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Disjoint hypercyclicity of weighted composition operators

ZAHRA KAMALI[1,*] and BAHMANN YOUSEFI[2]

[1]Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, Iran
[2]Department of Mathematics, Payame Noor University, P. O. Box 19395-3697, Tehran, Iran
*Corresponding author.
E-mail: zkamali@shirazu.ac.ir; b_yousefi@pnu.ac.ir

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Abstract. In this paper, we discuss about disjoint hypercyclicity of weighted composition operators on some function spaces of analytic functions on a plane domain.

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1. Introduction

Let X be a topological vector space and T a bounded linear operator on X. The T-orbit of a vector $x \in X$ is the set

$$O(x, T) := \{T^n(x) : n \in \mathbb{N} \cup \{0\}\}.$$

The operator T is said to be (weakly) hypercyclic if there exists a vector $x \in X$ such that $O(x, T)$ is (weakly) dense in X. Such a vector x is said to be (weakly) hypercyclic vector for T. The operator T is called (weakly) supercyclic if the set of scalar multiples of the elements of $O(T, x)$ is (weakly) dense. In this case, the vector x is called (weakly) supercyclic vector for T.

It is known that the direct sum of two hypercyclic operators need not be hypercyclic. de la Rosa and Read [5] gave an example of a hypercyclic operator whose direct sum $T \oplus T$ is not hypercyclic. But Salas gave an example of weighted shifts on c_0 or l_p $(1 \leq p < \infty)$ for which their direct sum is hypercyclic [9]. Finitely many hypercyclic operators acting on a common topological vector space are called disjoint if their direct sum has a hypercyclic vector on the diagonal of the product space.

DEFINITION 1.1

For $N \geq 2$, the operators $T_1, T_2, \dots, T_N$ are called disjoint hypercyclic or d-hypercyclic if the direct sum $T_1 \oplus T_2 \oplus \cdots \oplus T_N$ has a hypercyclic vector of the form $(x, x, \dots, x) \in X^N$.

In 2007, Bernal [1], and independently, Bès and Peris [2] initiated the study of the disjointness in hypercyclicity. In 2010, Martin in his dissertation [8] gave a characterization of disjointness among hypercyclic and supercyclic linear fractional composition operators on the classical Hardy space. For some sources on these topics, we refer to [1] through [15].

2. Disjointness among weighted composition operators on $H(\mathbb{D})$

Let $\mathbb{D}$ denote the open unit disk in the complex plane. In this section, we give some conditions under which weighted composition operators are disjoint hypercyclic on the space $H(\mathbb{D})$ of analytic functions on $\mathbb{D}$. If K is a compact subset of $\mathbb{D}$, for $f \in H(\mathbb{D})$, define $P_K(f) = \sup_{z\in K}|f(z)|$. Then $\{P_K : K \subseteq \mathbb{D}, K \text{ compact}\}$ is a family of seminorms that makes $H(\mathbb{D})$ a locally convex space. In fact, this topology is the topology of uniform convergence on compact subsets of unit disk, the so-called usual topology for $H(\mathbb{D})$, which is a F-space.

Each $w \in H(\mathbb{D})$ and holomorphic self-map φ of $\mathbb{D}$, induces a linear weighted composition operator $C_{w,\varphi} : H(\mathbb{D}) \to H(\mathbb{D})$ by $C_{w,\varphi}(f)((z) = M_w C_\varphi(f)(z) = w(z)f(\varphi(z))$ for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$, where M_w denotes the multiplication operator by w and C_φ is a composition operator. The mapping φ is called composition map and w is called the weight. For a positive integer n, the n-th iterate of φ is denoted by $\varphi^{[n]}$, and when φ is invertible $\varphi^{[-n]}$ is the n-th iterate of φ^{-1}, also $\varphi^{[0]}$ is the identity function. We note that

$$C^n_{w,\varphi}(f) = \prod_{j=0}^{n-1} w \circ \varphi^{[j]}(f \circ \varphi^{[n]})$$

for all f and $n \geq 1$.

