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A new characterization of $L_2(p)$ by NSE

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Abstract. In this paper we give a new characterization of simple group $L_2(p)$ with p a prime by both its order and $nse(L_2(p))$, the set of numbers of elements of $L_2(p)$ with the same order.

Keywords. Finite groups; numbers of elements with the same order; linear groups.

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1. Introduction

Throughout this paper, all groups are finite and G denotes a group. We denote by $\pi(G)$ the set of prime divisors of $|G|$, $\pi_e(G)$ the set of element orders of G. Let n be an integer. Then $\varphi(n)$ denotes the Euler function of n. G is called a simple K_n-group if G is simple with $|\pi(G)| = n$. The prime graph $GK(G)$ of a group G is defined as a graph with vertex set $\pi(G)$ and two distinct primes $p, q \in \pi(G)$ are adjacent if G contains an element of order pq. Moreover, the connected components of $GK(G)$ are denoted by π_i, $1 \leq i \leq t(G)$, where $t(G)$ is the number of connected components of G. In particular, we define by π_1 the component containing the prime 2 for a group of even order. Further unexplained notation is standard, and readers may refer to [4].

The motivation of this article is to investigate Thompson's problem related to algebraic number fields as follows (see Problem 12.37 of [5]). Write $M_t(G) := \{g \in G \mid g^t = 1\}$. G_1 and G_2 are of the same order type if and only if $|M_t(G_1)| = |M_t(G_2)|$, $t = 1, 2, \ldots$.

Thompson's problem. Suppose that G_1 and G_2 are of the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable?

Unfortunately, till now, no one could prove it completely, or even give a counterexample. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Set $nse(G) := \{m_k \mid k \in \pi_e(G)\}$. If groups G_1 and G_2 are of the same order type, then $|G_1| = |G_2|$ and $nse(G_1) = nse(G_2)$. So it is natural to investigate the Thompson's problem by $|G|$ and $nse(G)$.

Recall that a simple K_4-group G is a simple group satisfying $|\pi(G)| = 4$. The authors of [8] proved that all simple K_4-groups can be uniquely determined by $nse(G)$ and $|G|$. Also it is claimed in [7] that linear groups $L_2(2^a)$ can be determined by their orders and $nse(L_2(2^a))$ if $2^a - 1$ or $2^a + 1$ is a prime. In this paper, we continue to work on simple groups $L_2(p)$ and wish to prove that $L_2(p)$ is characterizable by their orders $|L_2(p)|$ and the set $nse(L_2(p))$ if p is a prime. Our main theorem is the following.

Theorem A. *Let G be a group and $p \geq 5$ be a prime. Then $G \cong L_2(p)$ if and only if $|G| = |L_2(p)|$ and $nse(G) = \{1, \varphi(d) \cdot p \cdot (p+1)/2, \varphi(s) \cdot p \cdot (p-1)/2, p^2 - 1\} = nse(L_2(p))$, where $1 < d \mid (p-1)/2$ and $1 < s \mid (p+1)/2$ are two integers.*

2. Preliminaries

In this section we give some lemmas which will be used in the sequel.

Lemma 2.1 (*Lemma* 2.6 *of* [6]). *Let G be a group. If $1 \neq n \in nse(G)$ and $2 \nmid n$, then the following statements hold*:

(1) $2 \mid |G|$;
(2) $m_2 = n$;
(3) *For any $2 < t \in \pi_e(G)$, $m_t \neq n$.*

Lemma 2.2 [3]. *Let G be a group and t be a positive integer dividing $|G|$. If $M_t(G) = \{g \in G | g^t = 1\}$, then $t \mid |M_t(G)|$.*

Lemma 2.3 (*Lemma* 2.3 *of* [7]). *Let G be a group and P be a cyclic Sylow p-subgroup of G. Assume further that $|P| = p^a$ and r is an integer such that $p^a r \in \pi_e(G)$. Then $m_{p^a r} = m_r(C_G(P)) m_{p^a}$. In particular, $\varphi(r) m_{p^a} \mid m_{p^a r}$.*

Recall that G is a 2-Frobenius group, if G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with K/H and H as Frobenius kernels.

Lemma 2.4 (*Theorem* 2 *of* [2]). *If G is a 2-Frobenius group of even order, then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \bigcup \pi(G/K) = \pi_1$, $|G/K| \mid |\mathrm{Aut}\ (K/H)|$, G/K and K/H are cyclic. In particular, $|G/K| < |K/H|$ and G is solvable.*

Lemma 2.5 (*Theorem of* [9]). *Let G be a group such that $t(G) \geq 2$. Then G has one of the following structures*:

(a) *G is a Frobenius or 2-Frobenius group,*
(b) *G has a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that $\pi(N) \cup \pi(G/G_1) \subseteq \pi_1$ and G_1/N is a nonabelian simple group.*

Lemma 2.6 (*Theorem* 1 *of* [1]). Let G be an unsolvable simple group of order g. If $p \mid g$, where $p > g^{1/3}$ is a prime, then G is isomorphic to $L_2(p)(p > 3)$ or $L_2(p-1)$, where $p > 3$ is a Fermat prime.

