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h-p Spectral element methods for three dimensional elliptic problems on non-smooth domains, Part-II: Proof of stability theorem

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Abstract. This is the second of a series of papers devoted to the study of h-p spectral element methods for three dimensional elliptic problems on non-smooth domains. The present paper addresses the proof of the main stability theorem. We assume that the differential operator is a strongly elliptic operator which satisfies Lax–Milgram conditions. The spectral element functions are non-conforming. The stability estimate theorem of this paper will be used to design a numerical scheme which give exponentially accurate solutions to three dimensional elliptic problems on non-smooth domains and can be easily implemented on parallel computers.

Keywords. Spectral element method; vertex singularity; edge singularity; vertex-edge singularity; modified coordinates; geometric mesh; quasi uniform mesh; stability estimate.

Mathematics Subject Classification. 65N35, 35J25.

1. Introduction

Many stationary phenomena in science and engineering are modelled by elliptic boundary value problems. It is well known that the regularity of solutions of these problems is severely affected by the presence of corners and edges in a three-dimensional domain Ω. There are three types of singularities caused by non-smoothness of domains in R^3: the vertex, the edge, and the vertex-edge singularities. In such cases, the standard numerical methods such as finite element method (FEM) and finite difference method (FDM) yield poor convergence results for numerical solutions of most of the practical problems. In order to have reliable and economical approximate solutions, it is desirable to find an efficient and accurate numerical technique.

This paper is the second of a series of papers devoted to the study of the h-p spectral element method for solving three dimensional elliptic problems on non-smooth domains using parallel computers. The first paper [5], which deals with the regularity of the solution in the neighbourhoods of vertices, edges and vertex-edges and description of the main stability theorem, will be published separately. The present paper develops the proof of the basic stability estimate of [5] based on the work in [13]. In the forthcoming work, we shall provide the numerical scheme, the parallel preconditioner, error estimates and the solution techniques based on the stability estimates. It is shown that the error decays exponentially with respect to the number of refinements in the geometric mesh and the number of degrees of freedom in each variable on each element. Theoretical results have been validated on parallel computers independently in [6].

The h-p version of the finite element method for solving three dimensional elliptic problems on non-smooth domains with exponential accuracy was proposed by Guo in [9, 12]. To overcome the singularities which arise along vertices and edges they used geometric meshes which are defined in neighbourhoods of vertices, edges and vertex-edges. We refer to [1–4, 10–12] for a detailed discussion on the regularity and the h-p FEM for three dimensional elliptic problems on non-smooth domains.

In [13], we proposed an exponentially accurate h-p spectral element method to solve elliptic boundary value problems on non-smooth domains in R^3. For Dirichlet problems, we use spectral element functions which are non-conforming and hence there are no common boundary values. For problems with mixed boundary conditions, the spectral element functions are essentially non-conforming except that they are continuous only at the wirebasket (union of vertices and edges) of the elements. Hence the cardinality of the set of common boundary values which is equal to the values of the function at the wirebasket of the elements is much smaller than the cardinality of the common boundary values for the standard finite element method.

To overcome the singularities which arise in the neighbourhoods of the vertices, vertex-edges and edges we use local systems of coordinates. These local coordinates are modified versions of spherical and cylindrical coordinate systems in their respective neighbourhoods. Away from these neighbourhoods standard cartesian coordinates are used. In each of these neighbourhoods, we use a geometrical mesh which becomes finer near the corners and edges. The geometrical mesh becomes a quasi-uniform mesh in the new system of coordinates.

We assume our spectral element functions to be a sum of tensor product of polynomials of variable degree bounded by W. Let N denote the number of layers in the geometric mesh imposed on each of the neighbourhoods of vertices, edges and vertex-edges. It is assumed that N is proportional to W. We remark that throughout the paper $\frac{1}{N}$ and W refer to h and p respectively for notational simplicity. We then define a quadratic form $\mathcal{V}^{N,W}(\{\mathcal{F}_u\})$ which measures the sum of the squares of the residuals in the partial differential equation and a fractional Sobolev norm of the residuals in the boundary conditions and enforce continuity across inter element boundaries by adding a term which measures the sum of the squares of the jump in the function and its derivatives at inter element boundaries in appropriate Sobolev norms suitably weighted to the functional being minimized. We prove that there is another quadratic form $\mathcal{U}^{N,W}(\{\mathcal{F}_u\})$ consisting of the weighted H^2 norms of the spectral element functions which is bounded by $\mathcal{V}^{N,W}(\{\mathcal{F}_u\})$ multiplied by a factor which grows logarithmically in W for problems with Dirichlet boundary conditions. For problems with mixed boundary conditions, this factor may grow rapidly as N^4 and thus the method is difficult to parallelize. To resolve this difficulty of parallelization, we can make the spectral element functions conforming at the wirebasket. We prove a stability theorem for mixed problems when the spectral element functions vanish on the wirebasket.

We remark that having obtained our approximate solution consisting of non-conforming spectral element functions we can make a correction to the approximate solution so that the corrected solution is conforming and the error between the corrected and exact solution is exponentially small in N in the H^1 norm over the whole domain. Throughout this paper, (x_1, x_2, x_3), (ρ, ϕ, θ) and (r, θ, x_3) denote the standard cartesian, spherical and cylindrical coordinates, respectively.

This paper is organized as follows. In §2, we quote the notations, definitions of various spaces, structure of the neighbourhoods of the vertices, edges and vertex-edges and the modified local systems of coordinates in these neighbourhoods from [13]. In §3 and §4, we derive estimates on the second order derivatives and the lower order derivatives of the solution. Estimates for terms in the interior and on the boundary of the polyhedron Ω are quoted in §5 and §6. In §7, we combine all the results of sections 3, 4, 5 and 6 to complete the proof of the main stability estimate.

2. Preliminaries

Let Ω denote a polyhedron in R^3, as shown in figure 1(a). We shall denote the boundary of Ω by $\partial\Omega$. Let Γ_i, $i \in \mathcal{I} = \{1, 2, \ldots, I\}$, be the faces of the polyhedron. Let $\mathcal{D}$ be a subset of $\mathcal{I}$ and $\mathcal{N} = \mathcal{I} \setminus \mathcal{D}$. We impose Dirichlet and Neumann boundary conditions on the faces Γ_i, $i \in \mathcal{D}$ and Γ_j, $j \in \mathcal{N}$, respectively. Further, let $\partial\Omega = \Gamma^{[0]} \cup \Gamma^{[1]}$, $\Gamma^{[0]} = \bigcup_{i \in \mathcal{D}} \bar{\Gamma}_i$ and $\Gamma^{[1]} = \bigcup_{i \in \mathcal{D}} \bar{\Gamma}_i$.

We consider an elliptic boundary value problem posed on Ω with mixed Neumann and Dirichlet boundary conditions:

$$\begin{aligned} Lw &= F \quad \text{in } \Omega, \\ w &= g^{[0]} \quad \text{for } x \in \Gamma^{[0]}, \\ \left(\frac{\partial w}{\partial n}\right)_A &= g^{[1]} \quad \text{for } x \in \Gamma^{[1]}, \end{aligned} \tag{2.1}$$

where n denotes the outward normal and $\left(\frac{\partial w}{\partial n}\right)_A$ is the usual conormal derivative. It is assumed that the differential operator

$$Lw(x) = \sum_{i,j=1}^{3} -(a_{ij} w_{x_j})_{x_i} + \sum_{i=1}^{3} b_i w_{x_i} + cw \tag{2.2}$$

is a strongly elliptic differential operator which satisfies the Lax–Milgram conditions. Moreover, $a_{ij} = a_{ji}$ for all i, j and the coefficients of the differential operator are analytic. The data F, $g^{[0]}$ and $g^{[1]}$ are analytic on each open face and $g^{[0]}$ is continuous on $\bigcup_{i \in \mathcal{D}} \bar{\Gamma}_i$.

In [5, 13], we divided the domain Ω into a regular region, a set of vertex neighbourhoods, a set of edge neighbourhoods and a set of vertex-edge neighbourhoods. A set of modified coordinates have been defined in these neighoubrhoods which are modifications of the standard spherical and cylindrical coordinates. Here, we briefly recall the notations, definitions and the description of these neighbourhoods. For a detailed structure of these neighbourhoods in various regions of the polyhedron Ω, we refer to [13].

Let S_j, $j \in \mathcal{J} = \{1, 2, \ldots, J\}$ be the edges and A_k, $k \in \mathcal{K} = \{1, 2, \ldots, K\}$ be the vertices of the polyhedron. We shall also denote an edge by e, where $e \in \mathcal{E} = \{S_1, S_2, \ldots, S_J\}$, the set of edges, and a vertex by v, where $v \in \mathcal{V} = \{A_1, A_2, \ldots, A_K\}$, the set of vertices. Now consider a vertex v and let e denote one of the edges passing through it, which we assume to coincide with the x_3-axis. Let ϕ denote the angle at which $x = (x_1, x_2, x_3)$ makes with the x_3-axis. By Ω^v, we denote the vertex neighbourhood of the vertex v (figure 1(b)) defined by

$$\Omega^v = \left(B_{\rho_v}(v) \Bigg\backslash \bigcup_{e \in \mathcal{E}^v} \overline{\mathcal{V}_{\rho_v, \phi_v}(v, e)} \right) \bigcap \Omega,$$

where $B_{\rho_v}(v) = \{x : \mathrm{dist}(x, v) < \rho_v\}$ and $\mathcal{V}_{\rho_v,\phi_v}(v, e) = \{x \in \Omega : 0 < \mathrm{dist}(x, v) < \rho_v,\ 0 < \phi < \phi_v\}$, where ϕ_v is a constant. For every vertex v, ρ_v and ϕ_v are chosen so small that $B_{\rho_v}(v) \cap B_{\rho_{v'}}(v') = \emptyset$ if the vertices v and v' are distinct and $\mathcal{V}_{\rho_v,\phi_v}(v, e') \bigcap \mathcal{V}_{\rho_v,\phi_v}(v, e'') = \emptyset$ if $e'$and e'' are distinct edges having v as a common vertex. Moreover, ρ_v and ϕ_v are chosen so that $\rho_v \sin(\phi_v) = Z$, a constant for all $v \in V$, the set of vertices.

Next, let e denote an edge, which we assume to coincide with the x_3-axis, whose end points are the vertices v and v'. Then we define the edge neighbourhood of the edge e denoted as Ω^e shown in figure 1(c) by

$$\Omega^e = \{x \in \Omega :\ \delta_v < x_3 < l_e - \delta_{v'},\ 0 < r < Z\},$$

where l_e is the length of the edge e, $\delta_v = \rho_v \cos(\phi_v)$, $\delta_{v'} = \rho_{v'} \cos(\phi_{v'})$ and $r = \sqrt{x_1{}^2 + x_2{}^2}$.

Now, by Ω^{v-e} we denote the vertex-edge neighbourhood of the vertex v and the edge e shown in figure 1(d) defined by

$$\Omega^{v-e} = \{x \in \Omega :\ 0 < \phi < \phi_v,\ 0 < x_3 < \delta_v = \rho_v \cos\phi_v\}.$$

Finally, Ω^r denotes the portion of the polyhedron Ω obtained after the closure of the vertex-neighbourhoods, edge neighbourhoods and vertex-edge neighbourhoods have been removed from it. Thus, let

$$\Delta = \left\{ \bigcup_{v \in \mathcal{V}} \bar{\Omega}^v \right\} \cup \left\{ \bigcup_{e \in \mathcal{E}} \bar{\Omega}^e \right\} \cup \left\{ \bigcup_{v-e \in \mathcal{V}-\mathcal{E}} \bar{\Omega}^{v-e} \right\}.$$

Then

$$\Omega^r = \Omega \setminus \Delta.$$

To overcome the singularities which arise in the neighbourhoods of the vertices, vertex-edges and edges we use local system of coordinates introduced in [13]. These local coordinates are modified versions of spherical and cylindrical coordinate systems in their respective neighbourhoods. Away from these neighbourhoods, standard Cartesian coordinates are used in the regular region of the polyhedron. Table 1 summarizes the system of coordinates used in various regions of the polyhedron Ω. For details, we refer to [13].

3. Estimates for the second derivatives of spectral element functions

3.1 *Estimates for the second derivatives in the interior*

We first obtain estimates for elements in the regular region Ω^r of Ω. Divide Ω^r into N_r elements, Ω^r_l for $1 \le l \le N_r$ consisting of curvilinear cubes, tetrahedrons and prisms. Consider an element Ω^r_l. Then Ω^r_l has n_l faces $\{\Gamma^r_{l,i}\}_{1 \le i \le n_l}$. Let $\partial\Gamma^r_{l,i}$ denote the boundary of the face $\Gamma^r_{l,i}$.

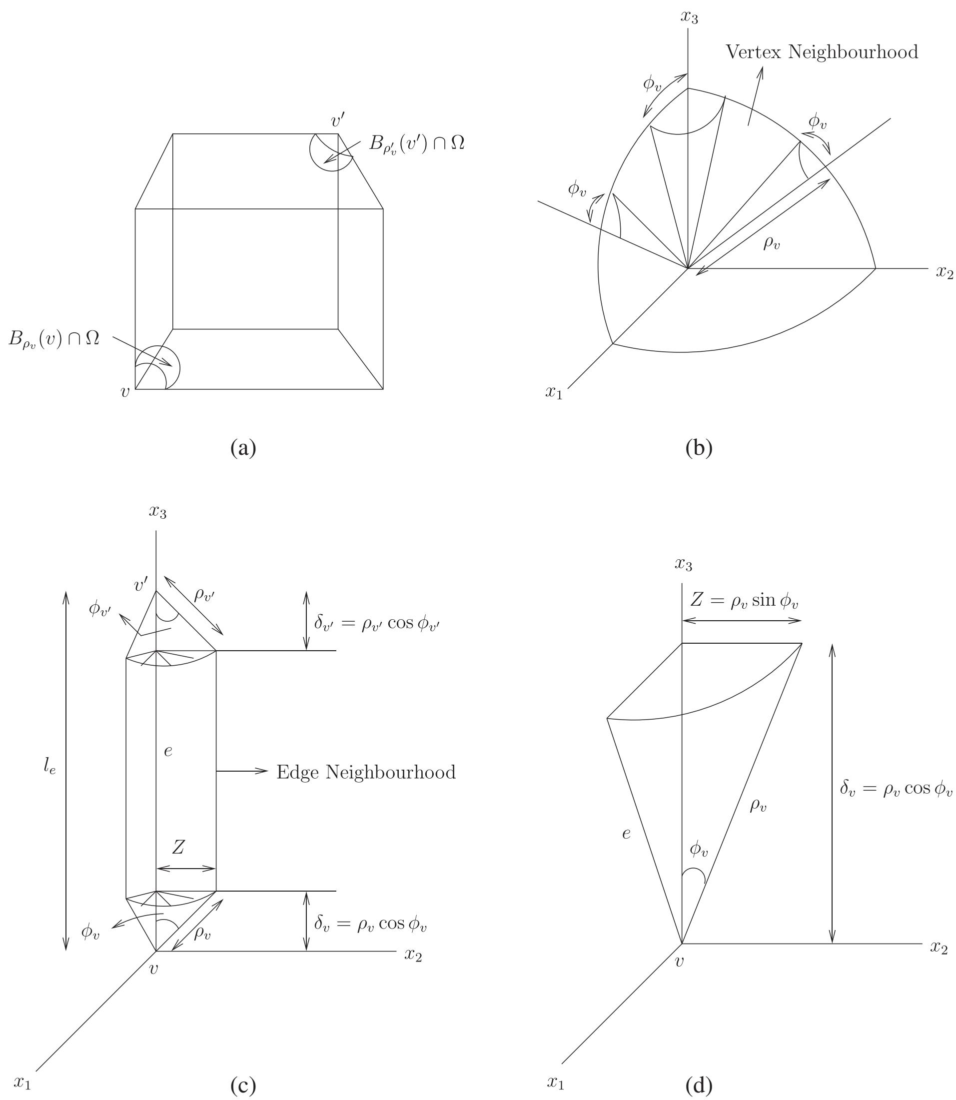


Figure 1. (**a**) Polyhedral domain Ω, (**b**) vertex neighbourhood Ω^v, (**c**) edge neighbourhood Ω^e, and (**d**) vertex-edge neighbourhood Ω^{v-e}.

Table 1. System of coordinates used in various parts of Ω.