The following proposition that is due to Yousefi and Rezaei limits the kind of maps that can produce hypercyclic weighted composition operators [11].

PROPOSITION 2.1

If $C_{w,\varphi}$ is hypercyclic on $H(\mathbb{D})$, then

(*i*) *φ has no fixed point in $\mathbb{D}$ and $w(z) \neq 0$ for every $z \in \mathbb{D}$;*
(*ii*) *φ is univalent.*

Hereafter, we may assume that all composition maps are univalent and also all weights are non-zero bounded functions on $\mathbb{D}$.

DEFINITION 2.2

For $N \geq 2$, we say the operators $T_1, \ldots, T_N$ in $L(X)$ are d-topologically transitive provided for every non-empty open subsets $V_0, \ldots, V_N$ of X, there exists $m \in \mathbb{N}$ so that $V_0 \cap T_1^{-m}(V_1) \cap \cdots \cap T_N^{-m}(V_N) \neq \emptyset$.

For the proof of the next theorem we need the following proposition.

PROPOSITION 2.3

Let $N \geq 2$ and $T_1, \ldots, T_N$ be operators in $L(X)$. Then $T_1, \ldots, T_N$ are d-topologically transitive if and only if the set of d-hypercyclic vectors for $T_1, \ldots, T_N$ is a dense G_δ.

Theorem 2.4. *Let $N \geq 2$ and $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ be hypercyclic weighted composition operators on $H(\mathbb{D})$. If for each compact set $K \subseteq \mathbb{D}$, there exists $n \geq 1$ such that the sets $K, \varphi_1^{[n]}(K), \ldots, \varphi_N^{[n]}(K)$ are pairwise disjoint, then the weighted composition operators $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ are d-hypercyclic on $H(\mathbb{D})$.*

Proof. We show that $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ are d-topologically transitive. Let $V_0, \ldots, V_N$ be non-empty open subsets of $H(\mathbb{D})$. There exist $\epsilon > 0$, compact subsets $K_0, \ldots, K_N$ of $\mathbb{D}$ and functions $f_0, \ldots, f_N \in H(\mathbb{D})$ such that

$$\{h \in H(\mathbb{D}) : \sup_{z \in K_j} |h(z) - f_j(z)| < \epsilon\} \subseteq V_j \quad (0 \leq j \leq N).$$

Put $K = \bigcup_{j=0}^{N} K_j$. Hence

$$\{h \in H(\mathbb{D}) : \sup_{z \in K} |h(z) - f_j(z)| < \epsilon\} \subseteq V_j \quad (0 \leq j \leq N).$$

Since K is compact, we can choose two closed discs B_1, B_2 in $\mathbb{D}$ such that $K \subseteq B_1 \subseteq B_2^0$. By hypothesis, there exists $n \geq 1$ so that $B_2, \varphi_1^{[n]}(B_2), \ldots, \varphi_N^{[n]}(B_2)$ are pairwise disjoint. Now, consider the map

$$R(z) = \begin{cases} f_0(z), & \text{if } z \in B_1 \\ \prod_{k=1}^{n} \frac{1}{w_j \circ \varphi_j^{[-k]}(z)} f_j \circ \varphi_j^{[-n]}(z), & \text{if } z \in \varphi_j^{[n]}(B_1) \quad (1 \leq j \leq N). \end{cases}$$

The complement of $B_1 \cup \varphi_1^{[n]}(B_1) \cup \cdots \cup \varphi_N^{[n]}(B_1)$ is connected, so by Runge's theorem, there exists a polynomial P such that

$$|P(z) - R(z)| < \min\left\{\epsilon, \frac{\epsilon}{\| w_j \|_\infty^n}, j = 1, \ldots, N\right\}$$

for all $z \in B_1 \cup \varphi_1^{[n]}(B_1) \cup \cdots \cup \varphi_N^{[n]}(B_1)$. So for all $z \in B_1$, we have

$$|f_0(z) - P(z)| < \epsilon$$

which implies that $P \in V_0$. Also, for all $z \in \varphi_j^{[-n]}(B_1)$ we get

