots]$, thus by (10) we have

$$f(\eta)g(\eta) = A^T E(\eta)E^T(\eta)B = A^T \tilde{B} E(\eta). \tag{11}$$

Remark 2. Equation (9) and (7) yield $f^{(i)}(\eta) = (f^{(i)}(\eta))^T = (A^T D^i E(\eta))^T = D^i E^T(\eta)A$, and since $f^{(j)},\ f^{(i)} \in \hat{C}[0, \infty)$, similar to (11) we will have

$$f^{(j)}(\eta)f^{(i)}(\eta) = A^T D^j E(\eta)D^i E^T(\eta)A = A^T D^j \tilde{A}_i E(\eta), \tag{12}$$

where D^0 is the identity matrix, $A_i = D^i A$ and $\tilde{A}_i$'s are as follows:

$$\tilde{A}_0 = \tilde{A},\ \ A_i = (-L)^i \begin{pmatrix} 0 \\ a_1 \\ 2^i a_2 \\ 3^i a_3 \\ \vdots \end{pmatrix},$$

$$\tilde{A}_i = (-L)^i \begin{pmatrix} 0 & a_1 & 2^i a_2 & 3^i a_3 & 4^i a_4 & \dots \\ 0 & 0 & a_1 & 2^i a_2 & 3^i a_3 & \dots \\ 0 & 0 & 0 & a_1 & 2^i a_2 & \dots \\ 0 & 0 & 0 & 0 & a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad i \geq 1,$$

$$(\tilde{A}_i)_{nm} = (-L)^i \begin{cases} (m-n)^i a_{m-n}\,, & m > n, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Also, for k-power of function $f(\eta)$, by induction, we have

$$\begin{aligned} f^k(\eta) &= f^2(\eta) f^{k-2}(\eta) = A^T \tilde{A} E(\eta) f^{k-2}(\eta) \\ &= A^T \tilde{A}^2 E(\eta) f^{k-3}(\eta) = \cdots = A^T \tilde{A}^{k-1} E(\eta) \end{aligned}$$

As is seen above, the product operational matrix is an upper triangular matrix which converts a nonlinear problem into a linear form.

4. Applications of the EFM

Here, we consider the nonlinear ordinary differential equations of the form

$$\sum_{j=0}^{k} d_j f^{(j)}(\eta) + \sum_{i,j=0}^{k} \mathrm{e}_{ij} f^{(i)}(\eta) f^{(j)}(\eta) = 0, \qquad \eta \in [0, \infty). \tag{14}$$

Combining former expansion

$$f(\eta) = \sum_{n=0}^{+\infty} a_n \mathrm{e}^{-nL\eta} \tag{15}$$

together with identity (12), gives an operational form for eq. (14) as follows:

$$\sum_{j=0}^{k} d_j A^T D^j E(\eta) + \sum_{i,j=0}^{k} e_{ij} A^T D^j \tilde{A}_i E(\eta) = 0\,. \tag{16}$$

Imposing inner product to the above equation by $E^T(\eta)$, results in

$$A^T \left[\sum_{j=0}^{k} d_j D^j + \sum_{i,j=0}^{k} e_{ij} D^j \tilde{A}_i \right] \langle E(\eta), E^T(\eta) \rangle_\omega$$

$$= A^T \left[\sum_{i=0}^{k} d_i D^i + \sum_{i,j=0}^{k} e_{ij} D^j \tilde{A}_i \right] H = \mathbf{0}^T.$$

It is known that the Hilbert matrix H is non-singular, so multiplying H^{-1} will reform to

$$A^T \left[\sum_{j=0}^{k} d_j D^j + \sum_{i,j=0}^{k} e_{ij} D^j \tilde{A}_i \right] = A^T \mathbf{B}^T = \mathbf{0}^T, \tag{17}$$

which does not involve the Hilbert matrix.

By using eqs (9), (10) and (13), $D^j \tilde{A}_i$ is computed as

$$D^j \tilde{A}_0 = D^j \tilde{A} = (-L)^j \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & 0 & 2^j a_0 & 2^j a_1 & 2^j a_2 & \dots \\ 0 & 0 & 0 & 3^j a_0 & 3^j a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}, \quad j \geq 1, \tag{18}$$

$$(D^j \tilde{A})_{nm} = (-L)^j \begin{cases} n^j a_{m-n}, & m \geq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} &D^0 \tilde{A}_i = \tilde{A}_i, \qquad i \geq 1, \\ &D^0 \tilde{A}_0 = \tilde{A}. \end{aligned} \tag{19}$$