Region	Coordinates	Type
Regular	x_1, x_2, x_3	Standard cartesian
Vertex neighbourhood	$x_1^v = \phi, x_2^v = \theta, x_3^v = \chi = \ln \rho$	Modified spherical
Edge neighbourhood	$x_1^e = \tau = \ln r, x_2^e = \theta, x_3^e = x_3$	Modified cylindrical
Vertex-edge neighbourhood	$x_1^{v-e} = \psi = \ln(\tan \phi), x_2^{v-e} = \theta,$ $x_3^{v-e} = \zeta = \ln x_3$	Hybrid

To proceed, we need to review some material in [8]. Let O be a bounded open subset of R^3 with a Lipschitz boundary ∂O. Assume in addition that ∂O is piecewise C^2. Let P be a point on ∂O in a neighbourhood of which ∂O is C^2. It is possible to find two curves of class C^2 in a neighbourhood of P, passing through P and being orthogonal there. Let us denote these curves by $\mathcal{C}_1, \mathcal{C}_2$ and by $\boldsymbol{\tau_1}, \boldsymbol{\tau_2}$ the unit tangent vectors to $\mathcal{C}_1, \mathcal{C}_2$ respectively and by s_1, s_2 the arc lengths along these curves. We assume that $\boldsymbol{\tau_1}, \boldsymbol{\tau_2}$ has the direct orientation at P. Let $\boldsymbol{\nu}$ be the unit normal at P defined as $\boldsymbol{\nu} = \boldsymbol{\tau_1} \times \boldsymbol{\tau_2}$. Then at P, $\mathcal{B}_P$, the second fundamental form at P is the bilinear form

$$(\boldsymbol{\zeta}, \boldsymbol{\eta})| \to -\sum_{j,k=1}^{2} \frac{\partial \boldsymbol{\nu}}{\partial s_j} \cdot \boldsymbol{\tau}_k \zeta_j \eta_k, \tag{3.1}$$

where $\boldsymbol{\zeta}, \boldsymbol{\eta}$ are the tangent vectors to ∂O at P and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2)$ in the basis $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$. In other words,

$$\mathcal{B}(\boldsymbol{\zeta}, \boldsymbol{\eta}) = -\frac{\partial \boldsymbol{\nu}}{\partial \boldsymbol{\zeta}} \cdot \boldsymbol{\eta}, \tag{3.2}$$

where $\frac{\partial}{\partial \boldsymbol{\zeta}}$ denotes differentiation in the $\boldsymbol{\zeta}$ direction.

Let $\mathbf{w}$ be a vector field defined in a neighbourhood of O. If P is a point on ∂O, then by $\mathbf{w}_\nu$ we shall denote the component of $\mathbf{w}$ in the direction of $\boldsymbol{\nu}$, while we shall denote by $\mathbf{w}_T$, the projection of $\mathbf{w}$ on the tangent hyperplane to ∂O, i.e.

$$\mathbf{w}_\nu = \mathbf{w} \cdot \boldsymbol{\nu}, \tag{3.3}$$

$$\mathbf{w}_T = \mathbf{w} - \mathbf{w}_\nu \boldsymbol{\nu} = \mathbf{w}_{\tau_1} \boldsymbol{\tau}_1 + \mathbf{w}_{\tau_2} \boldsymbol{\tau}_2. \tag{3.4}$$

Here, $\mathbf{w}_{\tau_i} = \mathbf{w} \cdot \boldsymbol{\tau}_i$ for $i = 1, 2$.

We shall denote by ∇_T the projection of the gradient vector on the tangent hyperplane

$$\nabla_T u = \nabla u - \frac{\partial u}{\partial \boldsymbol{\nu}} \boldsymbol{\nu} = \sum_{j=1}^{2} \frac{\partial u}{\partial s_j} \boldsymbol{\tau}_j. \tag{3.5}$$

Let $\mathbf{h}$ be a vector field defined on ∂O such that $\mathbf{h}$ is tangent to O except on a set of zero measure. Then

$$\mathrm{div}_T(\mathbf{h}) = \sum_{j=1}^{2} \left(\frac{\partial \mathbf{h}}{\partial s_j} \right) \cdot \boldsymbol{\tau}_j. \tag{3.6}$$

We now cite Theorem 3.1.1.2 of [8].

Theorem 3.1. *Let O be a bounded open subset of R^3 with a Lipschitz boundary ∂O. Assume, in addition that ∂O is piecewise C^2. Then for all $\mathbf{w} \in (H^2(O))^3$ we have*

$$\begin{aligned} \int_O (\mathrm{div}(\mathbf{w}))^2 dx - \sum_{i,j=1}^{3} \int_O \frac{\partial w_i}{\partial x_j} \frac{\partial w_j}{\partial x_i} dx &= \int_{\partial O} \{\mathrm{div}_T(\mathbf{w}_\nu \mathbf{w}_T) - 2\mathbf{w}_T \cdot \nabla_T \mathbf{w}_\nu\} d\sigma \\ &\quad - \int_{\partial O} \{(\mathrm{tr}\, \mathcal{B}) \mathbf{w}_\nu^2 + \mathcal{B}(\mathbf{w}_T, \mathbf{w}_T)\} d\sigma. \end{aligned} \tag{3.7}$$

Here, dx denotes a volume element and $d\sigma$ an element of surface area.

Consider an element Ω_l^r which is assumed to be a curvilinear cube as shown in figure 2. Let $\Gamma_{l,i}^r$ denote one of the faces of Ω_l^r. Let Q be a point inside $\Gamma_{l,i}^r$. The unit tangent vectors $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ and the unit normal vector $\boldsymbol{\nu}$ at Q are shown in figure 2. Consider a point $P \in \partial\Gamma_{l,i}^r$ and assume that P is not a vertex of Ω_l^r. Then we can define the vector $\boldsymbol{n}$ at P as the vector belonging to the tangent hyperplane which is orthogonal to the tangent vector to the curve $\partial\Gamma_{l,i}^r$ at P. Moreover, $\mathbf{n}$ is chosen to point out of $\Gamma_{l,i}^r$. Recall that A is the matrix $(a_{i,j})$. Define

$$\left(\frac{\partial u}{\partial X}\right)_A (P) = (X \cdot A\nabla u)(P), \quad \text{where } X = \mathbf{n}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\nu}. \tag{3.8}$$

Let s_1, s_2 denote arc lengths along $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ and s denote arc length measured along $\partial\Gamma_{l,i}^r$. Since the differential operator L is strongly elliptic, there exists a positive constant μ_0 such that

$$\sum_{i,j=1}^{3} a_{i,j}\zeta_i\zeta_j \geq \mu_0|\zeta|^2 . \tag{3.9}$$

Let us write

$$Mu = \sum_{i,j=1}^{3} (a_{i,j}u_{x_j})_{x_i} = \mathrm{div}(A\nabla_x u). \tag{3.10}$$

We can now prove the following result.

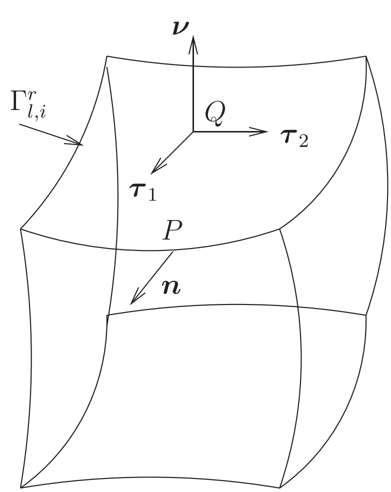


Figure 2. The element Ω_l^r.

Lemma 3.1. *Let $u \in H^3(\Omega_l^r)$. Then*

$$\frac{\mu_0^2}{2}\rho_v^2 \sin^2(\phi_v)\int_{\Omega_l^r}\sum_{i,j=1}^{3}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|^2 dx$$

$$\leq \rho_v^2 \sin^2(\phi_v)\int_{\Omega_l^r}|Lu|^2 dx - \rho_v^2\sin^2(\phi_v)\left\{\sum_i \oint_{\partial\Gamma_{l,i}^r}\left(\frac{\partial u}{\partial \boldsymbol{n}}\right)_A\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A ds\right.$$

$$\left. -2\sum_i\int_{\Gamma_{l,i}^r}\sum_{j=1}^{2}\left(\frac{\partial u}{\partial \boldsymbol{\tau}_j}\right)_A\frac{\partial}{\partial s_j}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A\right)d\sigma\right\} + C\int_{\Omega_l^r}\sum_{|\alpha|\leq 1}\left|D_x^\alpha u\right|^2 dx. \tag{3.11}$$

Here, C denotes a constant. Moreover, dx denotes a volume element, $d\sigma$ an element of surface area and ds an element of arc length.

Proof. The proof is similar to the proof of Lemmas 3.1 and 3.4 in [7]. We define the vector field $\mathbf{w}$ by $\mathbf{w} = A\nabla_x u$. Then using (3.8), we get

$$Mu = \operatorname{div}(\mathbf{w}), \quad \left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A = \mathbf{w}_\nu \quad \text{and} \quad \left(\frac{\partial u}{\partial \boldsymbol{\tau}_j}\right)_A = \mathbf{w}_{\tau_j}. \tag{3.12}$$

Applying Theorem 3.1,

$$\int_{\Omega_l^r}|Mu|^2 dx - \sum_{i,j=1}^{3}\int_{\Omega_l^r}\frac{\partial w_i}{\partial x_j}\frac{\partial w_j}{\partial x_i}dx = \left\{\sum_i\int_{\Gamma_{l,i}^r}\operatorname{div}_T(\mathbf{w}_\nu \mathbf{w}_T)d\sigma\right.$$

$$\left. -2\sum_i\int_{\Gamma_{l,i}^r}\sum_{j=1}^{2}\left(\frac{\partial u}{\partial \boldsymbol{\tau}_j}\right)_A\frac{\partial}{\partial s_j}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A\right)d\sigma\right\} - \sum_i\int_{\Gamma_{l,i}^r}\left\{(\operatorname{tr}\mathcal{B})\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A^2\right.$$

$$\left. + \mathcal{B}\left(\sum_{j=1}^{2}\left(\tfrac{\partial u}{\partial \boldsymbol{\tau}_j}\right)_A\boldsymbol{\tau}_j, \sum_{j=1}^{2}\left(\tfrac{\partial u}{\partial \boldsymbol{\tau}_j}\right)_A\boldsymbol{\tau}_j\right)\right\}d\sigma. \tag{3.13}$$

Now by Lemma 3.1.3.4 of [8] the following inequality holds for all $u \in H^2(\Omega)$:

$$\mu_0^2\sum_{i,j=1}^{3}\left|\frac{\partial^2 u}{\partial x_i\partial x_j}\right|^2 \leq \sum_{i,j=1}^{3}\frac{\partial w_i}{\partial x_j}\frac{\partial w_j}{\partial x_i} + 2\sum_{i,j,k,l=1}^{3}\left|a_{i,k}\frac{\partial^2 u}{\partial x_j \partial x_k}\frac{\partial a_{j,l}}{\partial x_i}\frac{\partial u}{\partial x_l}\right| dx$$

a.e. in Ω. Integrating we have

$$\mu_0^2\sum_{i,j=1}^{3}\int_{\Omega_l^r}\left|\frac{\partial^2 u}{\partial x_i\partial x_j}\right|^2 dx \leq \sum_{i,j=1}^{3}\int_{\Omega_l^r}\frac{\partial w_i}{\partial x_j}\frac{\partial w_j}{\partial x_i}dx$$

$$+ C\int_{\Omega_l^r}\sum_{i=1}^{3}\left|\frac{\partial u}{\partial x_i}\right|\sum_{i,j=1}^{3}\left|\frac{\partial^2 u}{\partial x_i\partial x_j}\right| dx.$$

The constant C in the above inequality depends on M, where M is a common bound for all C^1 norms of the $a_{i,j}$. Hence,

$$\frac{\mu_0^2}{2}\sum_{i,j=1}^{3}\int_{\Omega_l^r}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|^2 dx \le \sum_{i,j=1}^{3}\int_{\Omega_l^r}\frac{\partial w_i}{\partial x_j}\frac{\partial w_j}{\partial x_i}\,dx + C\int_{\Omega_l^r}\sum_{|\alpha|=1}|D_x^\alpha u|^2 dx. \tag{3.14}$$

Here, C denotes a generic constant. Now from Lemma 3.2 we have

$$\sum_i \int_{\Gamma_{l,i}^r}\mathrm{div}_T(\mathbf{w}_\nu \mathbf{w}_T)d\sigma = \sum_i \int_{\partial\Gamma_{l,i}^r}\mathbf{w}_\nu \mathbf{w}_n ds.$$

Here, $\mathbf{w}_n = \mathbf{w}\cdot\mathbf{n}$ and $\mathbf{n}$ is the vector depicted in figure 2 lying on the tangent hyperplane at the point P and orthogonal to the tangent vector to the curve $\partial\Gamma_{l,i}^r$. Now

$$Mu = Lu - \sum_{i=1}^{3} b_i u_{x_i} - cu. \tag{3.15}$$

Combining (3.13) and (3.14) and proceeding as in Lemma 3.4 in [7], we obtain

$$\begin{aligned}
&\frac{\mu_0^2}{2}\rho_v^2\sin^2(\phi_v)\int_{\Omega_l^r}\sum_{i,j=1}^{3}\left|\frac{\partial^2 u}{\partial x_i\partial x_j}\right|^2 dx \\
&\le \rho_v^2\sin^2(\phi_v)\int_{\Omega_l^r}|Lu|^2dx - \rho_v^2\sin^2(\phi_v)\left\{\sum_i\oint_{\partial\Gamma_{l,i}^r}\left(\frac{\partial u}{\partial \mathbf{n}}\right)_A\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A ds\right. \\
&\quad \left. -2\sum_i\int_{\Gamma_{l,i}^r}\sum_{j=1}^{2}\left(\frac{\partial u}{\partial \boldsymbol{\tau}_j}\right)_A\frac{\partial}{\partial s_j}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A\right)d\sigma\right\} \\
&\quad +C\int_{\Omega_l^r}\sum_{|\alpha|\le 1}\left|D_x^\alpha u\right|^2 dx + D\sum_i\int_{\Gamma_{l,i}^r}\sum_{|\alpha|=1}\left|D_x^\alpha u\right|^2 d\sigma.
\end{aligned} \tag{3.16}$$

Now for any $\epsilon > 0$, there exists a constant K_ϵ such that

$$\int_{\Gamma_{l,i}^r}\sum_{|\alpha|=1}\left|D_x^\alpha u\right|^2 d\sigma \le \epsilon\int_{\Omega_l^r}\sum_{|\alpha|=2}\left|D_x^\alpha u\right|^2 dx + K_\epsilon\int_{\Omega_l^r}\sum_{|\alpha|=1}\left|D_x^\alpha u\right|^2 dx.$$

Using this in (3.16) and choosing ϵ small enough (3.11) follows. □

We now prove the following result which we have used in the proof of Lemma 3.2.

Lemma 3.2. *Let $w \in H^2(\Omega_l^r)$ and $\Gamma_{l,k}^r$ denote one of the faces of Ω_l^r. Then*

$$\int_{\Gamma_{l,k}^r}\mathrm{div}_T(\mathbf{w}_\nu \mathbf{w}_T)d\sigma = \int_{\partial\Gamma_{l,k}^r}\mathbf{w}_\nu \mathbf{w}_n ds. \tag{3.17}$$

Here, $d\sigma$ denotes an element of surface area and ds an element of arc length.

Proof. We shall use geodesic coordinates to prove the result. For any point $P \in$ closure$(\Gamma^r_{l,k})$ there is an open subset U of R^2 containing $(0,0)$ such that $\pi : U \to R^3$ is an allowable surface patch for $\Gamma^r_{l,k}$ in a neighbourhood of P. Moreover, the first fundamental form of π is $d\zeta^2 + G(\zeta,\eta)d\eta^2$, where G is a smooth function on U such that $G(0,\eta) = 1$ and $G_\zeta(0,\eta) = 0$ whenever $(0,\eta) \in U$. Hence, for any $\epsilon > 0$ we can choose a fine enough triangulation of $\Gamma^r_{l,k}$ so that on each triangle there is a set of geodesic coordinates such that $|G_\zeta/G| \leq \epsilon$ for all $(\zeta,\eta) \in U_i$ for all i. Here, the curve corresponding to $\zeta = 0$ is chosen to be a geodesic. All the curves $\eta =$ constant are geodesics orthogonal to the curve $\zeta = 0$. This can always be done if the surface patch is small enough. Such a system of coordinates is called Fermi coordinates [14].

