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On the normality of orbit closures which are hypersurfaces

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Abstract. Let N be a quiver representation with non-zero admissible annihilator. In this paper, we prove the normality of the orbit closure $\bar{\mathcal{O}}_N$ when it is a hypersurface. The result thus gives new examples of normal orbit closures of quiver representations.

Keywords. Quiver representation; orbit closure; hypersurface; normality.

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1. Introduction

Throughout the paper, k denotes a fixed algebraically closed field of characteristic zero. *Quiver* is the terminology in representation theory for a directed graph. So a quiver consists of a set of vertices and a set of directed edges or arrows. We shall only consider finite quivers, i.e., those quivers whose set of vertices and the set of arrows are finite. A k-linear representation (representation, for short) of a quiver Q assigns a k-vector space to each vertex and a k-linear map to each arrow of Q. If we fix a natural number d_i for each vertex i of the quiver Q, then the representations of Q of dimension vector $\mathbf{d} = (d_i)$ form an affine space $\mathrm{rep}_Q(\mathbf{d})$. The linear algebraic group $\mathrm{GL}(\mathbf{d}) = \bigoplus \mathrm{GL}(d_i, k)$ acts on $\mathrm{rep}_Q(\mathbf{d})$ by conjugation, so that the $\mathrm{GL}(\mathbf{d})$-orbits in $\mathrm{rep}_Q(\mathbf{d})$ correspond bijectively to the isomorphism classes of representations of Q of dimension vector $\mathbf{d}$ (see §2 for a brief account of quiver representations).

It is an interesting task to study geometric properties of the Zariski closures – being affine varieties – of $\mathrm{GL}(\mathbf{d})$-orbits in $\mathrm{rep}_Q(\mathbf{d})$, as well as their connections with representation-theoretic properties of the quiver Q and its representations. Among the geometric properties of orbit closures that have attracted a lot of attention are the normality and the Cohen–Macaulayness. For instance, if Q is a Dynkin quiver of type A or D, then Bobiński and Zwara [2, 3] proved that $\mathrm{GL}(\mathbf{d})$-orbit closures in $\mathrm{rep}_Q(\mathbf{d})$ are normal and Cohen-Macaulay. If Q is the extended Dynkin quiver of type $\tilde{\mathbb{A}}_0$ (i.e., the quiver consisting of one vertex and one loop), then orbit closures are normal and Cohen–Macaulay by a classical result of Kraft and Procesi [4]. On the other hand, there exist orbit closures that are neither normal nor Cohen-Macaulay [8]. We refer to the survey [9] for more results on singularties of orbit closures of modules and quiver representations. In particular, it is shown in section 2 of [9] that in studying singularties of orbit closures, we may restrict our consideration to the quiver representations whose annihilators are admissible ideals (see §2 for definitions).

In [5, 6], we characterize the quiver representations for which the orbit closures are hypersurfaces. More precisely, we have the following two theorems (the first one holds in any characteristic).

Theorem 1.1 (Theorem 2.1 of [5]). *Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$ be a dimension vector. Let N be a representation in $\mathrm{rep}_Q(\mathbf{d})$ such that the annihilator $\mathrm{Ann}(N)$ is an admissible ideal in kQ. Then the orbit closure $\bar{\mathcal{O}}_N$ is a non-singular variety if and only if $\mathrm{Ann}(N) = 0$ and $\bar{\mathcal{O}}_N = \mathrm{rep}_Q(\mathbf{d})$.*

Theorem 1.2 (Theorem 1.4 of [6]). *Let Q be a quiver and $\mathbf{d} = (d_i)_{i \in Q_0}$ be a dimension vector. Let N be a representation in $\mathrm{rep}_Q(\mathbf{d})$ such that $\mathrm{Ann}(N)$ is an admissible ideal in kQ. Then $\bar{\mathcal{O}}_N$ is a singular hypersurface if and only if one of the following conditions hold:*

(A) $\mathrm{Ann}(N) = \langle \gamma^2 \rangle$, *where γ is a loop in Q at a vertex i with $d_i = 2$, and* $\mathrm{Ext}^1_{kQ/\langle \gamma^2 \rangle}(N, N) = 0$.
(B) $\mathrm{Ann}(N) = \langle \rho \rangle$, *where ρ is a relation in Q from a vertex i to a vertex j with $d_i = d_j = 1$, and* $\mathrm{Ext}^1_{kQ/\langle \rho \rangle}(N, N) = 0$.
(C) $\mathrm{Ann}(N) = 0$ *and* $\mathrm{Ext}^1_{kQ}(N, N) \cong k$.