$$D^j \tilde{A}_i = (-L)^{i+j} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & a_1 & 2^i a_2 & 3^i a_3 & 4^i a_4 & \dots \\ 0 & 0 & 0 & 2^j a_1 & 2^j 2^i a_2 & 2^j 3^i a_3 & \dots \\ 0 & 0 & 0 & 0 & 3^j a_1 & 3^j 2^i a_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad i, j \geq 1,$$

$$(D^j \tilde{A}_i)_{nm} = (-L)^{i+j} \begin{cases} n^j (m-n)^i a_{m-n}, & m > n, \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Now we can rewrite eq. (17) as

$$\mathbf{B}^T = e_{00} \tilde{A} + d_0 I + \sum_{j=1}^{k} D^j \left(e_{0j} \tilde{A} + d_j I \right) + \sum_{i=1}^{k} e_{i0} \tilde{A}_i + \sum_{i,j=1}^{k} e_{ij} D^j \tilde{A}_i.$$

The entries $\mathbf{b}_{nm}$ of $\mathbf{B}$ are obtained by using eqs (10), (13), (20) and (18) as

$$\mathbf{b}_{nm} = \begin{cases} \sum\limits_{j=0}^{k} (-nL)^j \left(a_0 e_{0j} + d_j \right), & n = m, \\ a_{n-m} \sum\limits_{i,j=0}^{k} (-L)^{i+j} m^j (n-m)^i e_{ij}, & n > m, \\ 0, & m > n. \end{cases} \tag{21}$$

Therefore, we can write

$$\sum_{m=0}^{\infty} \mathbf{b}_{nm} a_m = \sum_{m=0}^{n} \mathbf{b}_{nm} a_m = \mathbf{b}_{n0} a_0 + \mathbf{b}_{nn} a_n + \sum_{m=1}^{n-1} \mathbf{b}_{nm} a_m$$

$$= a_n\left(a_0\sum_{i=0}^{k}(-nL)^i e_{i0} + \mathbf{b}_{nn}\right) + \sum_{m=1}^{n-1}\mathbf{b}_{nm}a_m = 0,$$

$$a_n = -\frac{1}{a_0\sum_{i=0}^{k}(-nL)^i e_{i0} + \mathbf{b}_{nn}}\sum_{m=1}^{n-1}\mathbf{b}_{nm}a_m\ , \qquad n \geq 2. \tag{22}$$

As seen in the above equation, the coefficients a_n can be obtained iteratively from a_1 and L. The following conditions occur for a_0:

(i) If either d_0 or e_{00} is equal to zero, then $a_0 = 0$.
(ii) If both d_0, and e_{00} are nonzero, then

$$a_0 = -\frac{d_0}{e_{00}}.$$

(iii) Else ($d_0 = e_{00} = 0$),

$$a_0 = \frac{-\sum_{j=0}^{k}(-L)^j d_j}{\sum_{j=0}^{k}(-L)^j(e_{j0} + e_{0j})}.$$

As long as the series $\sum_{n=0}^{+\infty} a_n \mathrm{e}^{-nL\eta}$ converges, the general solution of eq. (14) will be obtained by replacing a_0 in eq. (22) . From the boundary conditions of a problem, determining parameters L and a_1 will give a particular solution.

4.1 *Error and stop analysis of EFM*

For an $N+1$-term truncation approximation of (15) namely

$$f_N(\eta) = \sum_{n=0}^{N} a_n \mathrm{e}^{-nL\eta},$$

we may check the error of the method from the residual function by using eq. (14) as follows:

$$R(\eta) = \sum_{j=0}^{k} d_j f_N^{(j)}(\eta) + \sum_{i,j=0}^{k} e_{ij} f_N^{(i)}(\eta) f_N^{(j)}(\eta). \tag{23}$$

Now compute the error

$$\|R(\eta)\|_\omega = \left(\int_0^{+\infty} |R(\eta)|^2\, \omega(\eta)\, \mathrm{d}\eta\right)^{1/2}, \tag{24}$$

as long as the desired approximation is reached.

As long as the series (15) converges, we can determine the valid N for an appropriate error, $\varepsilon > 0$, by applying the Cauchy condition

$$\| f_N(\eta) - f_M(\eta)\|_{\omega}^2 \leq \varepsilon, \quad N,\ M \geq N_0,\ \text{for some fixed integer number } N_0.$$

Let us consider $M = N - 1$, then one has

$$|a_N|^2 \int_0^{+\infty} \mathrm{e}^{-(2N+1)L\eta}\,\mathrm{d}\eta \leq \varepsilon \Longrightarrow \frac{{a_N}^2}{(2N+1)L} \leq \varepsilon$$
$$\Longrightarrow N \geq \frac{{a_N}^2}{2L\varepsilon} - \frac{1}{2}, \tag{25}$$

which is an essential criteria to stop computations. Obviously, such as spectral method, whatever the number of sentences of series increases, the approximated solution will be more accurate. By the way, we offer a criteria to stop computations in term of an arbitrary Cauchy error ε and L and a_n. According to eq. (25), we can derive that the approximated solutions with big L have faster convergence.