Now integrating over one such triangle we obtain

$$\begin{aligned}\int\limits_{\pi_i(U_i)} \mathrm{div}_T(\mathbf{w}_\nu \mathbf{w}_T)d\sigma &= \int\limits_{\pi_i(U_i)} \sum_{j=1}^{2} \frac{\partial}{\partial s_j}(\mathbf{w}_\nu(\mathbf{w}_T \cdot \boldsymbol{\tau}_j))d\sigma \\ &\quad - \int\limits_{\pi_i(U_i)} \sum_{j=1}^{2} \mathbf{w}_\nu\left(\mathbf{w}_T \cdot \frac{\partial \boldsymbol{\tau}_j}{\partial s_j}\right)d\sigma. \end{aligned} \tag{3.18}$$

Clearly,

$$\begin{aligned}\int\limits_{\pi_i(U_i)} \sum_{j=1}^{2} \frac{\partial}{\partial s_j}\big(\mathbf{w}_\nu(\mathbf{w}_T \cdot \boldsymbol{\tau}_j)\big)d\sigma &= \int\limits_{U_i}\left\{\frac{\partial}{\partial \zeta}(\mathbf{w}_\nu(\mathbf{w}_T \cdot \boldsymbol{\tau}_1))\sqrt{G}\right. \\ &\quad \left. + \frac{\partial}{\partial \eta}(\mathbf{w}_\nu(\mathbf{w}_T \cdot \boldsymbol{\tau}_2))\right\} d\zeta d\eta.\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\pi_i(U_i)} \sum_{j=1}^{2} \frac{\partial}{\partial s_j}\big(\mathbf{w}_\nu(\mathbf{w}_T \cdot \boldsymbol{\tau}_j)\big)d\sigma &= \int_{\partial U_i}\Big(\mathbf{w}_\nu(\mathbf{w}_T \cdot \boldsymbol{\tau}_1)\sqrt{G}d\eta \\ &\quad - \mathbf{w}_\nu(\mathbf{w}_T \cdot \boldsymbol{\tau}_2)d\zeta\Big) \\ &\quad - \int_{\pi_i(U_i)} \mathbf{w}_\nu(\mathbf{w}_T \cdot \boldsymbol{\tau}_1)\left(\frac{\partial\sqrt{G}}{\partial\zeta}\Big/\sqrt{G}\right)d\sigma.\end{aligned}$$

Now

$$\frac{dx}{ds} = x_\zeta \frac{d\zeta}{ds} + x_\eta \frac{d\eta}{ds} = \boldsymbol{\tau}_1 \frac{d\zeta}{ds} + \boldsymbol{\tau}_2\sqrt{G}\frac{d\eta}{ds}.$$

Hence, $\mathbf{n} = \boldsymbol{\tau}_1\sqrt{G}\frac{d\eta}{ds} - \boldsymbol{\tau}_2\frac{d\zeta}{ds}$ is the unit outward normal to $\partial\pi_i(U_i)$. And so

$$\begin{aligned}\int\limits_{\pi_i(U_i)} \sum_{j=1}^{2} \frac{\partial}{\partial s_j}(\mathbf{w}_\nu(\mathbf{w}_T \cdot \boldsymbol{\tau}_j))d\sigma &= \int\limits_{\partial\pi_i(U_i)} \mathbf{w}_\nu \mathbf{w}_n\, ds \\ &\quad - \int\limits_{\pi_i(U_i)} \mathbf{w}_\nu(\mathbf{w}_T \cdot \boldsymbol{\tau}_1)\left(\frac{\partial\sqrt{G}}{\partial\zeta}\Big/\sqrt{G}\right)d\sigma.\end{aligned}$$

Summing over all triangular elements of the form $\pi_i(U_i)$, we obtain using (3.18),

$$\begin{aligned}\int_{\Gamma^r_{l,k}} \mathrm{div}_T(\mathbf{w}_\nu \mathbf{w}_T) d\sigma &= \int_{\partial\Gamma^r_{l,k}} \mathbf{w}_\nu \mathbf{w}_n ds \\ &\quad - \sum_i \int_{\pi_i(U_i)} \mathbf{w}_\nu (\mathbf{w}_T \cdot \boldsymbol{\tau}_1) \left(\frac{\partial\sqrt{G}}{\partial\zeta} / \sqrt{G} \right) d\sigma \\ &\quad - \sum_i \int_{\pi_i(U_i)} \sum_{j=1}^{2} \mathbf{w}_\nu \left(\mathbf{w}_T \cdot \frac{\partial \boldsymbol{\tau}_j}{\partial s_j} \right) d\sigma. \end{aligned} \tag{3.19}$$

Now

$$\left| \sum_i \int_{\pi_i(U_i)} \mathbf{w}_\nu (\mathbf{w}_T \cdot \boldsymbol{\tau}_1) \left(\frac{\partial\sqrt{G}}{\partial\zeta} / \sqrt{G} \right) d\sigma \right| \le \epsilon \int_{\Gamma^r_{l,k}} |\mathbf{w}|^2 \, d\sigma \, . \tag{3.20a}$$

Next, at the point P the ζ parameter curves and the η parameter curves are geodesics. Hence, at P, $\frac{\partial \boldsymbol{\tau}_j}{\partial s_j} \cdot T' = 0$ for any vector T' which lies on the tangent plane at P. Thus, for any $\epsilon > 0$, we can choose a fine enough triangulation so that

$$\left| \sum_i \int_{\pi_i(U_i)} \mathbf{w}_\nu (\mathbf{w}_T \cdot \frac{\partial \boldsymbol{\tau}_j}{\partial s_j}) d\sigma \right| \le \epsilon \int_{\Gamma^r_{l,i}} (\mathbf{w}_\nu^2 + |\mathbf{w}_T|^2) d\sigma \le \epsilon \int_{\Gamma^r_{l,i}} |\mathbf{w}|^2 \, d\sigma. \tag{3.20b}$$

Now from (3.19) and (3.20) we obtain the result since ϵ is arbitrary. □

3.2 *Estimates for second derivatives in vertex neighbourhoods*

In figure 3, the intersection of Ω with a sphere of radius ρ_v with center at the vertex v is shown. On removing the vertex-edge neighbourhoods, which are shaded, we obtain the vertex neighbourhood Ω^v (figure 1(b)), where $v \in \mathcal{V}$ and $\mathcal{V}$ denotes the set of vertices.

Choose ρ_v and ϕ_v so that $\rho_v \sin(\phi_v) = Z$, a constant, for all $v \in V$. Let S^v denote the intersection of the closure of Ω^v with $B_{\rho_v}(v) = \{x : |x - v| \le \rho_v\}$. S^v is divided into triangular and quadrilateral elements S^v_j for $1 \le j \le I_v$, where I_v is a fixed constant. We impose a geometric mesh in the vertex neighbourhood Ω^v of the vertex v as shown in figure 3(b) (see [5, 13] for details).

Thus, Ω^v is divided into N_v curvilinear cubes and prisms $\{\Omega^v_l\}_{1 \le l \le N_v}$. Let (x^v_1, x^v_2, x^v_3) be the modified system of coordinates in the vertex neighbourhood (see table 1) and let $\tilde{\Omega}^v_l$ be the image of Ω^v_l in (x^v_1, x^v_2, x^v_3) coordinates. Now

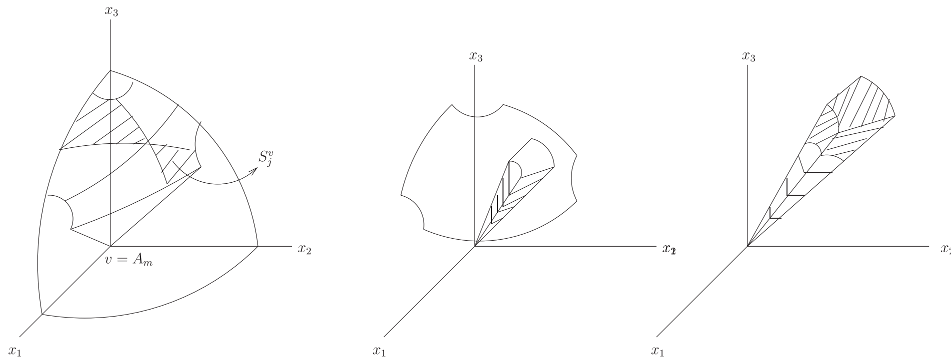
$$\int_{\Omega^v_l} \rho^2 |Lu|^2 dx = \int_{\tilde{\Omega}^v_l} e^{\mathcal{X}} \sin\phi |e^{2\mathcal{X}} Lu|^2 d\phi d\theta d\mathcal{X} \, . \tag{3.21}$$

Define

$$L^v u(x^v) = e^{\frac{\mathcal{X}}{2}} \sqrt{\sin\phi} (e^{2\mathcal{X}} Lu). \tag{3.22}$$

We have the relation

$$\rho \nabla_x u = Q^v \nabla_{x^v} u \, , \quad \text{where } Q^v = O^v P^v. \tag{3.23}$$



(a) (b)

Figure 3. Mesh imposed (**a**) on the spherical boundary S^v, and (**b**) on the vertex neighbourhood Ω^v.

Here O^v is the orthogonal matrix,

$$O^v = \begin{bmatrix} \cos\phi\cos\theta & -\sin\theta & \sin\phi\cos\theta \\ \cos\phi\sin\theta & \cos\theta & \sin\phi\sin\theta \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \quad \text{and} \quad P^v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.24a}$$

Define

$$A^v = (Q^v)^T A Q^v. \tag{3.24b}$$

Since $\phi_0 < \phi < \pi - \phi_0$, where ϕ_0 denotes a positive constant, so we have $\mu_0 I \le A^v \le \mu_1 I$ for some positive constants μ_0 and μ_1. Let $\tilde{\Omega}_l^v$ be a curvilinear cube and let its faces be denoted by $\{\tilde{\Gamma}_{l,i}^v\}$. We now prove the following result.

Lemma 3.3. *There exist positive constants C_v such that*

$$\begin{aligned} &\frac{\mu_0^2}{2}\sin^2(\phi_v)\int_{\tilde{\Omega}_l^v} e^{x_3^v}\sum_{r,s=1}^{3}\left|\frac{\partial^2 u}{\partial x_r^v \partial x_s^v}\right|^2 dx^v \\ &\le \sin^2(\phi_v)\int_{\tilde{\Omega}_l^v} |L^v u(x^v)|^2 dx^v \\ &- \sin^2(\phi_v)\left\{\sum_i \oint_{\partial\tilde{\Gamma}_{l,i}^v} e^{x_3^v}\sin(x_1^v)\left(\frac{\partial u}{\partial \boldsymbol{n}^v}\right)_{A^v}\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v} ds^v \right. \\ &\left. - 2\sum_i \int_{\tilde{\Gamma}_{l,i}^v} e^{x_3^v}\sin(x_1^v)\sum_{j=1}^{2}\left(\frac{\partial u}{\partial \boldsymbol{\tau}_j^v}\right)_{A^v}\frac{\partial}{\partial s_j^v}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v}\right) d\sigma^v \right\} \\ &+ C_v \int_{\tilde{\Omega}_l^v}\sum_{|\alpha|\le 1} e^{x_3^v}\left|D_{x^v}^{\alpha} u\right|^2 dx^v \end{aligned} \tag{3.25}$$

for $u \in H^3(\tilde{\Omega}_l^v)$. Here, dx^v denotes a volume element in x^v coordinates, $d\sigma^v$ an element of surface area and ds^v an element of arc length in x^v coordinates.

Proof. Let $\boldsymbol{f}$ denote a vector field. Then

$$\rho \,\mathrm{div}_x(\boldsymbol{f}) = \frac{e^{-2\mathcal{X}}}{\sin\phi}\mathrm{div}_{x^v}(e^{2\mathcal{X}}\sin\phi(Q^v)^T\boldsymbol{f}).$$

Take $\boldsymbol{f} = A\nabla_x u$. Therefore,

$$\int_{\Omega_l^v}|\rho\,\mathrm{div}_x(A\nabla_x u)|^2\,dx = \int_{\tilde{\Omega}_l^v}\frac{e^{-\mathcal{X}}}{\sin\phi}|\mathrm{div}_{x^v}(e^{\mathcal{X}}\sin\phi(Q^v)^T AQ^v\nabla_{x^v}u)|^2 dx^v. \tag{3.26}$$

Define

$$M^v u(x^v) = \mathrm{div}_{x^v}\left(e^{\mathcal{X}/2}\sqrt{\sin\phi}A^v\nabla_{x^v}u\right). \tag{3.27}$$

Then

$$\frac{e^{-\mathcal{X}/2}}{\sqrt{\sin\phi}}\mathrm{div}_{x^v}(e^{\mathcal{X}}\sin\phi A^v\nabla_{x^v}u) = M^v u(x^v) + \frac{1}{2}e^{\mathcal{X}/2}\sum_{j=1}^{3}\left(\sqrt{\sin\phi}\,a_{3,j}^v\frac{\partial u}{\partial x_j^v} + \frac{\cos\phi}{\sqrt{\sin\phi}}a_{1,j}^v\frac{\partial u}{\partial x_j^v}\right).$$

Here, A^v is as defined in (3.24b). Define the vector field $\mathbf{w}$ by

$$\mathbf{w} = e^{\mathcal{X}/2}\sqrt{\sin\phi}A^v\nabla_{x^v}u\,. \tag{3.28a}$$

Then

$$L^v u(x^v) = \mathrm{div}_{x^v}(\mathbf{w}) + \eta^v u(x^v),$$

where

$$\begin{aligned}\eta^v u(x^v) &= -\frac{1}{2}e^{\mathcal{X}/2}\sqrt{\sin\phi}\sum_{j=1}^{3}a_{3,j}^v\frac{\partial u}{\partial x_j^v} - \frac{1}{2}e^{\mathcal{X}/2}\frac{\cos\phi}{\sqrt{\sin\phi}}\sum_{j=1}^{3}a_{1,j}^v\frac{\partial u}{\partial x_j^v}\\ &\quad + \sum_{i=1}^{3}b_i^v\frac{\partial u}{\partial x_i^v} + c^v u\,.\end{aligned} \tag{3.28b}$$

Here,

$$\begin{aligned}\|b_i^v\|_{0,\infty,\tilde{\Omega}_l^v} &= O(e^{3\mathcal{X}/2}),\ \|c^v\|_{0,\infty,\tilde{\Omega}_l^v} = O(e^{5\mathcal{X}/2}) \quad \text{and}\\ &\quad \|e^{\mathcal{X}/2}\sqrt{\sin\phi}A^v\|_{1,\infty,\tilde{\Omega}_l^v} = O(e^{\mathcal{X}/2}).\end{aligned} \tag{3.29}$$

To obtain (3.25) we shall use Theorem 3.1 applied to the vector field $\mathbf{w}$ along with Lemma 3.2. Now

$$2\int_{\tilde{\Gamma}_{l,i}^v}\mathbf{w}_T\cdot\nabla_T\mathbf{w}_\nu\,d\sigma^v = 2\int_{\tilde{\Gamma}_{l,i}^v}\sum_{j=1}^{2}e^{\mathcal{X}/2}\sqrt{\sin\phi}\left(\frac{\partial u}{\partial\boldsymbol{\tau}_j^v}\right)_{A^v}\frac{\partial}{\partial s_j^v}\left(e^{\mathcal{X}/2}\sqrt{\sin\phi}\left(\frac{\partial u}{\partial\boldsymbol{\nu}^v}\right)_{A^v}\right)d\sigma^v$$

$$= 2\int_{\tilde{\Gamma}^v_{l,i}} \sum_{j=1}^{2} e^{\mathcal{X}} \sin\phi \left(\frac{\partial u}{\partial \boldsymbol{\tau}^v_j}\right)_{A^v} \frac{\partial}{\partial s^v_j}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v}\right) d\sigma^v$$

$$+2\int_{\tilde{\Gamma}^v_{l,i}} \sum_{j=1}^{2} e^{\mathcal{X}/2}\sqrt{\sin\phi} \left(\frac{\partial u}{\partial \boldsymbol{\tau}^v_j}\right)_{A^v} \left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v} \frac{\partial}{\partial s^v_j}\left(e^{\mathcal{X}/2}\sqrt{\sin\phi}\right) d\sigma^v. \tag{3.30}$$