Since a hypersurface is a complete intersection, it is Cohen–Macaulay. Our next aim is to prove the normality of the orbit closure which is a (singular) hypersurface. The main result of the paper states as follows.

Theorem 1.3. *Let Q be a quiver and $\mathbf{d} = (d_i)_{i \in Q_0}$ be a dimension vector. Let N be a representation in $\mathrm{rep}_Q(\mathbf{d})$ such that $\mathrm{Ann}(N)$ is a non-zero admissible ideal in kQ. If $\bar{\mathcal{O}}_N$ is a hypersurface, then $\bar{\mathcal{O}}_N$ is a normal variety.*

This theorem deals essentially with the cases (A) and (B) of Theorem 1.2. The normality of the orbit closure $\bar{\mathcal{O}}_N$ in the case (C) of Theorem 1.2 is an open question in general, and we shall handle it in a separate paper.

Since $\bar{\mathcal{O}}_N$ is an irreducible affine hypersurface, then, by a well-known criterion of Serre (see, for example, section III.8 of [7]), its normality is equivalent to the non-singularity in codimension 1, i.e., the singular locus $\mathrm{Sing}(\bar{\mathcal{O}}_N)$ is a closed subvariety of $\bar{\mathcal{O}}_N$ of codimension at least 2. This is our strategy for proving the normality of $\bar{\mathcal{O}}_N$.

In §2, we recall some notions on representations of quivers that are necessary. The proof of Theorem 1.3 is presented in §3. We illustrate Theorem 1.3 with two examples in §4. For basic background on the representation theory of algebras and quivers, we refer to [1].

2. Representations of quivers

Let $Q = (Q_0, Q_1; s, t : Q_1 \to Q_0)$ be a finite quiver, i.e., Q_0 is a finite set of vertices, Q_1 is a finite set of arrows $\alpha : s(\alpha) \to t(\alpha)$, where $s(\alpha)$ and $t(\alpha)$ denote the starting and terminating vertex of α, respectively. By an oriented path (path, for short) of length $r \geq 1$ in Q, we mean a sequence of arrows in Q_1:

$$\omega = \alpha_r \ldots \alpha_1,$$

such that $s(\alpha_{l+1}) = t(\alpha_l)$ for $l = 1, \dots, r-1$. In this situation we write $s(\omega) = s(\alpha_1)$ and $t(\omega) = t(\alpha_r)$, and say that ω is a path from $s(\alpha_1)$ to $t(\alpha_r)$. We agree to associate to each vertex $i \in Q_0$ a path ε_i in Q of length zero with $s(\varepsilon_i) = t(\varepsilon_i) = i$. We call a path ω of positive length with $s(\omega) = t(\omega)$ an oriented cycle. By a primitive cycle, we mean an oriented cycle which does not contain other oriented cycles as proper subpaths. A loop is an oriented cycle of length one.

The paths in Q form a k-linear basis of the path algebra kQ, in which the product of two paths ω and ρ is the path $\omega\rho$ if $s(\omega) = t(\rho)$, and is zero otherwise. A relation from a vertex i to a vertex j is a k-linear combination of paths from i to j of length at least two. In particular, a relation is an element in the vector space $\varepsilon_j \cdot kQ \cdot \varepsilon_i$. Given an element ρ in $\varepsilon_j \cdot kQ \cdot \varepsilon_i$, we denote by $\langle \rho \rangle$ the two-sided ideal in kQ generated by ρ.