5. Applications

The first problem considers viscous flow due to a stretching sheet with surface slip and suction studied by Wang [26] and the second problem arises from MHD flow over a non-linear stretching sheet [2, 8, 15, 19].

5.1 *A brief introduction to problem* 1

In this section, we shall obtain the solution for the model studied by Wang [26]. This model is an extension to Crane's solution [10] to include both suction and slip for two-dimensional stretching and axisymmetric stretching. Uniqueness of the solution was shown by Wang [26].

5.2 *Formulation*

Consider the velocity on the stretching surface at $z = 0$ to be

$$u = ax, \qquad v = (\mathfrak{m} - 1)ay, \qquad w = 0,$$

where $a > 0$ is the stretching rate, (u, v, w) is the velocity vector in the Cartesian (x, y, z) coordinates respectively. Two-dimensional stretching occurs when $\mathfrak{m} = 1$, and axisymmetric stretching occurs when $\mathfrak{m} = 2$. Axisymmetric stretching occurs on surfaces such as expanding balloons. The similarity transform is

$$u = axf'(\eta), \quad v = (\mathfrak{m} - 1)ayf'(\eta), \quad w = -\mathfrak{m}\sqrt{a\nu}\, f(\eta), \quad \eta = z\sqrt{\frac{a}{\nu}}, \tag{26}$$

where ν is the kinematic viscosity of the fluid. The continuity equation is satisfied and the Navier–Stokes equation becomes

$$f'''(\eta) + \mathfrak{m} f f''(\eta) - f'(\eta)^2 = 0. \tag{27}$$

Here we have used the fact that there is no lateral pressure gradient at infinity. On the surface, the velocity slip is assumed to be proportional to the local shear stress (Navier's condition)

$$u - ax = N\rho\nu\frac{\partial u}{\partial z} < 0, \qquad v - (\mathfrak{m}-1)ay = N\rho\nu\frac{\partial v}{\partial z} < 0, \tag{28}$$

where ρ is the density and N is a slip constant. Equation (28) can be rewritten as

$$f'(0) = 1 + \lambda f''(0), \tag{29}$$

where $\lambda = N\rho\sqrt{a\nu} > 0$ is a non-dimensional slip factor. Note that from eq. (28), $f''(0) < 0$. If there is suction velocity of $-w$ on the surface, the boundary condition is

$$f(0) = s, \tag{30}$$

where the suction factor is $s = w/(\mathfrak{m}\sqrt{a\nu})$. The third boundary condition is that there is no lateral velocity at infinity

$$\lim_{\eta\to\infty} f'(\eta) = 0. \tag{31}$$

We now use EFM to solve eqs (27) considering its conditions (29)–(31), and let $C = \lim_{\eta\to\infty} f(\eta)$.

5.3 *Applying EFM for solving the model*

In the first step of our analysis, we replace $f(\eta)$ and its derivatives for solving the model by exponential functions as follows:

$$\begin{aligned} f(\eta) &= A^T E(\eta), & f'(\eta) &= A^T D E(\eta),\\ f''(\eta) &= A^T D^2 E(\eta), & f'''(\eta) &= A^T D^3 E(\eta). \end{aligned} \tag{32}$$

Using eq. (10), we have

$$f(\eta)\, f''(\eta) = A^T D^2 E(\eta) E^T(\eta) A = A^T D^2 \tilde{A} E(\eta), \tag{33}$$

$$f'^2(\eta) = A^T D E(\eta) E^T(\eta) D A = A^T D \tilde{A}_1 E(\eta), \tag{34}$$

where $A_1 = DA$, and

$$D^2\tilde{A} = L^2 \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & a_0 & a_1 & \dots & a_{n-1} & \dots \\ 0 & 0 & 4a_0 & \dots & 4a_{n-2} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n^2 a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$D\tilde{A}_1 = L^2 \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & a_1 & 2a_2 & \dots & (n-1)a_{n-1} & \dots \\ 0 & 0 & 0 & 2a_1 & \dots & 2(n-2)a_{n-2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}.$$