And so using (3.28), (3.29) and (3.30) we obtain

$$\frac{\mu_0^2}{2}\sin^2(\phi_v)\int_{\tilde{\Omega}^v_l} e^{x^v_3} \sum_{r,s=1}^{3} \left|\frac{\partial^2 u}{\partial x^v_r \partial x^v_s}\right|^2 dx^v$$

$$\leq \sin^2(\phi_v)\int_{\tilde{\Omega}^v_l} |L^v u(x^v)|^2 dx^v$$

$$-\sin^2(\phi_v)\left\{\sum_i \oint_{\partial\tilde{\Gamma}^v_{l,i}} e^{x^v_3}\sin(x^v_1)\left(\frac{\partial u}{\partial \mathbf{n}^v}\right)_{A^v}\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v} ds^v\right.$$

$$\left. - 2\sum_i \int_{\tilde{\Gamma}^v_{l,i}} e^{x^v_3}\sin(x^v_1)\sum_{j=1}^{2}\left(\frac{\partial u}{\partial \boldsymbol{\tau}^v_j}\right)_{A^v}\frac{\partial}{\partial s^v_j}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v}\right) d\sigma^v\right\}$$

$$+ C_v\int_{\tilde{\Omega}^v_l}\sum_{|\alpha|\leq 1} e^{x^v_3}\left|D^\alpha_{x^v}u\right|^2 dx^v + D_v\sum_i\int_{\tilde{\Gamma}^v_{l,i}}\sum_{|\alpha|=1} e^{x^v_3}\left|D^\alpha_{x^v}u\right|^2 d\sigma^v.$$

Now for any $\epsilon > 0$ there exists a constant K_ϵ such that

$$\int_{\tilde{\Gamma}^v_{l,i}}\sum_{|\alpha|=1} e^{x^v_3}\left|D^\alpha_x u\right|^2 d\sigma^v \leq \epsilon\int_{\tilde{\Omega}^v_l}\sum_{|\alpha|=2} e^{x^v_3}\left|D^\alpha_x u\right|^2 dx^v$$

$$+ K_\epsilon\int_{\tilde{\Omega}^v_l}\sum_{|\alpha|=1} e^{x^v_3}\left|D^\alpha_x u\right|^2 dx^v.$$

Choosing ϵ small enough, (3.25) follows from the above equation. □

We now show that the boundary integrals in Lemmas 3.1 and 3.3 coincide when $\Gamma^v_{k,i} = \Gamma^r_{l,m}$ is a portion of the sphere $B_{\rho_v}(v) = \{x : |x - v| = \rho_v\}$, except that they have opposite signs. Let Q^v be the matrix defined in (3.23). Now, if $\mathbf{dx}$ is a tangent vector to a curve in x coordinates then its image in x^v coordinates is given by $\mathbf{dx}^v$, where

$$\mathbf{dx}^v = \frac{(Q^v)^T}{\rho}\mathbf{dx}\,. \tag{3.31}$$

Clearly, the first fundamental form ds^2 in x coordinates is

$$\begin{aligned} ds^2 &= \mathbf{dx}^T\mathbf{dx} = \rho^2(\mathbf{dx}^v)^T[(Q^v)^{-1}(Q^v)^{-T}]\mathbf{dx}^v \\ &= e^{2\mathcal{X}}(d\phi^2 + \sin^2\phi\, d\theta^2 + d\mathcal{X}^2)\,. \end{aligned} \tag{3.32}$$

Moreover, on $\Gamma^r_{l,m}$,

$$d\sigma = \rho_v^2 \sin\phi d\phi d\theta \,. \tag{3.33}$$

Choose, $\boldsymbol{\tau}_1^v = (1,0,0)^T$ and $\boldsymbol{\tau}_2^v = (0,1,0)^T$. These are then orthogonal unit tangent vectors on $\tilde{\Gamma}^v_{k,i}$ since $(ds^v)^2 = d\phi^2 + d\theta^2 + d\mathcal{X}^2$. Define

$$\boldsymbol{\tau}_1 = -(Q^v)^{-T}\boldsymbol{\tau}_1^v = -e_\phi, \qquad \boldsymbol{\tau}_2 = \frac{(Q^v)^{-T}}{\sin\phi}\boldsymbol{\tau}_2^v = e_\theta. \tag{3.34}$$

Let $\boldsymbol{\nu}^v = (0,0,1)^T$ denote the unit normal vector on $\tilde{\Gamma}^v_{k,i}$. Then

$$\boldsymbol{\nu} = -(Q^v)^{-T}\boldsymbol{\nu}^v \tag{3.35}$$

denotes the unit normal to $\Gamma^r_{l,m}$. Finally, let $\mathbf{ds}^v = (d\phi, d\theta, 0)^T$ denote a tangent vector field on $\tilde{\Gamma}^v_{k,i}$. Define

$$ds^v = \sqrt{d\phi^2 + d\theta^2}, \;\; \mathbf{ds} = \rho_v (Q^v)^{-T}\mathbf{ds}^v \;\text{ and }\; ds = \rho_v\sqrt{d\phi^2 + \sin^2(\phi) d\theta^2}. \tag{3.36}$$

Let $\mathbf{n}^v = \frac{(-d\theta, d\phi, 0)^T}{\sqrt{d\theta^2 + d\phi^2}}$ be the unit outward normal to $\partial\tilde{\Gamma}^v_{k,i}$. Define

$$\mathbf{m}^v = \left(-\sin\phi d\theta, \frac{d\phi}{\sin\phi}, 0\right)^T \Bigg/ \sqrt{d\phi^2 + \sin^2\phi d\theta^2}\,. \tag{3.37}$$

Then

$$\mathbf{n} = (Q^v)^{-T}\mathbf{m}^v \tag{3.38}$$

is the unit normal vector to $\partial\Gamma^r_{l,m}$. We now prove the following result.

Lemma 3.4. *Let $\Gamma^v_{k,i} = \Gamma^r_{l,m}$. Then the following identities hold*:

$$\begin{aligned} &\rho_v^2 \sin^2(\phi_v) \oint_{\partial\Gamma^r_{l,m}} \left(\frac{\partial u}{\partial \mathbf{n}}\right)_A \left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A ds \\ &\qquad = -\sin^2(\phi_v) \oint_{\partial\tilde{\Gamma}^v_{k,i}} e^{x_3^v}\sin(x_1^v)\left(\frac{\partial u}{\partial \mathbf{n}^v}\right)_{A^v}\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v} ds^v \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} &\rho_v^2 \sin^2(\phi_v) \int_{\Gamma^r_{l,m}} \sum_{j=1}^2 \left(\frac{\partial u}{\partial \boldsymbol{\tau}_j}\right)_A \frac{\partial}{\partial s_j}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A\right) d\sigma \\ &\qquad = -\sin^2(\phi_v)\int_{\tilde{\Gamma}^v_{k,i}} e^{x_3^v}\sin(x_1^v)\sum_{j=1}^2\left(\frac{\partial u}{\partial \boldsymbol{\tau}_j^v}\right)_{A^v} \frac{\partial}{\partial s_j^v}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v}\right) d\sigma^v. \end{aligned} \tag{3.40}$$

Proof. We first evaluate $\rho_v^2 \sin^2(\phi_v) \oint_{\partial\Gamma^r_{l,m}} \left(\frac{\partial u}{\partial \mathbf{n}}\right)_A \left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A \, ds$. By (3.23) and (3.35)

$$\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A = \frac{-\left((\boldsymbol{\nu}^v)^T (Q^v)^{-1} (Q^v)^{-T}\right)}{\rho_v} A^v \nabla_{x^v} u = \frac{-1}{\rho_v} \left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v}. \tag{3.41}$$

Now by (3.38)

$$\left(\frac{\partial u}{\partial \mathbf{n}}\right)_A = n^T A \nabla_x u = \frac{1}{\rho_v} ((\mathbf{m}^v)^T (Q^v)^{-1} (Q^v)^{-T} A^v \nabla_{x^v} u).$$

Using (3.35), (3.36), (3.37) and (3.38) we obtain

$$\left(\frac{\partial u}{\partial \mathbf{n}}\right)_A ds = \sin\phi \left(\frac{\partial u}{\partial \mathbf{n}^v}\right)_{A^v} ds^v. \tag{3.42}$$

Thus, from (3.41) and (3.42) we obtain (3.39). Finally, we evaluate the term

$$\rho_v^2 \sin^2(\phi_v) \int_{\Gamma^r_{l,m}} \sum_{j=1}^{2} \left(\frac{\partial u}{\partial \boldsymbol{\tau}_j}\right)_A \frac{\partial}{\partial s_j} \left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A\right) d\sigma\ .$$

Using (3.34) it is easy to show that

$$\left(\frac{\partial u}{\partial \boldsymbol{\tau}_1}\right)_A = \frac{-1}{\rho_v} \left(\frac{\partial u}{\partial \boldsymbol{\tau}_1^v}\right)_{A^v} \quad \text{and} \quad \left(\frac{\partial u}{\partial \boldsymbol{\tau}_2}\right)_A = \frac{\sin\phi}{\rho_v} \left(\frac{\partial u}{\partial \boldsymbol{\tau}_2^v}\right)_{A^v}. \tag{3.43}$$

Hence,

$$\frac{\partial}{\partial s_1} \left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A\right) = \frac{1}{\rho_v^2} \frac{\partial}{\partial s_1^v} \left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v}\right) \quad \text{and}$$

$$\frac{\partial}{\partial s_2} \left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A\right) = -\frac{1}{\rho_v^2 \sin\phi} \frac{\partial}{\partial s_2^v} \left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v}\right). \tag{3.44}$$

Moreover, from (3.33)

$$d\sigma = \rho_v^2 \sin\phi \, d\sigma^v. \tag{3.45}$$

Combining (3.43), (3.44) and (3.45) we obtain (3.40). □

3.3 *Estimates for second derivatives in vertex-edge neighbourhoods*

Figure 1(d) shows the vertex-edge neighbourhood Ω^{v-e} of the vertex v and the edge e. A geometric mesh is imposed on Ω^{v-e} as shown in figure 4 (see [5, 13] for details). To proceed further, let $(x_1^{v-e}, x_2^{v-e}, x_3^{v-e})$ be the modified coordinates introduced in the vertex-edge neighbourhood Ω^{v-e} (see table 1). Let $\tilde{\Omega}^{v-e}$ be the image of Ω^{v-e} in x^{v-e} coordinates. Thus, $\tilde{\Omega}^{v-e}$ is divided into $N^{v-e} = I^{v-e}(N+1)^2$ hexahedrons $\tilde{\Omega}_n^{v-e}$. Now

$$\nabla_{x^v} u = J^{v-e} \nabla_{x^{v-e}} u, \tag{3.46}$$

where

$$J^{v-e} = \begin{bmatrix} \sec^2\phi \cot\phi & 0 & -\tan\phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.47}$$

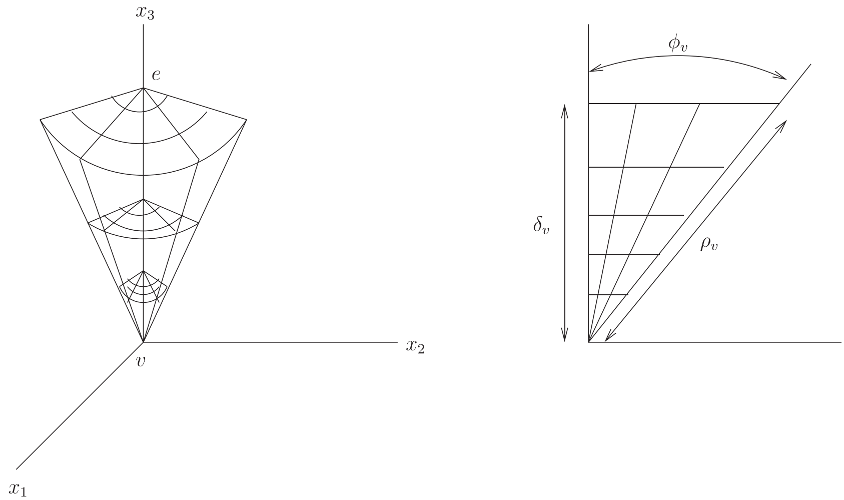


Figure 4. Geometric mesh imposed on Ω^{v-e}.

We now need to evaluate

$$\int_{\Omega_n^{v-e}} \rho^2 \sin^2\phi |Lu(x)|^2 dx \,.$$

Let $\hat{\Omega}_n^{v-e}$ denote the image of Ω_n^{v-e} in x^v coordinates. Then

$$\int_{\Omega_n^{v-e}} \rho^2 \sin^2\phi |Lu(x)|^2 dx = \int_{\hat{\Omega}_n^{v-e}} \sin^2\phi |L^v u(x^v)|^2 dx^v.$$

Let $\boldsymbol{f}$ denote a vector field. Then

$$\mathrm{div}_{x^v}(\boldsymbol{f}) = \frac{1}{\sin\phi\cos\phi}\mathrm{div}_{x^{v-e}}(\sin\phi\cos\phi(J^{v-e})^T\boldsymbol{f}). \tag{3.48}$$

Moreover,

$$\int_{\Omega_n^{v-e}} \rho^2 \sin^2\phi \mid Lu(x) \mid^2\ dx = \int_{\tilde{\Omega}_n^{v-e}} \sin^3\phi\cos\phi |L^v u(x^v)|^2\, dx^{v-e}.$$

Define

$$L^{v-e}u(x^{v-e}) = (\sin\phi)^{3/2}(\cos\phi)^{1/2}L^v u(x^v)\,. \tag{3.49}$$

Then using (3.27), (3.46) and (3.48) we can write

$$\begin{aligned} L^{v-e}u(x^{v-e}) \;=\; & (\sin\phi)^{1/2}(\cos\phi)^{-1/2} \\ & \mathrm{div}_{x^{v-e}}(e^{\mathcal{X}/2}(\sin\phi)^{3/2}\cos\phi(J^{v-e})^T A^v J^{v-e}\nabla_{x^{v-e}}u) \\ & +\sum_{i=1}^{2} b_i^{v-e}u_{x_i^{v-e}} + c^{v-e}u\,. \end{aligned} \tag{3.50}$$

Define

$$M^{v-e}u(x^{v-e}) = \mathrm{div}_{x^{v-e}}(e^{\mathcal{X}/2}(\cos\phi)^{1/2}\sin^2\phi(J^{v-e})^T A^v J^{v-e}\nabla_{x^{v-e}}u)$$

or

$$M^{v-e}u(x^{v-e}) = \mathrm{div}_{x^{v-e}}(e^{\zeta/2}A^{v-e}\nabla_{x^{v-e}}u),\quad \text{where } A^{v-e} = \sin^2\phi(J^{v-e})^T A^v J^{v-e}. \tag{3.51}$$

Using (3.25)

$$A^{v-e} = (K^{v-e})^T A K^{v-e}, \tag{3.52}$$

where

$$K^{v-e} = O^v R^{v-e} \qquad \text{and} \qquad R^{v-e} = \begin{bmatrix} \frac{1}{\cos\phi} & 0 & \frac{-\sin^2\phi}{\cos\phi} \\ 0 & 1 & 0 \\ 0 & 0 & \sin\phi \end{bmatrix}.$$

Now

$$(\tan\phi)^{1/2}\mathrm{div}_{x^{v-e}}(e^{\mathcal{X}/2}(\sin\phi)^{3/2}\cos\phi(J^{v-e})^T A^v J^{v-e}\nabla_{x^{v-e}}u) = M^{v-e}u(x^{v-e}) - \frac{1}{2}e^{\zeta/2}\sum_{j=1}^{3}\hat{a}_{1,j}^{v-e}\frac{\partial u}{\partial x_j^{v-e}}.$$

Hence, using (3.50),

$$L^{v-e}u(x^{v-e}) = M^{v-e}u(x^{v-e}) + \eta^{v-e}u(x^{v-e}). \tag{3.53}$$

Here,

$$\begin{aligned} \eta^{v-e}u(x^{v-e}) &= \frac{-1}{2}\,e^{\zeta/2}\sum_{j=1}^{3}\hat{a}_{1,j}^{v-e}\frac{\partial u}{\partial x_j^{v-e}} + (\sin\phi)^{3/2}(\cos\phi)^{1/2}\eta^v u(x^v) \\ &= e^{\zeta/2}\sum_{j=1}^{3}\hat{a}_{1,j}^{v-e}\frac{\partial u}{\partial x_j^{v-e}} + \sum_{i=1}^{3}\hat{b}_i^{v-e}\frac{\partial u}{\partial x_i^{v-e}} + \hat{c}^{v-e}u. \end{aligned}$$

Moreover, using (3.28b), (3.29) and (3.48), it can be shown that

$$\begin{aligned} \|\hat{b}_i^{v-e}\|_{0,\infty,\tilde{\Omega}_n^{v-e}} &= O(e^{\zeta/2}) \quad \text{for } i = 1, 2, \\ \|\hat{b}_3^{v-e}\|_{0,\infty,\tilde{\Omega}_n^{v-e}} &= O(e^{\zeta/2}\sin\phi) \qquad \text{and} \\ \|\hat{c}^{v-e}\|_{0,\infty,\tilde{\Omega}_n^{v-e}} &= O(e^{3\zeta/2}\sin^{\frac{3}{2}}\phi)\,. \end{aligned} \tag{3.54}$$

Note that the matrix A^{v-e} defined in (3.52) becomes singular as $\phi \to 0$. To overcome this problem, we introduce a new set of local variables $y = (y_1, y_2, y_3)$ in

$$\Omega_n^{v-e} = \{x : \phi_l^{v-e} < \phi < \phi_{l+1}^{v-e},\ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e},\ \delta_v(\mu_v)^k < x_3 < \delta_v(\mu_v)^{k-1}\}$$

defined by

$$\begin{aligned} y_1 &= x_1^{v-e}, \\ y_2 &= x_2^{v-e}, \\ y_3 &= \frac{x_3^{v-e}}{\sin\left(\phi_{l+1}^{v-e}\right)}\,. \end{aligned} \tag{3.55}$$

In making this transformation $\tilde{\Omega}_n^{v-e}$ is mapped to a hexahedron $\hat{\Omega}_n^{v-e}$ such that the length of the y_3 side becomes large as Ω_n^{v-e} approaches the edge of the domain Ω. It is important to note that the trace and embedding theorems in the theory of Sobolev spaces remain valid with a uniform constant for all the domains $\hat{\Omega}_n^{v-e}$. Now it is easy to see that

$$M^{v-e}u(x^{v-e}) = \mathrm{div}_y(e^{\zeta/2}A^y\nabla_y u), \quad \text{where } A^y = (N^y)^T A N^y. \tag{3.56}$$

Here, by (3.51) and (3.52),

$$N^y = O^v Q^y, \quad Q^y = \begin{bmatrix} \frac{1}{\cos(\phi)} & 0 & \frac{-\sin^2\phi}{\sin\left(\phi_{l+1}^{v-e}\right)\cos(\phi)} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\sin(\phi)}{\sin\left(\phi_{l+1}^{v-e}\right)} \end{bmatrix}.$$

Clearly, there exist positive constants μ_0 and μ_1 such that

$$\mu_0 I \le A^y \le \mu_1 I \tag{3.57}$$

for all elements Ω_n^{v-e}. Moreover, there exists a constant C such that $a_{i,j}^y$ and its derivatives with respect to y are uniformly bounded in $\hat{\Omega}_n^{v-e}$. Here, $a_{i,j}^y$ denotes the elements of the matrix A^y. Hence, we obtain the following result.