By a representation of Q, we mean a collection $V = (V_i, V_\alpha)$ of finite dimensional k-vector spaces V_i, $i \in Q_0$, together with linear maps $V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$, $\alpha \in Q_1$. The dimension vector of the representation V is the vector

$$\mathbf{dim}\, V = (\dim_k V_i) \in \mathbb{N}^{Q_0}.$$

A morphism $f : V \to W$ between two representations is a collection of linear maps $f_i : V_i \to W_i$, $i \in Q_0$, such that $f_{t(\alpha)} V_\alpha = W_\alpha f_{s(\alpha)}$ for each $\alpha \in Q_1$. The category of representations of Q is an abelian k-linear category, which is naturally equivalent to the category mod (kQ) of finite-dimensional left kQ-modules (see section III.1 of [1]).

For a path $\omega = \alpha_r \dots \alpha_1$ and a representation V of Q, we define

$$V_\omega = V_{\alpha_r} \circ \dots \circ V_{\alpha_1} : V_{s(\omega)} \to V_{t(\omega)}$$

and extend easily this definition to $V_\rho : V_i \to V_j$ for any ρ in $\varepsilon_j \cdot kQ \cdot \varepsilon_i$, where $i, j \in Q_0$, as ρ is a linear combination of paths ω with $s(\omega) = i$ and $t(\omega) = j$. We set the annihilator of the representation V to be

$$\mathrm{Ann}(V) = \left\{ \rho \in kQ \mid V_{\varepsilon_j \cdot \rho \cdot \varepsilon_i} = 0 \text{ for all } i, j \in Q_0 \right\},$$

which is a two-sided ideal in kQ. In fact, it is the annihilator of the kQ-module corresponding to V.

A two-sided ideal I in kQ is called admissible if $(\mathcal{R}_Q)^r \subseteq I \subseteq (\mathcal{R}_Q)^2$ for some integer $r \geq 2$, where $\mathcal{R}_Q$ denotes the two-sided ideal in kQ generated by arrows in Q. For such an ideal I, the category mod (kQ/I) of finite-dimensional left kQ/I-modules is equivalent to the full subcategory consisting of all the representations V of Q such that $\mathrm{Ann}(V) \supseteq I$. We shall identify these two categories.

Let $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$ be a dimension vector. The representations $V = (V_i, V_\alpha)$ of Q with $V_i = k^{d_i}$, $i \in Q_0$, form an affine space

$$\mathrm{rep}_Q(\mathbf{d}) = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_k(V_{s(\alpha)}, V_{t(\alpha)}) = \bigoplus_{\alpha \in Q_1} \mathbb{M}_{d_{t(\alpha)} \times d_{s(\alpha)}}(k),$$

where $\mathbb{M}_{d' \times d''}(k)$ stands for the space of $d' \times d''$-matrices with entries in k. The group

$$\mathrm{GL}(\mathbf{d}) = \bigoplus_{i \in Q_0} \mathrm{GL}(d_i, k)$$

acts regularly on $\mathrm{rep}_Q(\mathbf{d})$ by simultaneous conjugation

$$(g_i)_{i \in Q_0} * (V_\alpha)_{\alpha \in Q_1} = (g_{t(\alpha)} \cdot V_\alpha \cdot g_{s(\alpha)}^{-1})_{\alpha \in Q_1}.$$

The action of an element of $\mathrm{GL}(\mathbf{d})$ on $\mathrm{rep}_Q(\mathbf{d})$ corresponds to a change of basis at each vector space k^{d_i}, $i \in Q_0$, so that two representations in $\mathrm{rep}_Q(\mathbf{d})$ are isomorphic if and only if they belong to the same $\mathrm{GL}(\mathbf{d})$-orbit. Given a representation $W = (W_i, W_\alpha)$ of Q with $\mathbf{dim}W = \mathbf{d}$, we denote by $\mathcal{O}_W$ the $\mathrm{GL}(\mathbf{d})$-orbit in $\mathrm{rep}_Q(\mathbf{d})$ of representations isomorphic to W, and by $\bar{\mathcal{O}}_W$ the Zariski closure of $\mathcal{O}_W$ in $\mathrm{rep}_Q(\mathbf{d})$. Notice that $\mathcal{O}_W$ being the image of $\mathrm{GL}(\mathbf{d})$ under an orbit map, is irreducible. Thus $\bar{\mathcal{O}}_W$ is an irreducible affine variety.