The residual function is constructed by substituting eqs (32), (33) and (34) in eq. (27):

$$R(\eta) = A^T \left(D^3 + \mathfrak{m}\, D^2 \tilde{A} - \ D\tilde{A_1}\right) E(\eta) = L^2 A^T B^T E(\eta),$$

where entries of B have been obtained as follows:

$$b_{nm} = \begin{cases} 0, & (n = m = 0) \ \texttt{or} \ (m > n), \\ \mathfrak{m} m^2 a_0 - L m^3, & n = m \neq 0, \\ [\mathfrak{m} m^2 - m(n-m)] a_{n-m}\,, & n > m. \end{cases} \tag{35}$$

The residual function $R(\eta)$ should be equal to zero, therefore, one has

$$0 = \langle R(\eta), E^T(\eta)\rangle_{w(\eta)} = L A^T B^T H,$$

which yields $BA = 0$. Therefore, we can write

$$\begin{aligned} \sum_{m=0}^{\infty} b_{nm} a_m &= \sum_{m=1}^{n} b_{nm} a_m = 0 \Longrightarrow \mathfrak{m} n^2 a_0 a_n + \sum_{m=1}^{n-1} b_{nm} a_m \\ &= L n^3 a_n \Longrightarrow \sum_{m=1}^{n} [\mathfrak{m} m^2 - m(n-m)] a_{n-m} a_m \\ &= L n^3 a_n\,. \end{aligned} \tag{36}$$

Now when $n = m = 1$, and by assuming on coefficient a_n, say $a_1 \neq 0$, one has

$$\mathfrak{m}\, a_0 a_1 = L a_1 \ \Longrightarrow \ a_0 = \frac{L}{\mathfrak{m}}. \tag{37}$$

Subsequently, the other unknown coefficients $a_n,\ n \geq 2$ obtained from a_1 and L using eqs (36) and (37) are

$$\begin{aligned} \sum_{m=1}^{n-1} b_{nm} a_m &= n^2 a_n \left(L n - \mathfrak{m}\, \frac{L}{\mathfrak{m}}\right) = L n^2 a_n (n-1), \\ a_n &= \frac{1}{L n^2 (n-1)} \sum_{m=1}^{n-1} [(\mathfrak{m}+1) m^2 - n m] a_{n-m} a_m\,, \quad n \geq 2. \end{aligned} \tag{38}$$

The general solution for eq. (27) is now obtained from (37) and (38) in the following form:

$$f(\eta) = \frac{L}{\mathfrak{m}} + a_1 \mathrm{e}^{-L\eta} + \sum_{n=2}^{\infty} a_n \mathrm{e}^{-nL\eta}, \tag{39}$$

and hence $C = L/\mathfrak{m}$. Particular solution for this problem will be achieved by using the boundary conditions (29) and (30) to get the following equations:

$$\begin{cases} f(0) = \sum\limits_{m=0}^{n} a_m = s, \\ f'(0) = -L \sum\limits_{m=0}^{n} m a_m = 1 + \lambda L^2 \sum\limits_{m=0}^{n} m^2 a_m\,, \end{cases}$$

which will give us the special parameters L and a_1. The above system can simply be solved by the Newton method. Thus, all coefficients $a_n,\ n \geq 2$ are obtained iteratively. It is worth mentioning that the third boundary conditions (31), ($f'(\infty) = 0$) satisfy eq. (39).

Two-dimensional case

For $m = 1$, eq. (38) becomes

$$a_n = \frac{1}{Ln^2(n-1)} \sum_{m=1}^{n-1} (2m^2 - nm) a_{n-m} a_m\ , \qquad n \geq 2.$$

For a_2, one has

$$a_2 = \frac{1}{4L}(2-2){a_1}^2 = 0.$$

Consequently, for any $n \geq 2$, we have $a_n = 0$, and the exact solution by using the boundary conditions (29) and (30) can be obtained as

$$f(\eta) = L + (s - L)\mathrm{e}^{-L\eta}, \tag{40}$$

where L is the positive root of the following equation:

$$\lambda L^3 + (1 - \lambda s)L^2 - sL - 1 = 0. \tag{41}$$

The solutions exist for all s and all $\lambda \geq 0$.