$$\left|\prod_{k=1}^{n} \frac{1}{w_j \circ \varphi_j^{[-k]}(z)} f_j \circ \varphi_j^{[-n]}(z) - p(z)\right| < \frac{\epsilon}{\| w_j \|_\infty^n}, \quad (1 \leq j \leq N),$$

which implies that

$$|C_{w_j,\varphi_j}^{[n]} P(z) - f_j(z)| < \epsilon \quad (1 \leq j \leq N),$$

for all $z \in B_1$. Therefore $P \in V_0 \cap C_{w_1,\varphi_1}^{[-n]}(V_1) \cap \cdots \cap C_{w_N \varphi_N}^{[-n]}(V_N)$, and the theorem follows from Proposition 2.3. □

The holomorphic self-maps of the unit disk are divided into two classes, elliptic and non-elliptic functions. The elliptic type is an automorphism and has a fixed point in $\mathbb{D}$. The non-elliptic one has a unique fixed point $p \in \bar{\mathbb{D}}$, called the Denjoy–Wolff point of φ, which is known as attractive fixed point, that is the sequence of iterates of φ, $\{\varphi^{[n]}\}_n$ converges to p uniformly on compact subsets of $\mathbb{D}$ (see [4] for more details).

Remark 2.5. Let φ be a univalent self map of the disk with no interior fixed point. If there exists a compact subset K of $\mathbb{D}$ such that $\varphi^{[n]}(K) \cap K \neq \emptyset$ for all $n \in \mathbb{N}$, then there is a sequence $\{\zeta_n\}_n$ in K, such that for all $n \in \mathbb{N}$, $\varphi^{[n]}(\zeta_n) \in K$, so, $\varphi^{[n]}(\zeta_n) \to \alpha$, where α is an attractive fixed point of φ (in fact, α is Denjoy–Wolff point of φ [4]). Since K is compact we must have $\alpha \in K \subseteq \mathbb{D}$, which is a contradiction.

COROLLARY 2.6

Let $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ be hypercyclic weighted composition operators on $H(\mathbb{D})$, where $\varphi_1, \ldots, \varphi_N$ are linear fractional transformations of $\mathbb{D}$. If the attractive fixed points of $\varphi_1, \ldots, \varphi_N$ are all distinct, then $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ are d-hypercyclic on $H(\mathbb{D})$.

Proof. If for $1 \leq p \neq q \leq N$ and for some compact subset K of $\mathbb{D}$, $\varphi_p^{[n]}(K) = \varphi_q^{[n]}(K)$ for all $n \in \mathbb{N}$, then there exist two sequences $\{\zeta_n\}_n$ and $\{\zeta'_n\}_n$ of elements of K such that

$$\varphi_p^{[n]}(\zeta_n) = \varphi_q^{[n]}(\zeta'_n)$$

for all $n \in \mathbb{N}$. On the other hand, $\{\varphi_p^{[n]}(\zeta_n)\}_n$ and $\{\varphi_q^{[n]}(\zeta'_n)\}_n$ converge to the distinct attractive fixed point of φ_p and φ_q, respectively. This is a contradiction. $\square$

COROLLARY 2.7

Let $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ be hypercyclic weighted composition operators on $H(\mathbb{D})$. Suppose $\varphi_1, \ldots, \varphi_N$ are different linear fractional composition operators that have the same attractive fixed point α and the condition $\varphi'_p(\alpha) = \varphi'_q(\alpha) < 1$ does not hold for any $1 \leq p \neq q \leq N$. Then $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ are d-hypercyclic.