The algebra of polynomial functions on the affine space $\mathrm{rep}_Q(\mathbf{d})$ is

$$k[\mathrm{rep}_Q(\mathbf{d})] = k[X_{\alpha,p,q} \mid \alpha \in Q_1, 1 \le p \le d_{t(\alpha)}, 1 \le q \le d_{s(\alpha)}].$$

Here, $X_{\beta,p,q}$ maps a representation $W = (W_\alpha)$ to the (p,q)-entry of the matrix W_β. Let X_α stand for the $d_{t(\alpha)} \times d_{s(\alpha)}$-matrix whose (p,q)-entry is the variable $X_{\alpha,p,q}$, for any arrow $\alpha \in Q_1$. We define the $d_j \times d_i$-matrix X_ρ for $\rho \in \varepsilon_j \cdot kQ \cdot \varepsilon_i$, with entries in $k[\mathrm{rep}_Q(\mathbf{d})]$, in a similar way to that for representations of Q. For example, if $\rho = \beta\alpha$ is a path in Q from the vertex 1 to the vertex 3, as illustrated below:

$$\underset{1}{\bullet} \xrightarrow{\ \alpha\ } \underset{2}{\bullet} \xrightarrow{\ \beta\ } \underset{3}{\bullet}$$

and $d_1 = d_3 = 1$, $d_2 = 2$, then X_ρ is the polynomial $X_{\beta,1,1}X_{\alpha,1,1} + X_{\beta,1,2}X_{\alpha,2,1}$.

We shall need the following auxiliary result.

Lemma 2.1. *Let $\xi = \alpha_r \ldots \alpha_1$ be a path in the quiver Q such that $d_{t(\alpha_l)} \ge 2$ for $l = 1, \ldots, r-1$ and the arrows $\alpha_1, \ldots, \alpha_r$ are pairwise distinct. Then the entries of the matrix X_ξ are irreducible polynomials in $k[\mathrm{rep}_Q(\mathbf{d})]$. In particular, if $d_{s(\xi)} = d_{t(\xi)} = 1$, then the polynomial X_ξ is irreducible in $k[\mathrm{rep}_Q(\mathbf{d})]$.*

The proof of the lemma is straightforward. Indeed, we may assume that ξ is a path with $d_{s(\xi)} = d_{t(\xi)} = 1$, so that X_ξ is a polynomial (all the entries of X_ξ in general case are of this form). Suppose there is a non-trivial factorization $X_\xi = hg$ in $k[\mathrm{rep}_Q(\mathbf{d})]$. Let H (respectively, G) be the set of all arrows $\alpha \in Q_1$ such that some variable $X_{\alpha,p,q}$ appears in the polynomial h (respectively, g). Then clearly $H \cup G = \{\alpha_1, \ldots, \alpha_r\}$ and $H \cap G = \emptyset$. Since $d_{t(\alpha_l)} \ge 2$ for $l = 1, \ldots, r-1$, it follows that hg has a term which is not a term of X_ξ. Hence the lemma follows.

3. Proof of Theorem 1.3

Let N be a representation in $\mathrm{rep}_Q(\mathbf{d})$ such that $\mathrm{Ann}(N)$ is a non-zero admissible ideal in kQ. Assume that the orbit closure $\bar{\mathcal{O}}_N$ is a hypersurface. Then, by Theorems 1.1 and 1.2, $\bar{\mathcal{O}}_N$ is a singular variety and we have either case (A) or case (B) of Theorem 1.2.

Remark 3.1. It is shown in [6] that in the case (A), γ is a unique primitive cycle in Q, while in the case (B), either the quiver Q contains no oriented cycles, or ρ is a subpath of a unique primitive cycle that is not a loop in Q.