Figure 1 displays typical curves for $f'(\eta)$ which represents lateral velocity. It is seen that for increased slip λ the lateral velocity decreases near the surface but increases at larger distances. Figure 2 shows the normal velocity for suction ($s > 0$) represented by $f(\eta)$ which is always towards the stretching surface, but for injection ($s < 0$), the normal

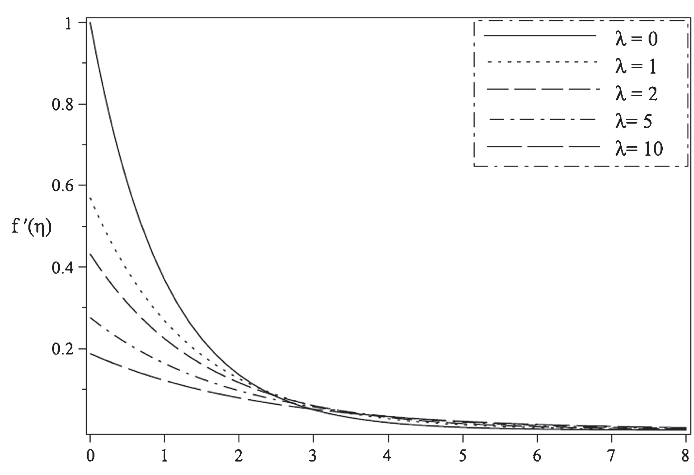

Figure 1. Graph of the function $f'(\eta)$ for two-dimensional case, $s = 0$.

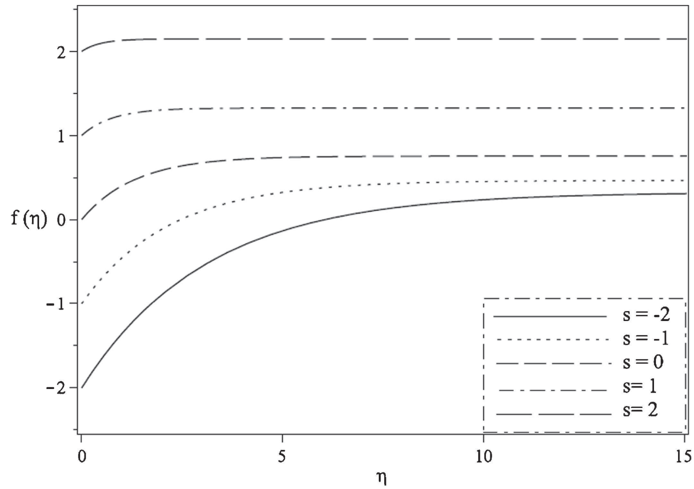

Figure 2. Graph of the function $f(\eta)$ for two-dimensional case, $\lambda = 1$.

velocity is zero at some finite distance η. Also note that the boundary layer thickness is larger for small s.

The axisymmetric case

For $m = 2$, eq. (38) becomes

$$a_n = \frac{1}{Ln^2(n-1)} \sum_{m=1}^{n-1} (3m^2 - nm) a_{n-m} a_m \,, \qquad n \geq 2.$$

The coefficients $a_n,\ n \geq 2$ are obtained by a_1 and L as follows

$$a_2 = \frac{{a_1}^2}{4L}, \quad a_3 = \frac{{a_1}^3}{12L^2}, \quad a_4 = \frac{17{a_1}^4}{576L^3}, \quad a_5 = \frac{61{a_1}^5}{5760L^4}, \quad a_6 = \frac{73{a_1}^6}{19200L^5}, \ldots$$

The initial value $f''(0)$ for some s and λ in comparison with solutions of the shooting method which is used and applied by the authors in ref. [26] are shown in Table 1. The final value C is given in Table 2. The residual errors of this problem for various values of λ and N have been displayed in Table 3. Also the rate of convergence using the residual error for the problems are obtained as follows

$$\text{rate} = \log_{\frac{m}{n}} \left(\frac{e_n}{e_m} \right),$$

where e_n denotes the error $\|R(\eta)\|_\omega$ for approximation $f_n(\eta)$. This table demonstrates the reliability and efficiency of the method.

5.4 *A brief introduction to problem* 2

In this section, we also wish to find the EFM solution for the model studied by Chaim [8]. This model considers the MHD flow of an incompressible viscous fluid over a stretching sheet.

Table 1. The initial value $f''(0)$, axisymmetric case.