Lemma 3.5. *Let $w^{v-e}(x_1^{v-e})$ be a smooth, positive weight function such that $w^{v-e}(x_1^{v-e})=1$ for all x_1^{v-e} such that*

$$x_1^{v-e} \ge \psi_1^{v-e} = \ln(\tan(\phi_1^{v-e})) \qquad \textit{and} \qquad \int_{-\infty}^{\psi_1^{v-e}} w^{v-e}(x_1^{v-e}) = 1.$$

Then there exists a positive constant C_{v-e} such that the estimate

$$\begin{aligned}
&\frac{\mu_0^2}{2}\int_{\tilde{\Omega}_n^{v-e}} e^{x_3^{v-e}}\left\{\sum_{i,j=1}^{2}\left(\frac{\partial^2 u}{\partial x_i^{v-e}\partial x_j^{v-e}}\right)^2 + \sum_{i=1}^{2}\sin^2\phi\left(\frac{\partial^2 u}{\partial x_i^{v-e}\partial x_3^{v-e}}\right)^2\right.\\
&\qquad\left. + \sin^4(\phi)\left(\frac{\partial^2 u}{\partial (x_3^{v-e})^2}\right)^2\right\} dx^{v-e}\\
&\le \int_{\tilde{\Omega}_n^{v-e}} \left| L^{v-e}u(x^{v-e})\right|^2 dx^{v-e}\\
&\qquad - \left\{\sum_k \oint_{\partial\tilde{\Gamma}_{n,k}^{v-e}} e^{x_3^{v-e}}\left(\frac{\partial u}{\partial \boldsymbol{n}^{v-e}}\right)_{A^{v-e}}\left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}} ds^{v-e}\right.\\
&\qquad\left. - 2\sum_k \int_{\tilde{\Gamma}_{n,k}^{v-e}} e^{x_3^{v-e}}\sum_{l=1}^{2}\left(\frac{\partial u}{\partial \boldsymbol{\tau}_l^{v-e}}\right)_{A^{v-e}}\frac{\partial}{\partial s_l^{v-e}}\left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}} d\sigma^{v-e}\right\}\\
&\qquad + C_{v-e}\left\{\int_{\tilde{\Omega}_n^{v-e}} e^{x_3^{v-e}}\left(\sum_{i=1}^{2}\left(\frac{\partial u}{\partial x_i^{v-e}}\right)^2 + \sin^2(\phi)\left(\frac{\partial u}{\partial x_3^{v-e}}\right)^2\right) dx^{v-e}\right.
\end{aligned}$$

$$+ \int\limits_{\tilde{\Omega}_n^{v-e}} e^{x_3^{v-e}} u^2 \, w^{v-e}(x_1^{v-e}) dx^{v-e} \Bigg\} \tag{3.58}$$

holds for all $\Omega_n^{v-e} \subseteq \Omega^{v-e}$.

Proof. Since the spectral element function $u(x^v)$ is a function of only x_3^{v-e} if $\Omega_n^{v-e} \subseteq \left\{x : 0 < \phi < \phi_1^{v-e}\right\}$, the result follows. Here, we have used the fact that the second fundamental form is identically zero. □

We now state the following result and refer to Appendix B of [13] for the proof.

Lemma 3.6. *Let* $\Gamma_{k,i}^v = \Gamma_{q,r}^{v-e}$. *Then the following identity holds.*

$$\begin{aligned}
&\sin^2(\phi_v) \oint_{\partial\tilde{\Gamma}_{k,i}^v} e^{x_3^v} \sin(x_1^v) \left(\frac{\partial u}{\partial \mathbf{n}^v}\right)_{A^v} \left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v} ds^v \\
&\quad -2\sin^2(\phi_v) \int_{\tilde{\Gamma}_{k,i}^v} e^{x_3^v} \sin(x_1^v) \sum_{j=1}^{2} \left(\frac{\partial u}{\partial \boldsymbol{\tau}_j^v}\right)_{A^v} \frac{\partial}{\partial s_j^v} \left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v}\right) d\sigma^v \\
&= -\oint_{\partial\tilde{\Gamma}_{q,r}^{v-e}} e^{x_3^{v-e}} \left(\frac{\partial u}{\partial \mathbf{n}^{v-e}}\right)_{A^{v-e}} \left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}} ds^{v-e} \\
&\quad +2\int_{\tilde{\Gamma}_{q,r}^{v-e}} e^{x_3^{v-e}} \sum_{j=1}^{2} \left(\frac{\partial u}{\partial \boldsymbol{\tau}_j^{v-e}}\right)_{A^{v-e}} \frac{\partial}{\partial s_j^{v-e}} \left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}}\right) d\sigma^{v-e}.
\end{aligned} \tag{3.59}$$

3.4 *Estimates for second derivatives in edge neighbourhoods*

Consider the edge e whose end points are v and v'. Figure 1c shows the edge neighbourhood Ω^e of the edge e. We impose a geometrical mesh on Ω^e shown in figure 5 as in [5, 13]. Let Ω_u^e denote an element

$$\Omega_u^e = \{x : r_j^e < r < r_{j+1}^e,\ \theta_k^e < \theta < \theta_{k+1}^e,\ Z_m^e < x_3 < Z_{m+1}^e\} \tag{3.60}$$

in the geometrical mesh. Let (x_1^e, x_2^e, x_3^e) be the modified coordinates introduced in the edge neighbourhood Ω^e (see table 1). Let $\tilde{\Omega}_u^e$ denote the image of Ω_u^e in x^e coordinate. Now

$$\nabla_x u = R^e \nabla_{x^e} u, \tag{3.61}$$

where

$$R^e = \begin{bmatrix} e^{-\tau}\cos\theta & -e^{-\tau}\sin\theta & 0 \\ e^{-\tau}\sin\theta & e^{-\tau}\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.62}$$

Let $\boldsymbol{f}$ denote a vector field. Then

$$\mathrm{div}_x(\boldsymbol{f}) = e^{-2\tau}\mathrm{div}_{x^e}(e^{2\tau}(R^e)^T \boldsymbol{f})\,. \tag{3.63}$$

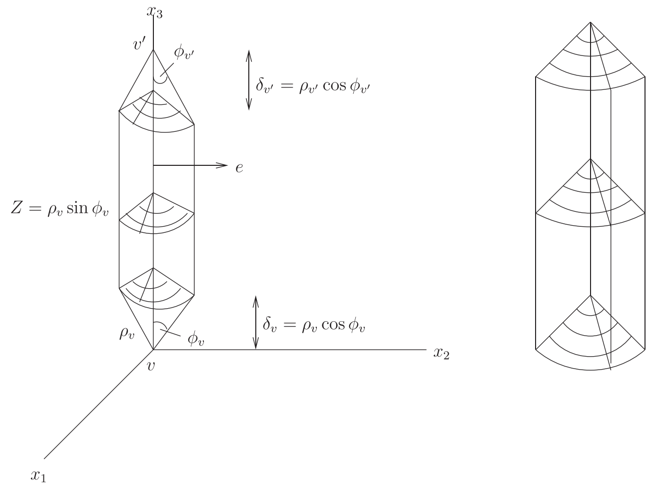


Figure 5. Geometrical mesh imposed on Ω^e.

We need to evaluate

$$\int_{\Omega_u^e} (\rho^2 \sin^2 \phi)|Lu(x)|^2 dx = \int_{\Omega_u^e} r^2 |Lu(x)|^2 dx = \int_{\tilde{\Omega}_u^e} |e^{2\tau} Lu(x)|^2 dx^e. \tag{3.64}$$

Let $M^e u(x^e) = e^{2\tau} Mu(x)$. Then

$$M^e u(x^e) = \mathrm{div}_{x^v}(e^{2\tau}(R^e)^T A R^e \nabla_{x^e} u) = \mathrm{div}_{x^e}\left(A^e \nabla_{x^e} u\right). \tag{3.65}$$

Here,

$$A^e = (S^e)^T A S^e \qquad \text{and} \qquad S^e = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & e^\tau \end{bmatrix}. \tag{3.66}$$

Hence,

$$e^{2\tau} Lu(x) = \mathrm{div}_{x^e}\left(A^e \nabla_{x^e} u\right) + \sum_{i=1}^{3} \hat{b}_i^e u_{x_i^e} + \hat{c}^e u\,. \tag{3.67}$$

Note that

$$\|\hat{b}^e\|_{0,\infty,\tilde{\Omega}^e} = O(e^\tau) \qquad \text{and} \qquad \|\hat{c}^e\|_{0,\infty,\tilde{\Omega}^e} = O(e^{2\tau}). \tag{3.68}$$

Now the matrix A^e becomes singular as the element Ω_u^e approaches the edge e. To overcome the singular nature of A^e as $r \to 0$, we introduce a set of local coordinates z in $\tilde{\Omega}_u^e$ defined by

$$\begin{aligned} z_1 &= x_1^e, \\ z_2 &= x_2^e, \\ z_3 &= \frac{x_3^e}{r_{j+1}^e}. \end{aligned} \tag{3.69}$$

Then $\tilde{\Omega}_u^e$ is mapped onto the hexahedron $\hat{\Omega}_u^e$ such that the length of the z_3 side becomes large as Ω_u^e approaches the edge of the domain Ω. Now it is easy to see that

$$M^e u(x^e) = M^z u(z) = \mathrm{div}_z\left(A^z \nabla_z u\right). \tag{3.70}$$

Here,

$$A^z = (T^z)^T A T^z, \quad T^z = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & \frac{e^\tau}{r^e_{j+1}} \end{bmatrix}. \tag{3.71}$$

Clearly, there exist positive constants μ_0 and μ_1 such that

$$\mu_0 I \leq A^z \leq \mu_1 I. \tag{3.72}$$

Moreover, there exists a constant C such that $a^z_{i,j}$ and its derivatives with respect to z are uniformly bounded in $\hat{\Omega}^e_u$. Here, $a^z_{i,j}$ denotes the elements of the matrix A^z. Hence, we obtain

Lemma 3.7. *Let $w^e(x^e_1)$ be a smooth,positive weight function such that $w^e(x^e_1) = 1$ for all $x^e_1 \geq \tau^e_1 = ln(r^e_1)$ and $\int_{-\infty}^{\tau^e_1} w^e(x^e_1)\, dx^e_1 = 1$. Then there exists a positive constant C_e such that*

$$\begin{aligned}
&\frac{\mu_0^2}{2}\int_{\tilde{\Omega}^e_u}\left\{\sum_{i,j=1}^{2}\left(\frac{\partial^2 u}{\partial x^e_i \partial x^e_j}\right)^2 + e^{2\tau}\sum_{i=1}^{2}\left(\frac{\partial^2 u}{\partial x^e_i \partial x^e_3}\right)^2 + e^{4\tau}\left(\frac{\partial^2 u}{\partial x^{e2}_3}\right)^2\right\} dx^e \\
&\leq \int_{\tilde{\Omega}^e_u} |L^e u(x^e)|^2\, dx^e - \left\{\sum_k \oint_{\partial\tilde{\Gamma}^e_{u,k}} \left(\frac{\partial u}{\partial \boldsymbol{n}^e}\right)_{A^e}\left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e} ds^e\right. \\
&\quad \left. - 2\sum_k \int_{\tilde{\Gamma}^e_{u,k}} \sum_{l=1}^{2}\left(\frac{\partial u}{\partial \boldsymbol{\tau}^e_l}\right)_{A^e} \frac{\partial}{\partial s^e_l}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e}\right\} d\sigma^e\right) \\
&\quad + C_e\left\{\int_{\tilde{\Omega}^e_u}\left(\sum_{i=1}^{2}\left(\frac{\partial u}{\partial x^e_i}\right)^2 + e^{2\tau}\left(\frac{\partial u}{\partial x^e_3}\right)^2\right) dx^e\right. \\
&\quad \left. + \int_{\tilde{\Omega}^e_u} u^2 w^e(x^e_1) dx^e\right\}
\end{aligned} \tag{3.73}$$

holds for all $\Omega^e_u \subseteq \Omega^e$.

Proof. Since the spectral element function $u(x^e)$ is a function of only x^e_3 if $\Omega^e_u \subseteq \{x : r < r^e_1\}$, the result follows. Here, we have used the fact that the second fundamental form is zero. □

Finally, we state the following results and refer to Appendix B of [13] for proofs.

Lemma 3.8. *Let $\Gamma^e_{u,k} = \Gamma^{v-e}_{n,l}$. Then*

$$\begin{aligned}
&\oint_{\partial\tilde{\Gamma}^{v-e}_{n,l}} e^{x^{v-e}_3}\left(\frac{\partial u}{\partial \mathbf{n}^{v-e}}\right)_{A^{v-e}}\left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}} ds^{v-e} \\
&\quad -2\sum_{j=1}^{2}\int_{\tilde{\Gamma}^{v-e}_{n,l}} e^{x^{v-e}_3}\left(\frac{\partial u}{\partial \boldsymbol{\tau}^{v-e}_j}\right)_{A^{v-e}} \frac{\partial}{\partial s^{v-e}_j}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}}\right) d\sigma^{v-e}
\end{aligned}$$

$$= -\oint_{\partial\tilde{\Gamma}^e_{u,k}} \left(\frac{\partial u}{\partial \mathbf{n}^e}\right)_{A^e} \left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e} ds^e$$

$$+ 2\sum_{j=1}^{2} \int_{\tilde{\Gamma}^e_{u,k}} \left(\frac{\partial u}{\partial \boldsymbol{\tau}^e_j}\right)_{A^e} \frac{\partial}{\partial s^e_j} \left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e}\right) d\sigma^e. \tag{3.74}$$

Lemma 3.9. *Let* $\Gamma^e_{u,k} = \Gamma^r_{l,j}$. *Then*

$$\oint_{\partial\tilde{\Gamma}^e_{u,k}} \left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e} \left(\frac{\partial u}{\partial \mathbf{n}^e}\right)_{A^e} s^e = -\rho_v^2 \sin^2(\phi_v) \oint_{\partial\Gamma^r_{l,j}} \left(\frac{\partial u}{\partial \mathbf{n}}\right)_A \left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A ds \tag{3.75}$$

and

$$\sum_{m=1}^{2} \int\limits_{\tilde{\Gamma}^e_{u,k}} \left(\frac{\partial u}{\partial \boldsymbol{\tau}^e_m}\right)_{A^e} \frac{\partial}{\partial s^e_m} \left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e} d\sigma^e$$

$$= -\rho_v^2 \sin^2(\phi_v) \sum_{m=1}^{2} \int\limits_{\Gamma^r_{l,j}} \left(\frac{\partial u}{\partial \boldsymbol{\tau}_m}\right)_A \frac{\partial}{\partial s_m} \left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A d\sigma. \tag{3.76}$$

4. Estimates for lower order derivatives

The estimates for the lower order derivatives are obtained as in [2].