Proof. For $1 \leq k \leq N$, let $\psi_k = T_\alpha \circ \varphi_k \circ T_\alpha^{-1}$, where $T_\alpha(z) = \frac{\alpha+z}{\alpha-z}$. Then ψ_k is a linear transformation of right half plane and is of the form

$$\psi_k(z) = \lambda_k z + b_k,$$

where $0 < \varphi'_k(\alpha) = \frac{1}{\lambda_k} \leq 1$ (page 6 of [3]) and Re $b_k \geq 0$. Let K' be a compact subset of right half plane and $1 \leq p \neq q \leq N$. Also suppose that $\psi_p^{[n]}(K') \cap \psi_q^{[n]}(K') \neq \emptyset$ for all $n \in \mathbb{N}$. Then there exist two sequences $\{\eta_n\}_n$ and $\{\eta'_n\}_n$ in K' such that $\psi_p^{[n]}(\eta_n) = \psi_q^{[n]}(\eta'_n)$ for all $n \in \mathbb{N}$. So

$$\eta_n = \psi_p^{[-n]}(\psi_q^{[n]}(\eta'_n)) = \left(\frac{\lambda_q}{\lambda_p}\right)^n \eta'_n + \frac{b_q}{\lambda_p^n} \Sigma_{j=0}^{n-1} \lambda_q^j - \frac{b_p}{\lambda_p} \Sigma_{j=0}^{n-1} \frac{1}{\lambda_p^j}.$$

By our assumption we have two cases: $\lambda_p = \lambda_q = 1$ or $\lambda_p \neq \lambda_q$. If $\lambda_p = \lambda_q = 1$, then passing through a subsequence, if necessary,

$$\eta_n = \psi_p^{[-n]}(\psi_q^{[n]}(\eta'_n)) = \eta'_n + (b_q - b_p)n \to \infty$$

as $n \to \infty$, which is a contradiction. Now, suppose $\lambda_p \neq \lambda_q$. Without loss of generality, we may assume $1 \leq \lambda_p < \lambda_q$. Hence passing through a subsequence

$$\psi_p^{[-n]}(\psi_q^{[n]}(\eta_n')) = \left(\frac{\lambda_q}{\lambda_p}\right)^n \left(\eta_n' + b_q \frac{1-\lambda_q^{-n}}{\lambda_q - 1} - \frac{b_p \lambda_p^{n-1}}{\lambda_q^n} \Sigma_{j=0}^{n-1} \frac{1}{\lambda_p^j}\right) \to \infty$$

as $n \to \infty$, which is also a contradiction. Thus in both cases $\psi_p^{[n]}(K') \cap \psi_q^{[n]}(K') = \emptyset$ for some $n \in \mathbb{N}$. So this is true for φ_p and φ_q and every compact subsets of the open unit disk and so the corollary follows from Theorem 2.4. □

3. Weakly *d*-supercyclicity and *d*-hypercyclicity on a Hilbert function space

In this section, we first introduce the concept of weakly d-supercyclic and then investigate this concept for weighted composition operators.

DEFINITION 3.1

For $N \geq 2$, we say that $T_1, \ldots, T_N$ are weakly d-supercyclic operators on a topological vector space X, if the projective orbit of their direct sums has a weakly supercyclic vector of the form $(x, \ldots, x) \in X^N$. The vector x is called a weakly d-supercyclic vector of $T_1, \ldots, T_N$.

Throughout this section $\mathcal{H}$ is a non-trivial Hilbert space of analytic functions on $\mathbb{D}$ such that $1, z \in \mathcal{H}$ and for each $\lambda \in \mathbb{D}$, the linear functional of point evaluation at λ given by $f \to f(\lambda)$ is bounded.

For any $\lambda \in \mathbb{D}$, let e_λ denote the linear functional of point evaluation at λ on $\mathcal{H}$, that is, $e_\lambda(f) = f(\lambda)$ for every f in $\mathcal{H}$. Since e_λ is a bounded linear functional, the Riesz representation theorem states that

$$e_\lambda(f) = \langle f, k_\lambda \rangle$$

for some $k_\lambda \in \mathcal{H}$. A well-known example of such spaces is the weighted Hardy space.