λ	Method	$s = 3$	$s = 2$	$s = 1$	$s = 0$	$s = -0.5$
0	EFM	−6.23939791	−4.34248658	−2.57031866	−1.17372073	−0.73825524
	Shooting	−6.2394	−4.3425	−2.5703	−1.1737	−0.7382
0.5	EFM	−1.50380385	−1.34626285	−1.06961564	−0.65052766	−0.46510104
	Shooting	−1.5038	−1.3463	−1.0696	−0.6505	−0.4651
1	EFM	−0.85785973	−0.80285739	−0.68841453	−0.46250964	−0.34824701
	Shooting	−0.8579 +	−0.8029	−0.6884	−0.4625	−0.3482
3	EFM	−0.31582568	−0.30785946	−0.28762542	−0.22312755	−0.18181655
	Shooting	−0.3158	−0.3079	−0.2876	−0.2231	−0.1818
10	EFM	−0.09836175	−0.09756638	−0.09531501	−0.08291164	−0.07231371
	Shooting	−0.09836	−0.09757	−0.09532	−0.08291	−0.07231

Table 2. The value of C as a function of s and λ, axisymmetric case.

λ	Method	$s = 3$	$s = 2$	$s = 1$	$s = 0$	$s = -0.5$
0	EFM	3.159264088	2.227357515	1.375936041	0.751497028	0.578063367
	Shooting	3.1592	2.2274	1.3759	0.7515	0.5781
0.5	EFM	3.040862590	2.078986410	1.198167041	0.617297138	0.490823463
	Shooting	3.0409	2.0790	1.1982	0.6173	0.4908
1	EFM	3.023528564	2.048266155	1.138889276	0.550950530	0.443101403
	Shooting	3.0235	2.0483	1.1389	0.5510	0.4431
3	EFM	3.008731587	2.018948310	1.064876361	0.432105781	0.352461316
	Shooting	3.0087	2.0190	1.0649	0.4321	0.3525
10	EFM	3.002728238	2.006067944	1.022963411	0.310654396	0.255606558
	Shooting	3.0027	2.0061	1.0230	0.3107	0.2556

Table 3. The residual errors $\|R(\eta)\|_\omega$ for the axisymmetric case with $s = -0.5$ and various values of λ and N.

λ	$N = 5$	$N = 10$	$N = 20$	$N = 40$	$N = 60$	$N = 80$	$N = 100$
0	0.65959	0.04582	0.00004	1.61×10^{-11}	5.52×10^{-18}	3.91×10^{-25}	2.19×10^{-32}
Rate	–	3.84	10.16	21.24	36.71	57.22	74.83
0.5	0.65240	0.07348	0.00018	2.02×10^{-10}	2.26×10^{-16}	6.26×10^{-23}	2.57×10^{-29}
Rate	–	3.15	8.67	19.76	33.80	52.48	65.90
1	0.69429	0.10072	0.00050	2.20×10^{-9}	5.04×10^{-15}	8.62×10^{-21}	1.52×10^{-26}
Rate	–	2.78	7.65	17.92	31.82	46.16	59.37
3	0.64456	0.18603	0.00684	8.58×10^{-7}	3.88×10^{-11}	1.59×10^{-15}	1.14×10^{-20}
Rate	–	1.79	4.76	12.96	24.67	35.12	53.08
5	0.59087	0.07152	0.09768	3.47×10^{-5}	1.29×10^{-8}	6.78×10^{-13}	5.39×10^{-16}
Rate	–	3.04	−0.45	11.46	19.48	34.25	31.98

5.5 *Formulation*

Let us consider the MHD flow of an incompressible viscous fluid over a stretching sheet at $y = 0$. The fluid is electrically conducting under the influence of an applied magnetic field $B(x)$ normal to the stretching sheet. The induced magnetic field is neglected. Let (u, v) be the fluid velocities in the (x, y) directions, then the resulting boundary layer equations are

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial f^2} - \frac{\sigma B^2(x)}{\rho}u, \end{aligned} \tag{42}$$

where ρ is the fluid density, σ is the electrical conductivity of the fluid and the external electric field and the polarization effects are negligible and $B(x)$ is assumed by Chaim [8] as

$$B(x) = B_0 x^{\frac{n-1}{2}}. \tag{43}$$

The boundary conditions corresponding to the non-linear stretching of a sheet are

$$u(x, 0) = cx^n, \quad v(x, 0) = 0, \quad \lim_{y\to\infty} u(x, y) = 0. \tag{44}$$

The similarity transformation is

$$\begin{aligned} \eta &= \sqrt{\frac{c(n+1)}{2\nu}}x^{\frac{n-1}{2}}y, \quad u = cx^n f'(\eta), \\ v &= -\sqrt{\frac{c\nu(n+1)}{2}}x^{\frac{n-1}{2}}\left[f(\eta) + \frac{n-1}{n+1}\eta f'(\eta)\right]. \end{aligned} \tag{45}$$