Lemma 4.1. *We can define a set of corrections* $\{\eta^r_l\}_{l=1,\dots,N_r}$, $\{\eta^v_l\}_{l=1,\dots,N_v}$ *for* $v \in \mathcal{V}$, $\{\eta^{v-e}_l\}_{l=1,\dots,N_{v-e}}$ *for* $v-e \in \mathcal{V}-\mathcal{E}$ *and* $\{\eta^e_l\}_{l=1,\dots,N_e}$ *for* $e \in \mathcal{E}$ *such that the corrected spectral element function* p *defined as*

$$\begin{aligned}
p^r_l &= u^r_l + \eta^r_l, && for\ l = 1,\dots,N_r,\\
p^v_l &= u^v_l + \eta^v_l, && for\ l = 1,\dots,N_v \quad and \quad v \in \mathcal{V},\\
p^{v-e}_l &= u^{v-e}_l + \eta^{v-e}_l, && for\ l = 1,\dots,N_{v-e} \quad and \quad v-e \in \mathcal{V}-\mathcal{E},\\
p^e_l &= u^e_l + \eta^e_l, && for\ l = 1,\dots,N_e \quad and \quad e \in \mathcal{E},
\end{aligned}$$

is conforming and $p \in H^1_0(\Omega)$ *i.e.* $p \in H^1(\Omega)$ *and* p *vanishes on* $\Gamma^{[0]}$. *Define*

$$\mathcal{U}^{N,W}_{(1)}(\{\mathcal{F}_s\}) = \sum_{l=1}^{N_r} \left\| s^r_l(x_1,x_2,x_3)\right\|^2_{1,\Omega^r_l} + \sum_{v\in\mathcal{V}} \sum_{l=1}^{N_v} \left\| s^v_l(x^v_1,x^v_2,x^v_3) e^{x^v_3/2}\right\|^2_{1,\tilde{\Omega}^v_l}$$

$$+ \sum_{v-e\in\mathcal{V}-\mathcal{E}} \left(\sum_{\substack{l=1\\ \mu(\tilde{\Omega}^{v-e}_l)<\infty}}^{N_{v-e}} \int_{\tilde{\Omega}^{v-e}_l} e^{x^{v-e}_3} \left(\sum_{i=1}^{2} \left(\frac{\partial s^{v-e}_l}{\partial x^{v-e}_i}\right)^2 \right.\right.$$

$$\left.\left. + \sin^2\phi \left(\frac{\partial s^{v-e}_l}{\partial x^{v-e}_3}\right)^2 + (s^{v-e}_l)^2 \right) dx^{v-e} \right.$$

$$
\left. + \sum_{\substack{l=1 \\ \mu(\tilde{\Omega}_l^{v-e})=\infty}}^{N_{v-e}} \int_{\tilde{\Omega}_l^{v-e}} e^{x_3^{v-e}} (s_l^{v-e})^2\, w^{v-e}(x^{v-e})\, dx^{v-e} \right)
$$

$$
+ \sum_{e\in\mathcal{E}} \left(\sum_{\substack{l=1 \\ \mu(\tilde{\Omega}_l^{e})<\infty}}^{N_e} \int_{\tilde{\Omega}_l^e} \left(\sum_{i=1}^{2} \left(\frac{\partial s_l^e}{\partial x_i^e} \right)^2 + e^{2\tau} \left(\frac{\partial s_l^e}{\partial x_3^e} \right)^2 + (s_l^e)^2 \right) dx^e \right.
$$

$$
\left. + \sum_{\substack{l=1 \\ \mu(\tilde{\Omega}_l^{e})=\infty}}^{N_e} \int_{\tilde{\Omega}_l^e} (s_l^e)^2\, w^e(x_1^e)\, dx^e \right). \tag{4.1}
$$

Then the estimate

$$
\mathcal{U}_{(1)}^{N,W}(\{\mathcal{F}_\eta\}) \leq C_W \mathcal{V}^{N,W}(\{\mathcal{F}_u\}) \tag{4.2}
$$

holds. Here, C_W is a constant, if the spectral element functions are conforming on the wirebasket WB of the elements, otherwise $C_W = C(\ln W)$, where C is a constant.

Proof. The proof is provided in Appendix C of [13]. □

Using Lemma 4.1 we obtain the following result.

Theorem 4.1. *The following estimate for the spectral element functions holds*:

$$
\mathcal{U}_{(1)}^{N,W}(\{\mathcal{F}_u\}) \leq K_{N,W} \mathcal{V}^{N,W}(\{\mathcal{F}_u\}). \tag{4.3}
$$

Here, $K_{N,W} = CN^4$, when the boundary conditions are mixed and $K_{N,W} = C(\ln W)^2$ when the boundary conditions are Dirichlet. If the spectral element functions vanish on the wirebasket WB of the elements then $K_{N,W} = C(\ln W)^2$, where C is a constant.

Proof. The proof is provided in Appendix C of [13]. □

5. Estimates for terms in the interior

5.1 *Estimates for terms in the interior of Ω^r*

Lemma 5.1. *Let Ω_m^r and Ω_p^r be elements in the regular region Ω^r of Ω and $\Gamma_{m,i}^r$ be a face of Ω_m^r and $\Gamma_{p,j}^r$ be a face of Ω_p^r such that $\Gamma_{m,i}^r = \Gamma_{p,j}^r$. Then for any $\epsilon > 0$ there exists a constant C_ϵ such that for W large enough,*

$$
\begin{aligned}
&\left| \int_{\partial\Gamma_{m,i}^r} \left(\left(\frac{\partial u_m^r}{\partial \boldsymbol{\nu}} \right)_A \left(\frac{\partial u_m^r}{\partial \mathbf{n}} \right)_A - \left(\frac{\partial u_p^r}{\partial \boldsymbol{\nu}} \right)_A \left(\frac{\partial u_p^r}{\partial \mathbf{n}} \right)_A \right) ds \right| \\
&\quad \leq C_\epsilon (\ln W)^2 \sum_{k=1}^{3} \left\| [u_{x_k}] \right\|_{1/2,\Gamma_{m,i}^r}^2 \\
&\qquad + \epsilon \sum_{1\leq|\alpha|\leq 2} (\|D_x^\alpha u_m^r\|_{0,\Omega_m^r}^2 + \|D_x^\alpha u_p^r\|_{0,\Omega_p^r}^2).
\end{aligned} \tag{5.1}
$$

Proof. The proof is similar to the proof of Lemma 3.3 in [7] and we refer to Appendix D of [13] for details.

Lemma 5.2. *Let Ω^r_m and Ω^r_p be elements in the regular region Ω^r of Ω and $\Gamma^r_{m,i}$ be a face of Ω^r_m and $\Gamma^r_{p,j}$ be a face of Ω^r_p such that $\Gamma^r_{m,i} = \Gamma^r_{p,j}$. Then for any $\epsilon > 0$, there exists a constant C_ϵ such that for W large enough,*

$$\left|\sum_{j=1}^{2}\left(\int_{\Gamma^r_{m,i}}\left(\frac{\partial u^r_m}{\partial \boldsymbol{\tau}_j}\right)_A \frac{\partial}{\partial s_j}\left(\frac{\partial u^r_m}{\partial \boldsymbol{\nu}}\right)_A d\sigma - \int_{\Gamma^r_{p,j}}\left(\frac{\partial u^r_p}{\partial \boldsymbol{\tau}_j}\right)_A \frac{\partial}{\partial s_j}\left(\frac{\partial u^r_p}{\partial \boldsymbol{\nu}}\right)_A d\sigma\right)\right|$$
$$\leq C_\epsilon (\ln W)^2 \sum_{k=1}^{3} \left\|[u_{x_k}]\right\|^2_{1/2,\Gamma^r_{p,j}}$$
$$+\epsilon \sum_{1\leq|\alpha|\leq 2} (\|D^\alpha_x u^r_m\|^2_{0,\Omega^r_m} + \|D^\alpha_x u^r_p\|^2_{0,\Omega^r_p}). \tag{5.2}$$

Proof. The proof is similar to the proof of Lemma 3.3 in [7] and we refer to Appendix D of [13]. □

5.2 *Estimates for terms in the interior of Ω^e*

Lemma 5.3. *Let Ω^e_m and Ω^e_p be elements in the edge neighbourhood Ω^e of Ω and $\Gamma^e_{m,i}$ be a face of Ω^e_m and $\Gamma^e_{p,j}$ be a face of Ω^e_p such that $\Gamma^e_{m,i} = \Gamma^e_{p,j}$ and $\mu(\tilde{\Gamma}^e_{m,i}) < \infty$. Then for any $\epsilon > 0$, there exists a constant C_ϵ such that for W large enough*

$$\left|\oint_{\partial\tilde{\Gamma}^e_{m,i}}\left(\left(\frac{\partial u^e_m}{\partial \mathbf{n}^e}\right)_{A^e}\left(\frac{\partial u^e_m}{\partial \boldsymbol{\nu}^e}\right)_{A^e} - \left(\frac{\partial u^e_p}{\partial \mathbf{n}^e}\right)_{A^e}\left(\frac{\partial u^e_p}{\partial \boldsymbol{\nu}^e}\right)_{A^e}\right)ds^e\right|$$
$$\leq C_\epsilon (\ln W)^2 (|||[u_{x^e_1}]|||^2_{\tilde{\Gamma}^e_{m,i}} + |||[u_{x^e_2}]|||^2_{\tilde{\Gamma}^e_{m,i}}$$
$$+|||G^e_{m,i}[u_{x^e_3}]|||^2_{\tilde{\Gamma}^e_{m,i}}) + \epsilon \sum_{k=m,p}\left(\int_{\tilde{\Omega}^e_k}\left(\sum_{i,j=1,2}\left(\frac{\partial^2 u^e_k}{\partial x^e_i \partial x^e_j}\right)^2\right.\right.$$
$$+e^{2\tau}\sum_{i=1}^{2}\left(\frac{\partial^2 u^e_k}{\partial x^e_k \partial x^e_3}\right)^2 + e^{4\tau}\left(\frac{\partial^2 u^e_k}{(\partial x^e_3)^2}\right)^2 + \sum_{i=1}^{2}\left(\frac{\partial u^e_k}{\partial x^e_i}\right)^2$$
$$\left.\left.+e^{2\tau}\left(\frac{\partial u^e_k}{\partial x^e_3}\right)^2\right)dx^e\right). \tag{5.3a}$$

Here, C_ϵ is a constant which depend on ϵ but is uniform for all $\tilde{\Gamma}^e_{m,i} \subseteq \tilde{\Omega}^e$ and $G^e_{m,i} =$ $\sup_{x^e\in\tilde{\Gamma}^e_{m,i}}(e^\tau)$. *If $\mu(\tilde{\Gamma}^e_{m,i}) = \infty$, then for any $\epsilon > 0$, for W, N large enough,*

$$\left|\oint_{\partial\tilde{\Gamma}^e_{m,i}}\left(\left(\frac{\partial u^e_m}{\partial \mathbf{n}^e}\right)_{A^e}\left(\frac{\partial u^e_m}{\partial \boldsymbol{\nu}^e}\right)_{A^e} - \left(\frac{\partial u^e_p}{\partial \mathbf{n}^e}\right)_{A^e}\left(\frac{\partial u^e_p}{\partial \boldsymbol{\nu}^e}\right)_{A^e}\right)ds^e\right|$$
$$\leq \epsilon\left(\int_{\tilde{\Omega}^e_m}(u^e_m)^2 w^e(x^e_1)dx^e + \int_{\tilde{\Omega}^e_m}(u^e_m)^2 w^e(x^e_1)dx^e\right) \tag{5.3b}$$

provided $W = O(e^{N^\alpha})$ with $\alpha < 1/2$.

The proof is provided in Appendix D of [13].

Lemma 5.4. *Let Ω^e_m and Ω^e_p be elements in the edge neighbourhood Ω^e of Ω and $\Gamma^e_{m,i}$ be a face of Ω^e_m and $\Gamma^e_{p,j}$ be a face of Ω^e_p such that $\Gamma^e_{m,i} = \Gamma^e_{p,j}$ and $\mu(\tilde{\Gamma}^e_{m,i}) < \infty$. Then for any $\epsilon > 0$, there exists a constant C_ϵ such that for W large enough,*

$$\left|\int_{\tilde{\Gamma}^e_{m,i}} \sum_{l=1}^{2}\left(\left(\frac{\partial u^e_m}{\partial \boldsymbol{\tau}^e_l}\right)_{A^e} \frac{\partial}{\partial s^e_l}\left(\left(\frac{\partial u^e_m}{\partial \boldsymbol{\nu}^e}\right)_{A^e}\right) - \left(\frac{\partial u^e_p}{\partial \boldsymbol{\tau}^e_l}\right)_{A^e} \frac{\partial}{\partial s^e_l}\left(\left(\frac{\partial u^e_p}{\partial \boldsymbol{\nu}^e}\right)_{A^e}\right)\right) d\sigma^e\right|$$
$$\leq C_\epsilon (\ln W)^2 \left(|||[u_{x^e_1}]|||^2_{\tilde{\Gamma}^e_{m,i}} + |||[u_{x^e_2}]|||^2_{\tilde{\Gamma}^e_{m,i}} + |||G^e_{m,i}[u_{x^e_3}]|||^2_{\tilde{\Gamma}^e_{m,i}}\right)$$
$$+\epsilon \sum_{k=m,p}\left(\int_{\tilde{\Omega}^e_k}\left(\sum_{i,j=1,2}\left(\frac{\partial^2 u^e_k}{\partial x^e_i \partial x^e_j}\right)^2 + e^{2\tau}\sum_{i=1}^{2}\left(\frac{\partial^2 u^e_k}{\partial x^e_i \partial x^e_3}\right)^2\right.\right.$$
$$\left.\left. +e^{4\tau}\left(\frac{\partial^2 u^e_k}{\left(\partial x^e_3\right)^2}\right)^2 + \sum_{i=1}^{2}\left(\frac{\partial u^e_k}{\partial x^e_i}\right)^2 + e^{2\tau}\left(\frac{\partial u^e_k}{\partial x^e_3}\right)^2\right) dx^e\right). \tag{5.4a}$$

If $\mu(\tilde{\Gamma}^e_{m,i}) = \infty$, then for any $\epsilon > 0$, for W, N large enough,

$$\left|\int_{\tilde{\Gamma}^e_{m,i}} \sum_{l=1}^{2}\left(\left(\frac{\partial u^e_m}{\partial \boldsymbol{\tau}^e_l}\right)_{A^e} \frac{\partial}{\partial s^e_l}\left(\left(\frac{\partial u^e_m}{\partial \boldsymbol{\nu}^e}\right)_{A^e}\right) - \left(\frac{\partial u^e_p}{\partial \boldsymbol{\tau}^e_l}\right)_{A^e} \frac{\partial}{\partial s^e_l}\left(\left(\frac{\partial u^e_p}{\partial \boldsymbol{\nu}^e}\right)_{A^e}\right)\right) d\sigma^e\right|$$
$$\leq \epsilon\left(\int_{\tilde{\Omega}^e_m}(u^e_m)^2 w^e(x^e_1)\, dx^e + \int_{\tilde{\Omega}^e_m}(u^e_m)^2 w^e(x^e_1) dx^e\right) \tag{5.4b}$$

provided $W = O(e^{N^\alpha})$ with $\alpha < 1/2$.

The proof is provided in Appendix D of [13].

We now state estimates for terms in the interior of vertex neighbourhoods and vertex-edge neighbourhoods, the proofs of which are similar to those for Lemmas 5.1 to 5.4.