Theorem 3.2. *If φ_1, φ_2 have their attractive fixed points α, β in $\mathbb{D}$, then $C_{w_1,\varphi_1}, C_{w_2,\varphi_2}$ are not weakly d-supercyclic on $\mathcal{H}$.*

Proof. Let f be a weakly d-supercyclic vector for $C_{w_1,\varphi_1}, C_{w_2,\varphi_2}$. Since the constant function 1 is in $\mathcal{H}$, (e_α, e_β) is a continuous map from $\mathcal{H} \oplus \mathcal{H}$ onto $\mathbb{C} \times \mathbb{C}$. Therefore,

$$\begin{aligned}&\{((\lambda C^n_{w_1,\varphi_1} f)(\alpha), (\lambda C^n_{w_2,\varphi_2} f)(\beta)) : n \geq 0, \lambda \in \mathbb{C}\} \\ &\quad = \{(\lambda (w_1(\alpha))^n f(\alpha), \lambda (w_2(\beta))^n f(\beta)) : n \geq 0, \lambda \in \mathbb{C}\}\end{aligned}$$

is dense in $\mathbb{C} \times \mathbb{C}$, hence $f(\alpha)$, $f(\beta)$ should be non-zero. Let $g(z) = 1$ and $\epsilon > 0$. Without loss of generality, we may assume that $| w_1(\alpha) | \leq | w_2(\beta) |$. Put

$$U = \{h \in \mathcal{H} : |\langle h - g, k_\alpha \rangle| < \epsilon\}$$

$$V = \{h \in \mathcal{H} :| \langle h, k_\beta \rangle |< \epsilon\}.$$

Since $U \times V$ is a weak neighborhood of $(g, 0)$, there exist $n \geq 1$ and a scalar $\lambda \in \mathbb{C}$ such that

$$|\langle \lambda C^n_{w_1,\varphi_1} f, k_\alpha\rangle - g(\alpha)| = |\ \lambda(w_1(\alpha))^n f(\alpha) - g(\alpha)\ | < \epsilon$$

and

$$|\langle \lambda C^n_{w_2,\varphi_2} f, k_\beta\rangle| = |\ \lambda(w_2(\beta))^n f(\beta)\ | < \epsilon. \qquad (*)$$

Note that $w_2(\beta) \neq 0$. Hence we have

$$|g(\alpha)| \leq \epsilon + |\lambda(w_1(\alpha))^n f(\alpha)| = \ \epsilon + \left|\frac{w_1(\alpha)}{w_2(\beta)}\right|^n \left|\frac{f(\alpha)}{f(\beta)}\right| \lambda(w_2(\beta))^n f(\beta)\ | \ .$$

Note that $|\frac{w_1(\alpha)}{w_2(\beta)}| \leq 1$. So for every $\varepsilon > 0$, by $(*)$ we get $|g(\alpha)| \leq \epsilon(1 + C)$, where $C = |\lambda \frac{f(\alpha)}{f(\beta)}|$. It is a contradiction with the fact that $g(\alpha) \neq 0$. So $C_{w_1,\varphi_1}, C_{w_2,\varphi_2}$ cannot be weakly d-supercyclic. □

Theorem 3.3. *Let $\mathcal{H} \subseteq H^\infty$, $C_{w_1,\varphi_1}, C_{w_2,\varphi_2}$ are weakly d-supercyclic on $\mathcal{H}$, and $\alpha \in \mathbb{D}$ is a fixed point of φ_1. Then the sequence $\{\Pi^{n-1}_{j=0} w_2 \circ \varphi_2^{[j]}(\beta)/(w_1(\alpha))^n\}_n$ is unbounded for every $\beta \in \mathbb{D} - \{\alpha\}$.*

Proof. Let f be a weakly d-supercyclic vector for $C_{w_1,\varphi_1}, C_{w_2,\varphi_2}$. Put $g(z) = z - \alpha$ and let $\varepsilon > 0$. Since

$$\{h \in \mathcal{H} : |\langle h - g, k_\alpha\rangle| < \varepsilon\} \times \{h \in \mathcal{H} : |\langle h - 0, k_\beta\rangle| < \varepsilon\}$$

is a weak neighborhood of $(g, 0)$, there exist $n \geq 1$ and $\lambda \in \mathbb{C}$ such that

$$|\lambda C^n_{w_1,\varphi_1} f(\alpha) - g(\alpha)| = |\lambda(w_1(\alpha))^n f(\alpha)| < \varepsilon \qquad (**)$$

and

$$|\lambda C^n_{w_2,\varphi_2} f(\beta) - g(\beta)| = |\lambda \Pi^{n-1}_{j=0} w_2 \circ \varphi_2^{[j]}(\beta) f(\varphi_2^{[n]}(\beta)) - g(\beta)| < \varepsilon.$$