The continuity equation is automatically satisfied. Using eqs (43)–(45), eq. (42) reduces to the non-linear differential equation with boundary conditions of the form

$$f'''(\eta) + f(\eta)f''(\eta) - \beta f'(\eta)^2 - Mf'(\eta) = 0, \tag{46}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \tag{47}$$

where

$$\beta = \frac{2n}{n+1}, \qquad M = \frac{2\sigma B_0^2}{\rho c(1+n)}. \tag{48}$$

5.6 *Applying EFM for solving the model* 2

As with the previous section, the entries of B will be deduced as follows:

$$b_{nm} = \begin{cases} 0, & (n = m = 0) \text{ or } (m > n) \\ Mm - L^2m^3, & n = m \neq 0 \\ L[(n-m)^2 - \beta m(n-m)]a_{n-m}, & n > m. \end{cases} \tag{49}$$

Now when $n = m = 1$, and by assuming $a_1 \neq 0$, one has

$$La_0a_1 + (M - L^2)a_1 = 0 \implies a_0 = \frac{L^2 - M}{L}. \tag{50}$$

Consequently, the other unknown coefficients $a_n,\ n \geq 2$ by using eqs (22) and (50), will be found using

$$\sum_{m=1}^{n-1} b_{nm} a_m = n(n-1)(nL^2+M)\, a_n,$$

$$a_n = \frac{L}{n(n-1)(nL^2+M)} \sum_{m=1}^{n-1} [(n-m)^2 - \beta m(n-m)] a_{n-m} a_m. \qquad (51)$$

Table 4. Comparison of the numerical value of $f''(0)$, obtained by EFM.

β	M	Croco transformation [8]	Shooting method [8]	Modified ADM [15]	HPM [19]	EFM	$\|R(\eta)\|_\omega$
1.5	0	−1.14902	−1.14860	−1.14902	−1.1486	−1.1485932051	1.38×10^{-11}
	1	−1.5253	−1.52527	−1.5253	−1.5252	−1.5252747637	7.06×10^{-15}
	5	−2.94150	−2.94144	−2.94142	−2.5161	−2.5161546412	1.72×10^{-19}
	10	−3.69567	−3.6956	−3.6956	−3.3663	−3.3663148778	5.82×10^{-22}
	50	−7.32561	−7.3256	−7.3256	−7.1647	−7.1647094332	3.16×10^{-24}
	100	10.1816	−10.1816	−10.1816	−10.0664	−10.066439086	1.45×10^{-28}
5	0	−1.90433	−1.9025	−1.9031	−1.9025	−1.9025048006	1.02×10^{-10}
	1	−2.15344	−2.1529	−2.1529	−2.1529	−2.1528632536	1.35×10^{-11}
	5	−2.94150	−2.94144	−2.94142	−2.9414	−2.9414368451	3.80×10^{-18}
	10	−3.69567	−3.6956	−3.6956	−3.6956	−3.6956559936	3.74×10^{-20}
	50	−7.32561	−7.3256	−7.3256	−7.3256	−7.3256096800	6.45×10^{-19}
	100	−10.1816	−10.1816	−10.1816	−10.1816	−10.181629758	8.27×10^{-29}

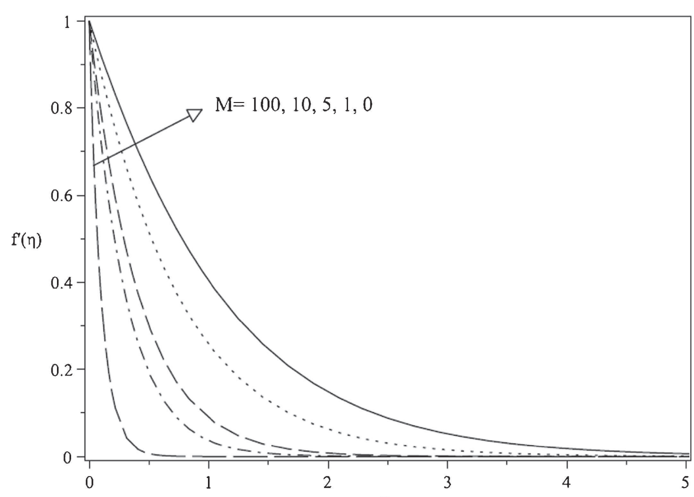

Figure 3. Variation of $f'(\eta)$ for different values of M with $\beta = 0.5$.