Remark 3.2. In the case (B), since $d_i = d_j = 1$, so N_ω is a scalar for any path ω from i to j. Assume that Q contains no oriented cycles. Then the vector space $\varepsilon_j \cdot kQ \cdot \varepsilon_i$ is of dimension at most 2, i.e., there are at most two paths from i to j in Q. Indeed, assume there are two paths ω and ξ from i to j. Then there exist scalars a and b, not both zero, such that

$$N_{a\omega+b\xi} = aN_\omega + bN_\xi = 0.$$

This implies $a\omega + b\xi \in \mathrm{Ann}(N) = \langle\rho\rangle$. Hence, up to a scalar, $\rho = a\omega + b\xi$, and clearly there is no more path from i to j.

As usual, the zero set in the affine space $\mathbb{A}^n = \mathbb{A}^n(k)$ of a subset T of $k[X_1, \ldots, X_n]$ is denoted by $Z(T)$. We shall use several times the following simple fact.

Lemma 3.3. *The zero set of two coprime polynomials in $k[X_1, \ldots, X_n]$ is of codimension 2 in the affine space $\mathbb{A}^n$.*

PROPOSITION 3.4

Let $\omega = \alpha_r \ldots \alpha_2\alpha_1$ be a path in the quiver Q of length $r \geq 2$ such that $d_{s(\omega)} = d_{t(\omega)} = 1, d_{t(\alpha_l)} \geq 2$ for $l = 1, \ldots, r-1$ and the arrows α_l are pairwise distinct. Then the singular locus of $Z(X_\omega)$ is of codimension at least 4 in $\mathrm{rep}_Q(\mathbf{d})$. *In particular, $Z(X_\omega)$ is a normal variety.*

Proof. The singular locus $\mathrm{Sing}(Z(X_\omega))$ of $Z(X_\omega)$ is given by the following equations in $\mathrm{rep}_Q(\mathbf{d})$:

$$X_\omega = 0, \qquad \frac{\partial X_\omega}{\partial X_{\alpha_l, p_l, q_l}} = X_{\alpha_r\ldots\alpha_{l+1},1,p_l} \cdot X_{\alpha_{l-1}\ldots\alpha_1,q_l,1} = 0,$$

for $l = 1, \ldots, r$ and $1 \leq p_l \leq d_{t(\alpha_l)}$, $1 \leq q_l \leq d_{s(\alpha_l)}$. Here, $X_{\alpha_r\ldots\alpha_{l+1},1,p_l}$ denotes the $(1, p_l)$-entry of the matrix $X_{\alpha_r\ldots\alpha_{l+1}}$, and analogously for $X_{\alpha_{l-1}\ldots\alpha_1,q_l,1}$.

Let $\mathcal{Z}$ be an irreducible component of $\mathrm{Sing}(Z(X_\omega))$. The vanishing of partial differentials of X_ω with respect to the variables $X_{\alpha_r,1,q_r}$ for $1 \leq q_r \leq d_{s(\alpha_r)}$ implies that $\mathcal{Z}$ is contained in the zero set of all the entries of the matrix $X_{\alpha_{r-1}\ldots\alpha_1}$, which is denoted by $Z(X_{\alpha_{r-1}\ldots\alpha_1})$. Let $1 \leq s \leq r-1$ be the least number such that $\mathcal{Z} \subseteq Z(X_{\alpha_s\ldots\alpha_1})$. Then the vanishing of partial differentials with respect to the variables X_{α_s,p_s,q_s} for $p \leq d_{t(\alpha_s)}, q \leq d_{s(\alpha_s)}$, leads to the inclusion

$$\mathcal{Z} \subseteq Z(X_{\alpha_r\ldots\alpha_{s+1}}) \cup Z(X_{\alpha_{s-1}\ldots\alpha_1}).$$

Since $\mathcal{Z}$ is irreducible, it follows from the choice of s that $\mathcal{Z} \subseteq Z(X_{\alpha_r\ldots\alpha_{s+1}})$. Hence

$$\mathcal{Z} \subseteq Z(X_{\alpha_r\ldots\alpha_{s+1}}) \cap Z(X_{\alpha_s\ldots\alpha_1}).$$