Put $a_n = \Pi^{n-1}_{j=0} w_2 \circ \varphi_2^{[j]}(\beta)/(w_1(\alpha))^n$. Assume on the contrary that there is a constant C such that $|\ a_n\ | \leq C$ for all n. Thus by using (**), we get

$$\begin{aligned} |g(\beta)| &< \varepsilon + |\lambda \Pi^{n-1}_{j=0} w_2 \circ \varphi_2^{[j]}(\beta) f(\varphi_2^{[n]}(\beta)| \\ &< \varepsilon + \left|\lambda(w_1(\alpha))^n f(\alpha)\right| \left|\frac{a_n}{f(\alpha)}\right| \left| f(\varphi_2^{[n]}(\beta))\right| \\ &\leq \varepsilon + \frac{\varepsilon C C_1}{|f(\alpha)|}, \end{aligned}$$

where f is bounded by C_1. Note that the last inequality follows from $(*)$. Since $\varepsilon > 0$ is arbitrary, $g(\beta) = 0$, which is a contradiction. This completes the proof. □

COROLLARY 3.4

Suppose $\mathcal{H} \subseteq H^{\infty}$ and φ_1 has a fixed point α in $\mathbb{D}$. If there is a point $\beta \in \mathbb{D} - \{\alpha\}$ and a positive integer N such that $|w_2(\varphi_2^{[n]}(\beta))| \leq |w_1(\alpha)|$ for all $n \geq N$, then $C_{w_1,\varphi_1}, C_{w_2,\varphi_2}$ are not weakly d-supercyclic.

Recall that a multiplier of $\mathcal{H}$ is an analytic function φ on $\mathbb{D}$ such that $\varphi\mathcal{H} \subseteq \mathcal{H}$. The set of all multipliers of $\mathcal{H}$ is denoted by $M(\mathcal{H})$. If φ is a multiplier, then the multiplication operator M_{φ}, defined by $M_{\varphi} f = \varphi f$, is bounded on $\mathcal{H}$. It is known that every multiplier is a bounded holomorphic function on $\mathbb{D}$. The following definition is a generalization of the well-known hypercyclicity criterion to the setting of disjointness.

Theorem 3.5 (*d*-Hypercyclicity criterion). *Suppose X is a topological vector space and $T_1, T_2, \ldots, T_N$ are bounded linear operator on X. If there exist an increasing sequence of positive integers $\{n_k\}$ and dense subsets $X_0, X_1, \ldots, X_N$ of X and mappings $S_{m,k} : X_m \to X$ where $k \in \mathbb{N}$, $1 \leq m \leq N$, such that*

(*i*) $T_m^{n_k} \to 0$ *pointwise on* X_0 *as* $k \to \infty$,
(*ii*) $S_{m,k} \to 0$ *pointwise on* X_m *as* $k \to \infty$ *and*
(*iii*) $(T_i^{n_k} S_{m,k} - \delta_{i,m} Id_{X_m}) \to 0$ *pointwise on* $X_m (1 \leq i \leq N)$, *then* $T_1, T_2, \ldots, T_N$ *are d-hypercyclic.*

Note that if the operators $T_1, T_2, \ldots, T_N$ are d-hypercyclic, then each of them must be hypercyclic.

PROPOSITION 3.6

Let φ_1, φ_2 be non-constant multipliers of $\mathcal{H}$. If $M_{\varphi_1}^*, M_{\varphi_2}^*$ are d-hypercylic operators, then φ_1/φ_2 must be non-constant.