Using the boundary conditions (47), the particular solution of this problem can be obtained by solving the following equations to find L and a_1:

$$\begin{cases} f(0) = \sum\limits_{m=0}^{n} a_m = 0, \\ f'(0) = -L \sum\limits_{m=0}^{n} m a_m = 1. \end{cases}$$

For the special case of $\beta = 1$, one has

$$a_2 = \frac{L}{4L^2 + 2M}(1-1){a_1}^2 = 0.$$

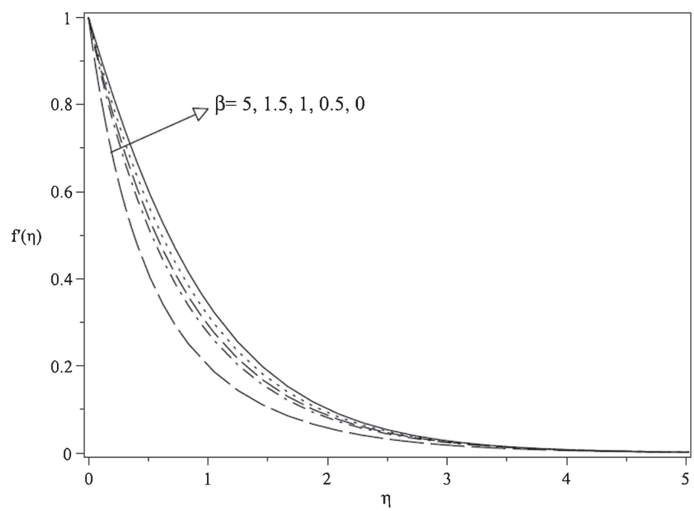

Figure 4. Variation of $f'(\eta)$ for different values of β with $M = 0.5$.

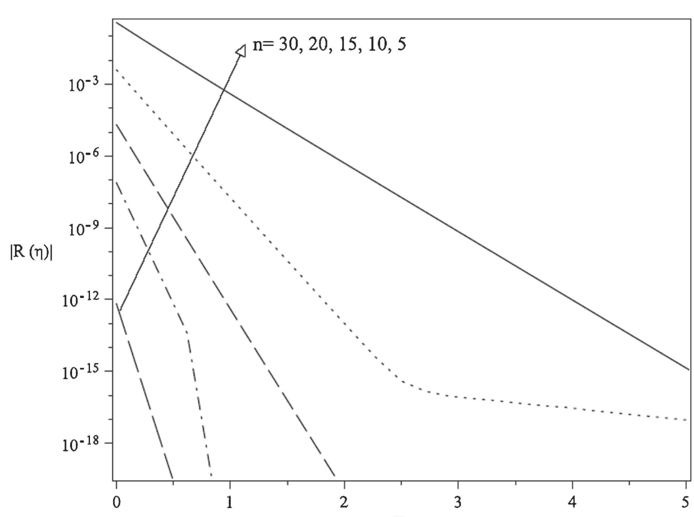

Figure 5. $|R(\eta)|$ for various iterations with $\beta = 5$ and $M = 0.5$.

Subsequently, for any $n \geq 2$, we have $a_n = 0$. Thus the solution is obtained using EFM as

$$f(\eta) = a_0 + a_1 \exp(-L\eta).$$

Using the problem conditions (47), a_1 and L obtained are

$$a_1 = \frac{-1}{\sqrt{M+1}}, \qquad L = \sqrt{M+1}.$$

which are reduced to the exact solution of the problem

$$f(\eta) = \frac{1 - \exp(-\sqrt{M+1}\eta)}{\sqrt{M+1}}.$$

Table 4 clearly reveals that the new solution method shows excellent agreement with the existing solutions in the literatures [8, 15, 19]. Also from the last column of this table, the residual error $\|R(\eta)\|_\omega$ demonstrates the rapid convergence and efficiency of the method. In figures 3 and 4, the variations of $f'(\eta)$ approximated by the new method for some typical problem's parameters are plotted. Figure 5 illustrates function $|R(\eta)|$ for various iterations. It is evident that this method is convergent according to norm-∞.

6. Conclusion

Solutions based on the exponential functions methods (EFM) have been presented for nonlinear differential equations in semi-interval arising from two models of viscous flow due to a stretching sheet. The first model is Crane's solution extended to include both suction and slip for two-dimensional stretching and axisymmetric stretching for which the uniqueness of the solution was shown by Wang. The other is the MHD flow over a non-linear stretching sheet. The method presented in this paper uses a set of functions which solve the problems on the whole domain without requiring small parameters, truncating it to a finite domain, imposing the asymptotic condition, and transforming the domain of the problem.

These functions are proposed to provide an effective but simple way to improve the convergence of the solution by an iterative method. The validity of the method is based on the assumption that it converges by increasing the number of iterations. Comparisons made among the numerical solutions, demonstrate rapid convergence and numerical efficiency of the new method for these kind of problems.

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