We define three subquivers Γ, Γ' and Γ'' of Q such that $\Gamma_0 = \Gamma'_0 = \Gamma''_0 = Q_0$, $\Gamma'_1 = \{\alpha_1, \ldots, \alpha_s\}$, $\Gamma''_1 = \{\alpha_{s+1}, \ldots, \alpha_r\}$ and $\Gamma_1 = Q_1 \setminus (\Gamma'_1 \cup \Gamma''_1)$. By Lemma 2.1, the entries of the matrix $X_{\alpha_s\ldots\alpha_1}$ are irreducible polynomials, and thus are mutually coprime. Since $d_{t(\alpha_s)} \geq 2$, the set $\mathcal{V} = Z(X_{\alpha_s\ldots\alpha_1})$ is of codimension at least 2 in $\mathrm{rep}_{\Gamma'}(\mathbf{d})$, by Lemma 3.3. We get an analogous claim for the set $\mathcal{W} = Z(X_{\alpha_r\ldots\alpha_{s+1}})$ in $\mathrm{rep}_{\Gamma''}(\mathbf{d})$. Since

$$Z(X_{\alpha_r\ldots\alpha_{s+1}}) \cap Z(X_{\alpha_s\ldots\alpha_1}) = \mathcal{V} \times \mathcal{W} \times \mathrm{rep}_\Gamma(\mathbf{d}),$$

so $\mathcal{Z}$ is of codimension at least 4 in $\mathrm{rep}_Q(\mathbf{d})$. As a consequence, $\mathrm{Sing}(Z(X_\omega))$ is of codimension at least 3 in $Z(X_\omega)$. Therefore, by Serre's criterion, $Z(X_\omega)$ is a normal variety. $\square$

Proof of Theorem 1.3 Let $I(\bar{\mathcal{O}}_N)$ denote the defining ideal of the variety $\bar{\mathcal{O}}_N$ in $k[\mathrm{rep}_Q(\mathbf{d})]$. In the case (A), by Corollary 3.12 of [6], we have

$$I(\bar{\mathcal{O}}_N) = (X_{\gamma,1,1} + X_{\gamma,2,2}, X_{\gamma,1,1}X_{\gamma,2,2} - X_{\gamma,1,2}X_{\gamma,2,1}).$$

So $\bar{\mathcal{O}}_N$ is the zero set of the polynomial $X^2_{\gamma,1,1} + X_{\gamma,1,2}X_{\gamma,2,1}$ in the affine space

$$\mathrm{rep}_Q^{\mathrm{trace}}(\mathbf{d}) = \{(V_\alpha)_{\alpha\in Q_1} \in \mathrm{rep}_Q(\mathbf{d}) \mid V_{\gamma,1,1} + V_{\gamma,2,2} = 0\}.$$

It follows easily that $\mathrm{codim}_{\bar{\mathcal{O}}_N} \mathrm{Sing}(\bar{\mathcal{O}}_N) = 2$. Therefore $\bar{\mathcal{O}}_N$ is a normal variety.

In the case (B), it follows from the proof of Proposition 3.10 and Corollary 3.15 in [6] that $I(\bar{\mathcal{O}}_N) = (X_\rho)$. By definition, ρ is a linear combination of paths from i to j of length at least 2. If ρ is a single path, then the claim follows from Proposition 3.4. In particular, this is the case when Q contains an oriented cycle (see Remark 3.1). Hence, in view of Remark 3.2, we may assume that $\rho = \tau(a\cdot\rho_1 + b\cdot\rho_2)\omega$, where $a, b \in k^*$, and the paths τ, ρ_1, ρ_2 and ω have no arrow in common. Let

$$\rho_1 = \alpha_m \dots \alpha_1, \quad \rho_2 = \beta_n \dots \beta_1, \quad \tau = \gamma_r \dots \gamma_1.$$