5.3 *Estimates for terms in the interior of Ω^v*

Lemma 5.5. *Let Ω^v_m and Ω^v_p be elements in the vertex neighbourhood Ω^v of Ω and $\Gamma^v_{m,i}$ be a face of Ω^v_m and $\Gamma^v_{p,j}$ be a face of Ω^v_p such that $\Gamma^v_{m,i} = \Gamma^v_{p,j}$. Then for any $\epsilon > 0$, there exists a constant C_ϵ such that for W large enough,*

$$\left|\oint_{\partial\tilde{\Gamma}^v_{m,i}} e^{x^v_3}\sin(x^v_1)\left(\left(\frac{\partial u^v_m}{\partial \mathbf{n}^v}\right)_{A^v}\left(\frac{\partial u^v_m}{\partial \boldsymbol{\nu}^v}\right)_{A^v} - \left(\frac{\partial u^v_p}{\partial \mathbf{n}^v}\right)_{A^v}\left(\frac{\partial u^v_p}{\partial \boldsymbol{\nu}^v}\right)_{A^v}\right) ds^v\right|$$
$$\leq C_\epsilon (\ln W)^2 \sum_{k=1}^{3} R^v_{m,i}\left\|[u_{x_k}]\right\|_{1/2,\tilde{\Gamma}^v_{m,i}}$$
$$+\epsilon \sum_{1\leq|\alpha|\leq 2}(\|e^{x^v_3/2}D^\alpha_{x^v}u^v_m\|^2_{0,\tilde{\Omega}^v_m} + \|e^{x^v_3/2}D^\alpha_{x^v}u^v_p\|^2_{0,\tilde{\Omega}^v_p}). \tag{5.5}$$

Here, $R^v_{m,i} = \sup_{x^v \in \tilde{\Gamma}^v_{m,i}} (e^{x^v_3})$. *If* $\mu(\tilde{\Gamma}^v_{m,i}) = \infty$, *then the integral in the right-hand side of* (5.5) *is zero.*

Lemma 5.6. *Let* Ω^v_m *and* Ω^v_p *be elements in the vertex neighbourhood* Ω^v *of* Ω *and* $\Gamma^v_{m,i}$ *be a face of* Ω^v_m *and* $\Gamma^v_{p,j}$ *be a face of* Ω^v_p *such that* $\Gamma^v_{m,i} = \Gamma^v_{p,j}$. *Then for any* $\epsilon > 0$, *there exists a constant* C_ϵ *such that for* W *large enough,*

$$\left|\sum_{l=1}^{2}\left(\int_{\Gamma^v_{m,i}} e^{x^v_3}\sin(x^v_1)\left\{\left(\frac{\partial u^v_m}{\partial \boldsymbol{\tau}^v_l}\right)_{A^v}\frac{\partial}{\partial s^v_l}\left(\left(\frac{\partial u^v_m}{\partial \boldsymbol{\nu}}\right)_{A^v}\right)d\sigma^v\right.\right.\right.$$

$$\left.\left.\left.-\int_{\Gamma^v_{p,j}}\left(\frac{\partial u^v_p}{\partial \boldsymbol{\tau}^v_l}\right)_{A^v}\frac{\partial}{\partial s^v_l}\left(\left(\frac{\partial u^v_p}{\partial \boldsymbol{\nu}}\right)_{A^v}\right)d\sigma^v\right\}\right)\right|$$

$$\leq C_\epsilon(\ln W)^2\sum_{k=1}^{3} R^v_{m,i}\,\left\|[u_{x_k}]\right\|_{1/2,\tilde{\Gamma}^v_{m,i}}$$

$$+\epsilon\sum_{1\leq|\alpha|\leq 2}(\|e^{x^v_3/2}D^\alpha_{x^v}u^v_m\|^2_{0,\tilde{\Omega}^v_m}+\|e^{x^v_3/2}D^\alpha_{x^v}u^v_p\|^2_{0,\tilde{\Omega}^v_p}). \tag{5.6}$$

If $\mu(\tilde{\Gamma}^v_{m,i}) = \infty$, *then the integral in the right-hand side of* (5.6) *is zero.*

5.4 *Estimates for terms in the interior of* Ω^{v-e}

Lemma 5.7. *Let* Ω^{v-e}_m *and* Ω^{v-e}_p *be elements in the vertex-edge neighbourhood* Ω^{v-e} *of* Ω *and* $\Gamma^{v-e}_{m,i}$ *be a face of* Ω^{v-e}_m *and* $\Gamma^{v-e}_{p,j}$ *be a face of* Ω^{v-e}_p *such that* $\Gamma^{v-e}_{m,i} = \Gamma^{v-e}_{p,j}$. *Then for any* $\epsilon > 0$, *there exists a constant* C_ϵ *such that for* W *large enough,*

$$\left|\oint_{\partial\tilde{\Gamma}^{v-e}_{m,i}} e^{x^{v-e}_3}\left(\left(\frac{\partial u^{v-e}_m}{\partial \mathbf{n}^{v-e}}\right)_{A^{v-e}}\left(\frac{\partial u^{v-e}_m}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}}-\left(\frac{\partial u^{v-e}_p}{\partial \mathbf{n}^{v-e}}\right)_{A^{v-e}}\left(\frac{\partial u^{v-e}_p}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}}\right)ds^{v-e}\right|$$

$$\leq C_\epsilon(\ln W)^2(|||[u_{x^{v-e}_1}]|||^2_{\tilde{\Gamma}^{v-e}_{m,i}}+|||[u_{x^{v-e}_2}]|||^2_{\tilde{\Gamma}^{v-e}_{m,i}}+|||E^{v-e}_{m,i}[u_{x^{v-e}_3}]|||^2_{\tilde{\Gamma}^{v-e}_{m,i}})$$

$$+\epsilon\sum_{k=m,p}\left(\int_{\tilde{\Omega}^{v-e}_k}\left(\sum_{i,j=1,2}\left(\frac{\partial^2 u^{v-e}_k}{\partial x^{v-e}_i\partial x^{v-e}_j}\right)^2+\sin^2\phi\sum_{i=1}^{2}\left(\frac{\partial^2 u^{v-e}_k}{\partial x^{v-e}_i\partial x^{v-e}_3}\right)^2\right.\right.$$

$$\left.\left.+\sin^4\phi\left(\frac{\partial^2 u^{v-e}_k}{(\partial x^{v-e}_3)^2}\right)^2+\sum_{i=1}^{2}\left(\frac{\partial u^{v-e}_k}{\partial x^{v-e}_i}\right)^2+\sin^2\phi\left(\frac{\partial u^{v-e}_k}{\partial x^{v-e}_3}\right)^2\right)dx^{v-e}\right). \tag{5.7a}$$

Here, C_ϵ *is a constant which depend on* ϵ *but is uniform for all* $\tilde{\Gamma}^{v-e}_{m,i} \subseteq \tilde{\Omega}^{v-e}$, *and* $E^{v-e}_{m,i} = \sup_{x^{v-e}\in\tilde{\Gamma}^{v-e}_{m,i}}(\sin\phi)$.

If $\mu(\tilde{\Gamma}^{e}_{m,i}) = \infty$, *then for any* $\epsilon > 0$,

$$\left| \oint_{\partial\tilde{\Gamma}^{v-e}_{m,i}} e^{x_3^{v-e}} \left(\left(\frac{\partial u_m^{v-e}}{\partial \mathbf{n}^{v-e}} \right)_{A^{v-e}} \left(\frac{\partial u_m^{v-e}}{\partial \boldsymbol{\nu}^{v-e}} \right)_{A^{v-e}} - \left(\frac{\partial u_p^{v-e}}{\partial \mathbf{n}^{v-e}} \right)_{A^{v-e}} \left(\frac{\partial u_p^{v-e}}{\partial \boldsymbol{\nu}^{v-e}} \right)_{A^{v-e}} \right) ds^{v-e} \right|$$

$$\leq \epsilon \left(\sum_{k=m,p} \int_{\tilde{\Omega}_k^{v-e}} (u_k^{v-e})^2 w^{v-e}(x_1^{v-e}) dx^{v-e} \right) \tag{5.7b}$$

for W, N *large enough provided* $W = O(e^{N^\alpha})$ *with* $\alpha < 1/2$.

Lemma 5.8. *Let* Ω_m^{v-e} *and* Ω_p^{v-e} *be elements in the edge neighbourhood* Ω^{v-e} *of* Ω *and* $\Gamma_{m,i}^{v-e}$ *be a face of* Ω_m^{v-e} *and* $\Gamma_{p,j}^{v-e}$ *be a face of* Ω_p^{v-e} *such that* $\Gamma_{m,i}^{v-e} = \Gamma_{p,j}^{v-e}$. *Then for any* $\epsilon > 0$, *there exists a constant* C_ϵ *such that for* W *large enough*,

$$\left| \int_{\tilde{\Gamma}^{v-e}_{m,i}} e^{x_3^{v-e}} \sum_{l=1}^{2} \left(\left(\frac{\partial u_m^{v-e}}{\partial \boldsymbol{\tau}_l^{v-e}} \right)_{A^{v-e}} \frac{\partial}{\partial s_l^{v-e}} \left(\left(\frac{\partial u_m^{v-e}}{\partial \boldsymbol{\nu}^{v-e}} \right)_{A^{v-e}} \right) \right.\right.$$
$$\left.\left. - \left(\frac{\partial u_p^{v-e}}{\partial \boldsymbol{\tau}_l^{v-e}} \right)_{A^{v-e}} \frac{\partial}{\partial s_l^{v-e}} \left(\left(\frac{\partial u_p^{v-e}}{\partial \boldsymbol{\nu}^{v-e}} \right)_{A^{v-e}} \right) \right) d\sigma^{v-e} \right|$$
$$\leq C_\epsilon (\ln W)^2 (|||[u_{x_1^{v-e}}]|||^2_{\tilde{\Gamma}^{v-e}_{m,i}} + |||[u_{x_2^{v-e}}]|||^2_{\tilde{\Gamma}^{v-e}_{m,i}} + |||E^{v-e}_{m,i}[u_{x_3^{v-e}}]|||^2_{\tilde{\Gamma}^{v-e}_{m,i}})$$
$$+ \epsilon \sum_{k=m,p} \left(\int_{\tilde{\Omega}_k^{v-e}} \left(\sum_{i,j=1,2} \left(\frac{\partial^2 u_k^{v-e}}{\partial x_i^{v-e} \partial x_j^{v-e}} \right)^2 + \sin^2\phi \sum_{i=1}^{2} \left(\frac{\partial^2 u_k^{v-e}}{\partial x_i^{v-e} \partial x_3^{v-e}} \right)^2 \right.\right.$$
$$+ \sin^4\phi \left(\frac{\partial^2 u_k^{v-e}}{(\partial x_3^{v-e})^2} \right)^2 + \sum_{i=1}^{2} \left(\frac{\partial u_k^{v-e}}{\partial x_i^{v-e}} \right)^2$$
$$\left.\left. + \sin^2\phi \left(\frac{\partial u_k^{v-e}}{\partial x_3^{v-e}} \right)^2 \right) dx^{v-e} \right). \tag{5.8a}$$

If $\mu(\tilde{\Gamma}^{e}_{m,i}) = \infty$, *then for any* $\epsilon > 0$,

$$\left| \int_{\tilde{\Gamma}^{v-e}_{m,i}} e^{x_3^{v-e}} \sum_{l=1}^{2} \left(\left(\frac{\partial u_m^{v-e}}{\partial \boldsymbol{\tau}_l^{v-e}} \right)_{A^{v-e}} \frac{\partial}{\partial s_l^{v-e}} \left(\left(\frac{\partial u_m^{v-e}}{\partial \boldsymbol{\nu}^{v-e}} \right)_{A^{v-e}} \right) \right.\right.$$
$$\left.\left. - \left(\frac{\partial u_p^{v-e}}{\partial \boldsymbol{\tau}_l^{v-e}} \right)_{A^{v-e}} \frac{\partial}{\partial s_l^{v-e}} \left(\left(\frac{\partial u_p^{v-e}}{\partial \boldsymbol{\nu}^{v-e}} \right)_{A^{v-e}} \right) \right) d\sigma^{v-e} \right|$$

$$\leq \epsilon \left(\sum_{k=m,p} \int_{\tilde{\Omega}_k^{v-e}} (u_k^{v-e})^2 w^{v-e}(x_1^{v-e})\, dx^{v-e} \right) \tag{5.8b}$$

for W, N large enough, provided $W = O(e^{N^\alpha})$ with $\alpha < 1/2$.

6. Estimates for terms on the boundary

6.1 *Estimates for terms on the boundary of Ω^r*

To simplify the presentation we assume the face constituting part of the boundary of Ω lies on the $x_2 - x_3$ plane. The contributions from the boundary which have to be estimated will then consist of terms from the regular region, the vertex region, the vertex-edge region and the edge region as shown in figure 6. We first examine how to estimate the terms on the boundary of Ω for the regular region, and terms from the other regions can be estimated similarly.

Lemma 6.1. *Let $\Gamma^r_{m,j}$ be part of the boundary of the element Ω^r_m which lies on the $x_2 - x_3$ axis. Define the contributions from $\Gamma^r_{m,j}$ by*

$$\begin{aligned}(BT)^r_{m,j} \;=\; \rho_v^2 \sin^2(\phi_v) \Bigg(&- \oint_{\partial\Gamma^r_{m,j}} \left(\frac{\partial u}{\partial \mathbf{n}}\right)_A \left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A ds \\ &+ 2 \int_{\Gamma^r_{m,j}} \sum_{j=1}^{2} \left(\frac{\partial u}{\partial \boldsymbol{\tau}_j}\right)_A \frac{\partial}{\partial s_j}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)_A\right) d\sigma \Bigg).\end{aligned} \tag{6.1}$$

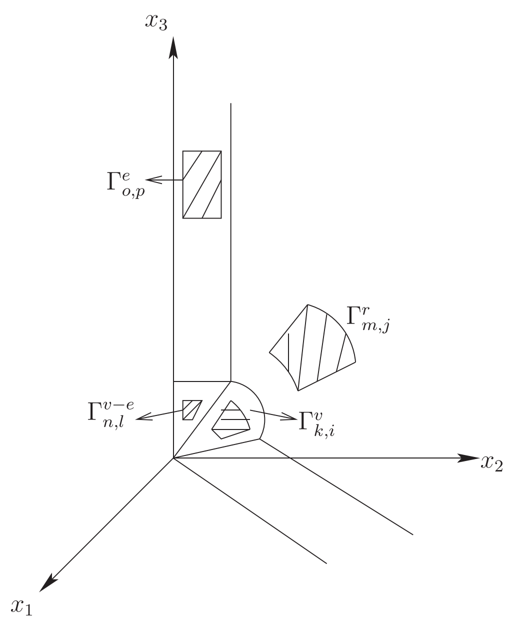


Figure 6. Boundary terms.