Proof. Assume that, to reach a contradiction, $\varphi_2 = \lambda\varphi_1$ for some $\lambda \in \mathbb{C}$. Let $f \neq 0$ be a d-hypercyclic vector for $M_{\varphi_1}, M_{\lambda\varphi_1}$. There exist subsequences (n_k) and (m_k) of integers for which $(M_{\varphi_1}^{*n_k} f, M_{\lambda\varphi_1}^{*n_k} f) \to (f, 0)$ and $(M_{\varphi_1}^{*m_k} f, M_{\lambda\varphi_1}^{*m_k} f) \to (0, f)$. We know that $M_{\lambda\varphi_1}^{*n} = \bar{\lambda}^n M_{\varphi_1}^{*n}$ for all $n \in \mathbb{N}$. So $M_{\varphi_1}^{*n_k} f \to f$ and $\bar{\lambda}^{n_k} M_{\varphi_1}^{*n_k} f \to 0$. So we must have $|\lambda| < 1$. On the other hand, since $M_{\varphi_1}^{*m_k} f \to 0$ and $\bar{\lambda}^{m_k} M_{\varphi_1}^{*m_k} f \to f$, so $|\lambda| \geq 1$. This is a contradiction. □

By the following theorem, Godefroy and Shapiro [6] characterized the hypercyclicity of M_{φ}^* on $\mathcal{H}$.

Theorem 3.7. *Suppose φ is a non-constant multiplier of $\mathcal{H}$. Then the operator M_{φ}^* is hypercyclic, if $\varphi(\mathbb{D})$ intersects the unit circle. Conversely, if every bounded function φ in $H^{\infty}(\mathbb{D})$ is a multiplier of $\mathcal{H}$, with $\| M_{\varphi} \| = \| \varphi \|_{\infty}$ and the operator M_{φ}^* is hypercyclic, then $\varphi(\mathbb{D})$ intersects the unit circle.*

We mean by $\mathbb{D}$, $\partial\mathbb{D}$ the unit disk and unit circle, respectively.

Theorem 3.8. *Let φ_1 and φ_2 be non-constant multipliers of $\mathcal{H}$ and suppose $M_{\varphi_1}^*$ and $M_{\varphi_2}^*$ are hypercyclic. If $\varphi_1(\varphi_2^{-1}(\mathbb{D})) \cap \partial\mathbb{D} \neq \emptyset$ and $\varphi_2(\varphi_1^{-1}(\mathbb{D})) \cap \partial\mathbb{D} \neq \emptyset$, then $M_{\varphi_1}^*$ and $M_{\varphi_2}^*$ are d-hypercyclic.*

Proof. Put

$$V_0 = \{z \in \Omega : |\varphi_1(z)| < 1, |\varphi_2(z)| < 1\}$$
$$V_1 = \{z \in \Omega : |\varphi_1(z)| > 1, |\varphi_2(z)| < 1\}$$