3. Proof of Theorem A

Proof. If $G \cong L_2(p)$, then we obviously see that $|G| = |L_2(p)| = p(p^2-1)/2$. Further, it follows by II, Theorem 8.2, 8.3, 8.4, 8.5 of [4] that $nse(G) = nse(L_2(p)) = \{1, \varphi(d) \cdot$

$p\cdot(p+1)/2, \varphi(s)\cdot p\cdot(p-1)/2, p^2-1\}$, where $1 < d \mid (p-1)/2$ and $1 < s \mid (p+1)/2$ are two integers. Now we prove the converse.

It follows by Lemma 2.1 that $2 \in \pi(G)$. Moreover, $m_2 = p(p-1)/2$ or $p(p+1)/2$. Let P be a Sylow p-subgroup of G. Then P is a cyclic group of order p as $|G| = |L_2(p)| = p(p^2-1)/2$. Further, Lemma 2.2 implies that $p \mid (1+m_p)$, leading to $m_p = p^2-1$.

We claim that $pr \notin \pi_e(G)$ for every prime $r \in \pi(G)$ distinct from p. Otherwise, assume that there exists some prime $r \in \pi(G)$ such that $r \neq p$ and $pr \in \pi_e(G)$. Then Lemma 2.3 indicates that $(r-1)m_p \mid m_{pr}$. If $m_{pr} = \varphi(d)\cdot p\cdot(p+1)/2$ for some $1 < d \mid (p-1)/2$, then $(r-1)(p^2-1) \mid \varphi(d)\cdot p\cdot(p+1)$, which implies that $(p-1) \mid \varphi(d)$, a contradiction. The same argument gives $m_{pr} \neq \varphi(s)\cdot p\cdot(p-1)/2$. Hence $m_{pr} = p^2-1$. It follows that $(r-1)(p^2-1) \mid (p^2-1)$, yielding $r = 2$. On the other hand, Lemma 2.2 indicates that $2p \mid (1+m_2+m_p+m_{2p}) = 2p^2-1+m_2$. Note that $p \mid m_2$. This contradiction shows that $t(G) \geq 2$.

Suppose first that G is a Frobenius group with Frobenius kernel K and Frobenius complement H. We easily see that $t(G) = 2$. Moreover, there exists no element of order pr for every prime $r \in \pi(G)$, and we obtain that either $|H| = p$ or $|K| = p$. Assume that the former holds. Then $|H| \mid (|K|-1) = (p^2-1)/2-1$, a contradiction. This yields that $|K| = p$, implying $m_p = p-1$, again a contradiction. Suppose that G is a 2-Frobenius group. By applying Lemma 2.4, we see that G is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2 = \{p\}$, $\pi(H)\cup\pi(G/K) = \pi_1$, G/K and K/H are cyclic and $|G/K| \mid |\mathrm{Aut}(K/H)|$. Then $|K/H| = p$. Let $t = |G/K|$. Then $|H| = (p+1)\cdot(p-1)/2t$. Since $t = |G/K| \mid (p-1)$, we have $(p+1)/2 \mid |H|$. Note that H is nilpotent. Then all elements of order $s \mid (p+1)/2$ are all contained in H. This implies that $m_s \leq |H|-1$, that is, $m_s \leq (p^2-1)/2t$. If $m_s = \varphi(s)\cdot p(p+1)/2$, then $\varphi(s)p \leq (p-1)/t$, a contradiction. If $m_s = \varphi(d)\cdot p(p-1)/2$, then $\varphi(s)p \leq (p+1)/t$, also a contradiction. Similarly, $m_s = p^2-1$ is impossible.

Therefore, it follows from Lemma 2.5 that G has a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$ such that M/N is a non-abelian simple group and $\pi(N)\cup\pi(G/M) \subseteq \pi_1$. Hence $p \mid |M/N|$. It is clear that $|G| < p^3$. Then Lemma 2.6 shows that $M/N \cong L_2(p)$ or $L_2(p-1)$. If $M/N \cong L_2(p)$, then $|M/N| = |G|$. Hence $N = 1$ and $G = M \cong L_2(p)$. If $M/N \cong L_2(p-1)$, then $(p-2)(p-1)p \mid p(p-1)(p+1)$. Moreover, $\frac{p+1}{p-2} = 1 + \frac{3}{p-2}$ is an integer, leading to $p = 5$. As a result, $G \cong L_2(5) \cong L_2(4) \cong M/N$. Hence $N = 1$ and thus $G \cong L_2(5)$, and the proof is complete. $\square$

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