If Dirichlet boundary conditions are imposed on $\Gamma^r_{m,j}$, then for any $\epsilon > 0$, there exist constants C_ϵ and K_ϵ such that

$$|(BT)^r_{m,j}| \leq C_\epsilon (\ln W)^2 \|u^r_m\|^2_{3/2,\Gamma^r_{m,j}} + K_\epsilon \sum_{|\alpha|=1} \|D^\alpha_x u^r_m\|^2_{0,\Omega^r_m} + \epsilon \sum_{|\alpha|=2} \|D^\alpha_x u^r_m\|^2_{0,\Omega^r_m}. \tag{6.2}$$

If Neumann boundary conditions are imposed on $\Gamma^r_{m,j}$, then for any $\epsilon > 0$ there exists a constant C_ϵ such that

$$|(BT)^r_{m,j}| \leq C_\epsilon (\ln W)^2 \left\| \left(\frac{\partial u^r_m}{\partial \boldsymbol{\nu}}\right)_A \right\|^2_{1/2,\Gamma^r_{m,j}} + \epsilon \sum_{1\leq|\alpha|\leq 2} \|D^r_x u^r_m\|^2_{2,\Omega^r_m}. \tag{6.3}$$

Proof. The proof is provided in Appendix D of [13]. □

6.2 *Estimates for terms on the boundary of Ω^e*

Lemma 6.2. *Let $\Gamma^e_{m,j}$ be part of the boundary of the element Ω^e_m which lies on the $x_2 - x_3$ axis. Define the contributions from $\Gamma^e_{m,j}$ by*

$$(BT)^e_{m,j} = -\oint_{\partial\tilde{\Gamma}^e_{m,j}} \left(\frac{\partial u}{\partial \mathbf{n}^e}\right)_{A^e} \left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e} ds^e - 2\int_{\tilde{\Gamma}^e_{m,j}} \sum_{l=1}^{2} \left(\frac{\partial u}{\partial \boldsymbol{\tau}^e_l}\right)_{A^e} \frac{\partial}{\partial s^e_l}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e}\right) d\sigma^e. \tag{6.4}$$

If Dirichlet boundary conditions are imposed on $\Gamma^e_{m,j}$ and $\mu(\tilde{\Gamma}^e_{m,j}) < \infty$, then for any $\epsilon > 0$, there exists constants C_ϵ, K_ϵ such that for W large enough,

$$\begin{aligned}|(BT)^e_{m,j}| \leq\ & C_\epsilon (\ln W)^2 \left(\|u^e_m\|^2_{0,\tilde{\Gamma}^e_{m,i}} + \left|\left|\left| \left(\frac{\partial u^e_m}{\partial x^e_1}\right) \right|\right|\right|^2_{\tilde{\Gamma}^e_{m,j}} + \left|\left|\left| G^e_{m,j} \left(\frac{\partial u^e_m}{\partial x^e_3}\right) \right|\right|\right|^2_{\tilde{\Gamma}^e_{m,j}} \right) \\ & + K_\epsilon \int_{\tilde{\Omega}^e_m} \left(\sum_{i=1}^{2} \left(\frac{\partial u^e_m}{\partial x^e_i}\right)^2 + e^{2\tau} \left(\frac{\partial u^e_m}{\partial x^e_3}\right)^2 \right) dx^e \\ & + \epsilon \int_{\tilde{\Omega}^e_m} \left(\sum_{i,j=1,2} \left(\frac{\partial^2 u^e_m}{\partial x^e_i \partial x^e_j}\right)^2 + e^{2\tau} \sum_{i=1}^{2} \left(\frac{\partial^2 u^e_m}{\partial x^e_i \partial x^e_3}\right)^2 + e^{4\tau} \left(\frac{\partial^2 u^e_m}{(\partial x^e_3)^2}\right)^2 \right) dx^e \end{aligned} \tag{6.5}$$

If Neumann boundary conditions are imposed on $\Gamma^e_{m,j}$ and $\mu(\tilde{\Gamma}^e_{m,j}) < \infty$, then for any $\epsilon > 0$ there exists a constant C_ϵ such that

$$|(BT)^e_{m,j}| \leq C_\epsilon (\ln W)^2 \left|\left|\left| \left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e} \right|\right|\right|^2_{\tilde{\Gamma}^e_{m,j}} + \epsilon \left(\int_{\tilde{\Omega}^e_m} \left(\sum_{i,j=1}^{2} \left(\frac{\partial^2 u^e_m}{\partial x^e_i \partial x^e_j} \right)^2 \right.\right.$$
$$+ e^{2\tau} \sum_{i=1}^{2} \left(\frac{\partial^2 u^e_m}{\partial x^e_i \partial x^e_3} \right)^2 + e^{4\tau} \left(\frac{\partial^2 u^e_m}{(\partial x^e_3)^2} \right)^2 + \sum_{i=1}^{2} \left(\frac{\partial u^e_m}{\partial x^e_i} \right)^2$$
$$\left.\left. + e^{2\tau} \left(\frac{\partial u^e_m}{\partial x^e_3} \right)^2 \right) dx^e \right). \tag{6.6}$$

If $\mu(\tilde{\Gamma}^e_{m,j}) = \infty$, then for any $\epsilon > 0$ for N, W large enough,

$$|(BT)^e_{m,j}| \leq \epsilon \int_{\tilde{\Omega}^e_m} (u^e_m)^2 w^e(x^e_1) dx^e,$$

provided $W = O(e^{N^\alpha})$ for $\alpha < 1/2$.

Proof. The proof is provided in Appendix D of [13]. □

We now state estimates for terms on the boundary of vertex neighbourhoods and vertex-edge neighbourhoods, the proofs of which are similar to those for Lemmas 6.1 and 6.2.

6.3 *Estimates for terms on the boundary of Ω^v*

Lemma 6.3. *Let $\Gamma^v_{m,j}$ be part of the boundary of the element Ω^v_m which lies on the $x_2 - x_3$ axis. Define the contributions from $\Gamma^v_{m,j}$ by*

$$(BT)^v_{m,j} = \sin^2(\phi_v) \left(- \oint_{\partial\tilde{\Gamma}^v_{m,j}} e^{x^v_3} \sin(x^v_1) \left(\frac{\partial u}{\partial \mathbf{n}^v}\right)_{A^v} \left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v} ds^v \right.$$
$$\left. -2 \int_{\tilde{\Gamma}^v_{m,j}} e^{x^v_3} \sin(x^v_1) \sum_{l=1}^{2} \left(\frac{\partial u}{\partial \boldsymbol{\tau}^v_l}\right)_{A^v} \frac{\partial}{\partial s^v_l} \left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v} \right) d\sigma^v \right). \tag{6.7}$$

If Dirichlet boundary conditions are imposed on $\Gamma^v_{m,j}$ and $\mu(\tilde{\Gamma}^v_{m,j}) < \infty$, then for any $\epsilon > 0$, there exists constants C_ϵ and K_ϵ such that

$$|(BT)^v_{m,j}| \leq C_\epsilon (\ln W)^2 R^v_{m,j} \|u^v_m\|^2_{3/2,\tilde{\Gamma}^v_{m,j}}$$
$$+ K_\epsilon \sum_{|\alpha|=1} \|e^{x^v_3/2} D^\alpha_{x^v} u^v_m\|^2_{0,\tilde{\Omega}^e_m} + \epsilon \sum_{|\alpha|=2} \|e^{x^v_3/2} D^\alpha_{x^v} u^v_m\|^2_{0,\tilde{\Omega}^v_m}. \tag{6.8}$$

If Neumann boundary conditions are imposed on $\Gamma^v_{m,j}$, then

$$|(BT)^v_{m,j}| \leq C_\epsilon (\ln W)^2 R^v_{m,j} \left\| \left(\frac{\partial u^v_m}{\partial \boldsymbol{\nu}^v}\right)_{A^v} \right\|^2_{1/2,\tilde{\Gamma}^v_{m,j}} + \epsilon \sum_{1 \leq |\alpha| \leq 2} e^{x^v_3/2} \|D^\alpha_{x^v} u^v_m\|^2_{0,\tilde{\Omega}^v_m}. \tag{6.9}$$

If $\mu(\tilde{\Gamma}^v_{m,j}) = \infty$, then $(BT)^v_{m,j} = 0$.

6.4 *Estimates for terms on the boundary of* Ω^{v-e}

Lemma 6.4. *Let* $\Gamma_{m,j}^{v-e}$ *be part of the boundary of the element* Ω_m^{v-e} *which lies on the* $x_2 - x_3$ *axis. Define the contributions from* $\Gamma_{m,j}^{v-e}$ *by*

$$\begin{aligned}(BT)_{m,j}^{v-e} &= -\oint_{\partial\tilde{\Gamma}_{m,j}^{v-e}} e^{x_3^{v-e}} \left(\frac{\partial u}{\partial \mathbf{n}^{v-e}}\right)_{A^{v-e}} \left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}} ds^{v-e} \\ &\quad -2\int_{\tilde{\Gamma}_{m,j}^{v-e}} e^{x_3^{v-e}} \sum_{l=1}^{2}\left(\frac{\partial u}{\partial \boldsymbol{\tau}_l^{v-e}}\right)_{A^{v-e}} \frac{\partial}{\partial s_l^{v-e}}\left(\left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}}\right) d\sigma^{v-e}. \end{aligned} \tag{6.10}$$

If Dirichlet boundary conditions are imposed on $\Gamma_{m,j}^{v-e}$ *and* $\mu(\tilde{\Gamma}_{m,j}^{v-e}) < \infty$, *then for any* $\epsilon > 0$, *there exists constants* C_ϵ *and* K_ϵ *such that*

$$\begin{aligned}|(BT)_{m,j}^{v-e}| &\leq C_\epsilon (\ln W)^2 \left(\|u_m^{v-e}\|_{0,\tilde{\Gamma}_{m,i}^{v-e}}^2 + \left|\left|\left| \left(\frac{\partial u_m^{v-e}}{\partial x_1^{v-e}}\right) \right|\right|\right|_{\tilde{\Gamma}_{m,j}^{v-e}}^2 \right. \\ &\quad \left. + \left|\left|\left| E_{m,j}^{v-e}\left(\frac{\partial u_m^{v-e}}{\partial x_3^{v-e}}\right) \right|\right|\right|_{\tilde{\Gamma}_{m,j}^{v-e}}^2 \right) \\ &\quad + K_\epsilon \int_{\tilde{\Omega}_m^{v-e}} \left(\sum_{i=1}^{2} \left(\frac{\partial u_m^{v-e}}{\partial x_i^{v-e}}\right)^2 + \sin^2\phi \left(\frac{\partial u_m^{v-e}}{\partial x_3^{v-e}}\right)^2 \right) dx^{v-e} \\ &\quad + \epsilon \int_{\tilde{\Omega}_m^{v-e}} \left(\sum_{i,j=1,2} \left(\frac{\partial^2 u_m^{v-e}}{\partial x_i^{v-e}\partial x_j^{v-e}}\right)^2 + \sin^2\phi \sum_{i=1}^{2}\left(\frac{\partial^2 u_m^{v-e}}{\partial x_i^{v-e}\partial x_3^{v-e}}\right)^2 \right. \\ &\quad \left. + \sin^4\phi \left(\frac{\partial^2 u_m^{v-e}}{\left(\partial x_3^{v-e}\right)^2}\right)^2 \right) dx^{v-e}. \end{aligned} \tag{6.11}$$

If Neumann boundary conditions are imposed on $\Gamma_{m,j}^{v-e}$ *and* $\mu(\tilde{\Gamma}_{m,j}^{v-e}) < \infty$, *then for any* $\epsilon > 0$, *there exists a constant* C_ϵ *such that*

$$|(BT)_{m,j}^{v-e}| \leq C_\epsilon (\ln W)^2 \left|\left|\left| \left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}} \right|\right|\right|_{\tilde{\Gamma}_{m,j}^{v-e}}^2$$

$$+\epsilon \left(\int_{\tilde{\Omega}_m^{v-e}} \left(\sum_{i,j=1}^{2} \left(\frac{\partial^2 u_m^{v-e}}{\partial x_i^{v-e}\partial x_j^{v-e}}\right)^2 \right.\right. \tag{6.12}$$

$$+\sin^2\phi \sum_{i=1}^{2}\left(\frac{\partial^2 u_m^{v-e}}{\partial x_i^{v-e}\partial x_3^{v-e}}\right)^2 + \sin^4\phi \left(\frac{\partial^2 u_m^{v-e}}{(\partial x_3^{v-e})^2}\right)^2$$

$$+\sum_{i=1}^{2}\left(\frac{\partial u_m^{v-e}}{\partial x_i^{v-e}}\right)^2 \tag{6.13}$$

$$\left.\left. +\sin^2\phi \left(\frac{\partial u_m^{v-e}}{\partial x_3^{v-e}}\right)^2 \right) dx^{v-e} \right). \tag{6.14}$$

If $\mu(\tilde{\Gamma}^{v-e}_{m,j}) = \infty$, *then for any* $\epsilon > 0$, *for* N, W *large enough,*

$$|(BT)^{v-e}_{m,j}| \le \epsilon \int_{\tilde{\Omega}^{v-e}_m} (u^{v-e}_m)^2 w^{v-e}(x^{v-e}_1) dx^{v-e}$$

provided $W = O(e^{N^\alpha})$ *for* $\alpha < 1/2$.

7. Proof of the stability theorem

We are now in a position to prove the stability estimates of [5]. We cite these again.

Theorem 7.1 (Theorem 4.1 of [5]). *Consider the elliptic boundary value problem* (2.1). *Suppose the boundary conditions are Dirichlet. Then*

$$\mathcal{U}^{N,W}(\{\mathcal{F}_u\}) \le C(\ln W)^2 \mathcal{V}^{N,W}(\{\mathcal{F}_u\})$$

provided $W = O(e^{N^\alpha})$ *for* $\alpha < 1/2$.

Theorem 7.2 (Theorem 4.2 of [5]). *If the boundary conditions for the elliptic boundary value problem* (2.1) *are mixed, then*

$$\mathcal{U}^{N,W}(\{\mathcal{F}_u\}) \le C N^4 \mathcal{V}^{N,W}(\{\mathcal{F}_u\})$$

provided $W = O(e^{N^\alpha})$ *for* $\alpha < 1/2$.

Theorem 7.3 (Theorem 4.3 of [5]). *If the spectral element functions* $(\{\mathcal{F}_u\})$ *are conforming on the wire basket* WB *and vanish on* WB, *then*

$$\mathcal{U}^{N,W}(\{\mathcal{F}_u\}) \le C(\ln W)^2 \mathcal{V}^{N,W}(\{\mathcal{F}_u\}) \tag{7.1}$$

provided $W = O(e^{N^\alpha})$ *for* $\alpha < 1/2$.

Proof. Define

$$\begin{aligned}
\mathcal{U}^{N,W}_{(2)}(\{\mathcal{F}_u\}) = & \sum_{l=1}^{N_r} \int_{\Omega^r_l} \sum_{i,j=1}^{3} \left(\frac{\partial^2 u^r_l}{\partial x_i \partial x_j} \right)^2 dx \\
& + \sum_{v\in\mathcal{V}} \sum_{\substack{l=1,\\ \mu(\tilde{\Omega}^v_l)<\infty}}^{N_v} \int_{\tilde{\Omega}^v_l} e^{x^v_3} \sum_{i,j=1}^{3} \left(\frac{\partial^2 u^v_l}{\partial x^v_i \partial x^v_j} \right)^2 dx^v \\
& + \sum_{v-e\in\mathcal{V}-\mathcal{E}} \sum_{\substack{l=1,\\ \mu(\tilde{\Omega}^{v-e}_l)<\infty}}^{N_{v-e}} \int_{\tilde{\Omega}^{v-e}_l} e^{x^{v-e}_3} \left(\sum_{i,j=1,2} \left(\frac{\partial^2 u^{v-e}_l}{\partial x^{v-e}_i \partial x^{v-e}_j} \right)^2 \right. \\
& \left. + \sin^2\phi \sum_{i=1}^{2} \left(\frac{\partial^2 u^{v-e}_l}{\partial x^{v-e}_i \partial x^{v-e}_3} \right)^2 + \sin^4\phi \left(\frac{\partial^2 u^{v-e}_l}{(\partial x^{v-e}_3)^2} \right)^2 \right) dx^{v-e}
\end{aligned}$$

$$+\sum_{e\in\mathcal{E}}\sum_{\substack{l=1,\\ \mu(\tilde{\Omega}_l^e)<\infty}}^{N_e}\int_{\tilde{\Omega}_l^e}\left(\sum_{i,j=1,2}\left(\frac{\partial^2 u_l^e}{\partial x_i^e\partial x_j^e}\right)^2+e^{2\tau}\sum_{i=1}^{2}\left(\frac{\partial^2 u_l^e}{\partial x_i^e\partial x_3^e}\right)^2\right.$$
$$\left.+e^{4\tau}\left(\frac{\partial^2 u_l^e}{(\partial x_3^e)^2}\right)^2\right)dx^e.$$

Then combining the results in sections 3, 4, 5 and 6, we obtain that for any $\epsilon>0$, there exist constants C_ϵ and K_ϵ such that

$$\begin{aligned}\mathcal{U}_{(2)}^{N,W}(\{\mathcal{F}_u\}) \;\le\; & C_\epsilon(\ln W)^2\mathcal{V}^{N,W}(\{\mathcal{F}_u\})+\epsilon\,\mathcal{U}_{(2)}^{N,W}(\{\mathcal{F}_u\})\\ &+K_\epsilon\,\mathcal{U}_{(1)}^{N,W}(\{\mathcal{F}_u\}.\end{aligned}\tag{7.2}$$

Here, $\mathcal{U}_{(1)}^{N,W}(\{\mathcal{F}_u\})$ is as defined in (4.1). Hence, choosing ϵ small enough, we obtain

$$\mathcal{U}_{(2)}^{N,W}(\{\mathcal{F}_u\}) \;\le 2(C_\epsilon(\ln W)^2\mathcal{V}^{N,W}(\{\mathcal{F}_u\})+K_\epsilon\,\mathcal{U}_{(1)}^{N,W}(\{\mathcal{F}_u\})).\tag{7.3}$$

At the same time using Theorem 4.1, we have

$$\mathcal{U}_{(1)}^{N,W}(\{\mathcal{F}_u\})\le K_{N,W}\mathcal{V}^{N,W}(\{\mathcal{F}_u\}).\tag{7.4}$$

Moreover,

$$\mathcal{U}^{N,W}(\{\mathcal{F}_u\})=\mathcal{U}_{(1)}^{N,W}(\{\mathcal{F}_u\})+\mathcal{U}_{(2)}^{N,W}(\{\mathcal{F}_u\}).\tag{7.5}$$

Combining (7.3), (7.4) and (7.5) the result follows. □

We have now established our main stability estimate theorem which will be the foundation stone for designing an efficient and exponentially accurate numerical scheme, parallel preconditioner and error estimates for elliptic boundary value problems on polyhedral domains containing singularities in the framework of the h-p spectral element method in the forthcoming work.

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