and

$$V_2 = \{z \in \Omega : |\varphi_1(z)| < 1, |\varphi_2(z)| > 1\}.$$

Note that since φ_1 is a non-constant analytic map, by open mapping theorem, $\varphi_1(\varphi_2^{-1}(\mathbb{D})$ is open and intersects the unit circle. So the open sets V_0, V_2 are both non-empty. Note that the sets $X_0 = \text{span}\{k_z : z \in V_0\}$, $X_1 = \text{span}\{k_z : z \in V_1\}$ and $X_2 = \text{span}\{k_z : z \in V_2\}$ are dense in $\mathcal{H}$. To see this, suppose that $f \in \mathcal{H}$ and $\langle f, k_z\rangle = 0$ for all $z \in V_0$. Then the zero set of f has a limit point in Ω and so $f \equiv 0$; i.e., $\bar{X_0} = \mathcal{H}$. Similarly $\bar{X_1} = \mathcal{H}$ and $\bar{X_2} = \mathcal{H}$. Note that $M^{*n}_{\varphi_1} k_z = \overline{\varphi_1(z)}^n k_z$, and $M^{*n}_{\varphi_2} k_z = \overline{\varphi_2(z)}^n k_z$. If $z \in V_0$, $|\varphi_1(z)| < 1$ and $|\varphi_2(z)| < 1$. By this fact and the linearly of $M^*_{\varphi_1}, M^*_{\varphi_2}$ we get that $||M^{*n}_{\varphi_1}|| \to 0$, $||M^{*n}_{\varphi_2}|| \to 0$ pointwise on X_0. Set $V_1' = \{k_z : z \in V_1\}$. First we suppose that V_1' is a linearly independent set. In this case we can define a linear map $S_1 : X_1 \to \mathcal{H}$ by extending the definition $S_1 k_z = \overline{\varphi_1(z)}^{-1} k_z$ $(z \in V_1)$ linearly to X_1. Since $|\varphi_1(z)| > 1$ for each $z \in V_1$, there is no possibility of dividing by zero, and moreover $S_1^n k_z = \overline{\varphi_1(z)}^{-n} k_z \to 0$ as $n \to \infty$. Clearly for each $z \in V_1$, $M^*_{\varphi_1} S_1 k_z = k_z$ and $M^{*n}_{\varphi_2} S_1^n k_z = \overline{\varphi_2(z)}^n \overline{\varphi_1(z)}^{-n} k_z \to 0$. Now, assume that V_1' is not linearly independent. In this case, we use the same method as one used by Godefroy and Shapiro in Theorem 4.5 of [6]. Consider a countable dense subset $F_1 = \{z_n \in \mathbb{D} : n \geq 1\}$, and by using induction choose a sequence $\{\lambda_n\}$ as follows: Take $\lambda_1 = z_1$, $F_2 = F_1 - \{z \in F_1 : k_z \in \text{span}\{k_{\lambda_1}\}\}$. Denote the first element of F_2 by λ_2 and let $F_3 = F_2 - \{z \in F_2 : k_z \in \text{span}\{k_{\lambda_1}, k_{\lambda_2}\}\}$. Continuing this process, we obtain a subset $L = \{\lambda_n : n \geq 1\}$ of V_1 for which the set $V_L = \{k_\lambda : \lambda \in L\}$ is linearly independent and dense in $\mathcal{H}$. Define $S_{1,n} : X_L \to \mathcal{H}$ by $S_{1,n} k_\lambda = \overline{\varphi_1(\lambda)}^{-n} k_\lambda$. Clearly $M^{*n}_{\varphi_1} S_{1,n} k_\lambda = k_\lambda$ for all $k_\lambda \in X_L$. Furthermore, $S_{1,n} \to 0$ as $n \to \infty$ pointwise on X_L. Now put $V_2' = \{k_z : z \in V_2\}$. The same process can be done for V_2' to obtain S_2^n or $S_{2,n}$, respectively where V_2' is linearly independent or not. So the condition of d-hypercyclicity criterion are satisfied and the proof is complete. □

COROLLARY 3.9

Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be non-constant multipliers of $\mathcal{H}$, and suppose that $M^*_{\varphi_1}, M^*_{\varphi_2}, \ldots, M^*_{\varphi_N}$ are hypercyclic. If $\varphi_i(\varphi_j^{-1}(\mathbb{D})) \cap \partial\mathbb{D} \neq \emptyset$ for all $1 \leq i, j \leq N$ with $i \neq j$, then $M^*_{\varphi_1}, M^*_{\varphi_2}, \ldots, M^*_{\varphi_N}$ are d-hypercyclic.

Example 3.10. Let $\varphi_1(z) = z + \frac{1}{2}$ and $\varphi_2(z) = z - 1$. Then $|\varphi_2(\frac{1}{2})| < 1$, $|\varphi_1(\frac{1}{2})| = 1$, and so $\varphi_1(\varphi_2^{-1}(\mathbb{D})) \cap \partial\mathbb{D} \neq \emptyset$. Similarly, since $|\varphi_2(0)| = 1$ and $|\varphi_1(0)| < 1$, we get $\varphi_2(\varphi_1^{-1}(\mathbb{D})) \cap \partial\mathbb{D} \neq \emptyset$, so by Theorem 3.5, $M^*_{\varphi_1}, M^*_{\varphi_2}$ are d-hypercyclic.

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