Let $\mathcal{Z}$ be an irreducible component of $\mathrm{Sing}(\bar{\mathcal{O}}_N)$. Analogously to the proof of Proposition 3.4, the vanishing of partial differentials of X_ρ with respect to the variables X_{α_m,p_m,q_m} implies that

$$\mathcal{Z} \subseteq Z(X_\tau) \quad \text{or} \quad \mathcal{Z} \subseteq Z(X_{\alpha_{m-1}\dots\alpha_1\omega}).$$

In the former case, we proceed as in the proof of Proposition 3.4 and obtain

$$\mathcal{Z} \subseteq Z(X_{\gamma_r\dots\gamma_{s+1}}) \cap Z(X_{\gamma_s\dots\gamma_1(a\cdot\rho_1+b\cdot\rho_2)\omega})$$

for some $0 \le s \le r-1$. Since the polynomial X_ρ is irreducible, we have $d_{s(\gamma_{s+1})} \ge 2$. Then it follows from Lemmas 2.1 and 3.3 that $\mathcal{Z}$ is of codimension at least 3 in $\mathrm{rep}_Q(\mathbf{d})$.

We consider the latter case, when $\mathcal{Z} \subseteq Z(X_{\alpha_{m-1}\dots\alpha_1\omega})$. If $\mathcal{Z} \subseteq Z(X_\omega)$, then we are in a similar situation as above. Thus assume $\mathcal{Z} \not\subseteq Z(X_\omega)$. Let $1 \le u \le m-1$ be the least number such that $\mathcal{Z} \subseteq Z(X_{\alpha_u\dots\alpha_1\omega})$. Then the vanishing of partial differentials of X_ρ with respect to the variables X_{α_u,p_u,q_u} implies that

$$\mathcal{Z} \subseteq Z(X_{\tau\alpha_m\dots\alpha_{u+1}}) \cap Z(X_{\alpha_u\dots\alpha_1\omega}). \tag{3.1}$$

In this case, however, $d_{t(\alpha_u)}$ may equal 1. Then we consider the path $\rho_2 = \beta_n \dots \beta_1$ and by using the same arguments, we obtain

$$\mathcal{Z} \subseteq Z(X_{\tau\beta_n\dots\beta_{v+1}}) \cap Z(X_{\beta_v\dots\beta_1\omega}) \tag{3.2}$$

for some $1 \le v \le n-1$. Let Γ, Γ' and Γ'' be subquivers of Q such that $\Gamma_0 = \Gamma'_0 = \Gamma''_0 = Q_0$. Γ'_1 consists of the arrows appearing in the paths ω, $\alpha_u \dots \alpha_1$ and $\beta_v \dots \beta_1$, Γ''_1 consists of the arrows appearing in the paths $\alpha_m \dots \alpha_{u+1}$, $\beta_n \dots \beta_{v+1}$ and τ, and finally $\Gamma_1 = Q_1 \setminus (\Gamma'_1 \cup \Gamma''_1)$. Then, by Lemma 3.3, the set

$$\mathcal{V} = Z(X_{\alpha_u\dots\alpha_1\omega}) \cap Z(X_{\beta_v\dots\beta_1\omega})$$

is of codimension at least 2 in $\mathrm{rep}_{\Gamma'}(\mathbf{d})$, where the entries of the corresponding matrices are considered as polynomials in $k[\mathrm{rep}_{\Gamma'}(\mathbf{d})]$. We get an analogous claim for the set

$$\mathcal{W} = Z(X_{\tau\alpha_m\dots\alpha_{u+1}}) \cap Z(X_{\tau\beta_n\dots\beta_{v+1}})$$

in $\operatorname{rep}_{\Gamma''}(\mathbf{d})$. Now it follows from (3.1) and (3.2) that

$$\mathcal{Z} \subseteq \mathcal{V} \times \mathcal{W} \times \operatorname{rep}_{\Gamma}(\mathbf{d}),$$

so $\mathcal{Z}$ is of codimension at least 4 in $\operatorname{rep}_Q(\mathbf{d})$.

To sum up, we have $\operatorname{codim}_{\operatorname{rep}_Q(\mathbf{d})} \mathcal{Z} \geq 3$. Since $\bar{\mathcal{O}}_N$ is of codimension 1 in $\operatorname{rep}_Q(\mathbf{d})$, we obtain $\operatorname{codim}_{\bar{\mathcal{O}}_N} \operatorname{Sing}(\bar{\mathcal{O}}_N) \geq 2$. Hence, by Serre's criterion, $\bar{\mathcal{O}}_N$ is a normal variety. $\square$

4. Examples

We give two examples of normal orbit closures of quiver representations to illustrate Theorem 1.3. Note that the quiver Q in Example 4.1 is a wild quiver, i.e., it is not a Dynkin or extended Dynkin quiver, while the one in Example 4.2 is an extended Dynkin quiver of type $\tilde{\mathbb{A}}_2$.

Example 4.1. In the case (A) of Theorem 1.2, the loop γ is a unique primitive cycle in the quiver Q, see Remark 3.1. Hence the simplest and typical example of this case is the representation

$$\begin{tikzcd} k^2 \arrow[loop right, "\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}"] \end{tikzcd}$$

of the quiver consisting of one vertex and a loop. The corresponding orbit closure is the normal variety of nilpotent 2×2-matrices.

For another example, let Q be the quiver

$$\begin{tikzcd} \bullet \arrow[r, "\alpha"] & \bullet \arrow[loop right, "\gamma"] \end{tikzcd}$$

and N be the representation

$$\begin{tikzcd} k \arrow[r, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & k^2 \arrow[loop right, "\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}"] \end{tikzcd}$$

of Q of dimension vector $\mathbf{d} = (1, 2)$. Here the linear maps N_α and N_γ are represented by their matrices in the canonical bases of k and k^2. Observe that $\operatorname{Ann}(N) = \langle \gamma^2 \rangle$ is an admissible ideal in kQ. Moreover, N is a projective $kQ/\langle \gamma^2 \rangle$-module (see Lemma III.2.4 of [1]). Thus $\operatorname{Ext}^1_{kQ/\langle \gamma^2 \rangle}(N, N) = 0$. Therefore $\bar{\mathcal{O}}_N$ is a hypersurface. By Theorem 1.3, it is a normal variety.

Example 4.2. Let Q be the quiver

$$\begin{tikzcd} & \bullet \arrow[dl, "\beta"'] & \\ \bullet \arrow[rr, "\gamma"'] & & \bullet \arrow[ul, "\alpha"'] \end{tikzcd}$$

and N be the representation

$$\begin{tikzcd} & k^3 \arrow[dl, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}"'] & \\ k^2 \arrow[rr, "{[0\ 1]}"'] & & k \arrow[ul, "\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}"'] \end{tikzcd}$$

of Q of dimension vector $\mathbf{d} = (3, 2, 1)$. Then $\mathrm{Ann}(N) = \langle\gamma\beta\alpha\rangle$ is an admissible ideal in kQ. Consequently, the orbit closure $\bar{\mathcal{O}}_N$ is contained in the irreducible subvariety $Z(X_{\gamma\beta\alpha})$ *of* $\mathrm{rep}_Q(\mathbf{d})$ of dimension 10. On the other hand, the stabilizer of N,

$$\mathrm{Stab}(N) = \{g \in \mathrm{GL}(\mathbf{d}) \mid g * N = N\},$$

is precisely the group of automorphisms of N. It is an open and dense subset of the space $\mathrm{End}(N)$ of endomorphisms of N. A simple calculation shows that the vector space $\mathrm{End}(N)$ is four-dimensional. Thus we get

$$\begin{aligned}\dim\bar{\mathcal{O}}_N = \dim\mathcal{O}_N &= \dim\mathrm{GL}(\mathbf{d}) - \dim\mathrm{Stab}(N)\\ &= \dim\mathrm{GL}(\mathbf{d}) - \dim\mathrm{End}(N) = 10.\end{aligned}$$

It follows that $\bar{\mathcal{O}}_N = Z(X_{\gamma\beta\alpha})$. Hence $\bar{\mathcal{O}}_N$ is a hypersurface, and by Theorem 1.3, it is normal.

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