

## Limit law of the iterated logarithm for $B$ -valued trimmed sums

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MS received 16 May 2013; revised 7 May 2014

**Abstract.** Given a sequence of i.i.d. random variables  $\{X, X_n; n \geq 1\}$  taking values in a separable Banach space  $(B, \|\cdot\|)$  with topological dual  $B^*$ , let  $X_n^{(r)} = X_m$  if  $\|X_m\|$  is the  $r$ -th maximum of  $\{\|X_k\|; 1 \leq k \leq n\}$  and  ${}^{(r)}S_n = S_n - (X_n^{(1)} + \cdots + X_n^{(r)})$  be the trimmed sums when extreme terms are excluded, where  $S_n = \sum_{k=1}^n X_k$ . In this paper, it is stated that under some suitable conditions,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{\|{}^{(r)}S_k\|}{\sqrt{k}} = \sigma(X) \quad \text{a.s.,}$$

where  $\sigma^2(X) = \sup_{f \in B_1^*} \mathbb{E} f^2(X)$  and  $B_1^*$  is the unit ball of  $B^*$ .

**Keywords.** Banach space; trimmed sums; the limit law of the iterated logarithm.

**2010 Mathematics Subject Classification.** 60F15, 60G50.

### 1. Introduction and main result

Suppose that  $(B, \|\cdot\|)$  is a separable Banach space with topological dual  $B^*$ , and  $(\Omega, \mathcal{U}, \mathbb{P})$  is a probability space. Let  $\{X, X_n; n \geq 1\}$  be a sequence of  $B$ -valued independent and identically distributed (i.i.d.) random variables, and denote the partial sums by  $S_n = \sum_{k=1}^n X_k$  for  $n \geq 1$ . For any integer  $r \geq 1$  and  $n \geq r$ , set  $X_n^{(r)} = X_j$  if  $X_j$  is the  $r$ -th maximum of  $\{\|X_k\|; 1 \leq k \leq n\}$  (0 if  $r > n$ ), and let  ${}^{(r)}S_n = S_n - (X_n^{(1)} + \cdots + X_n^{(r)})$  (0 if  $r > n$ ) be the trimmed sums when extreme terms are excluded. Notice that  ${}^{(0)}S_n$  is just  $S_n$ .

It is well known that the trimmed sums play an important role in probability and statistics, since trimming is a standard method to decrease the effect of large sample elements in statistical procedures, e.g., for constructing robust estimators and tests. A great amount of work has been done during the last decades on the limit theorems of  ${}^{(r)}S_n$ . One can refer to [6, 8, 9] for the strong law of large number, [2–6] for various kinds of laws of the iterated logarithm (LIL) and [10, 11] for the strong approximation theorems.

Among these classical limit theorems, the LIL is an important and profound result, since it describes the precise convergence rates. When LIL is involved, we always consider the limsup LIL or the liminf LIL (for example, the Hartman–Wintner’s LIL or the

Chung's LIL). Recently, Chen [1] gave a new type of LIL, which is called the limit LIL, for the partial sums of i.i.d. real-valued random variables, and his result reads as follows.

**Theorem A.** *Let  $\{X, X_n; n \geq 1\}$  be a sequence of i.i.d. real-valued random variables with  $EX = 0$  and  $EX^2 = \sigma^2$ , and we have*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{S_k}{\sqrt{k}} = \sigma \quad a.s.,$$

where  $\log t = \log(t \vee e)$  and  $\log \log t = \log(\log t)$ ,  $t \geq 0$ .

Later, Li and Liang [7] established a version of the limit LIL in Banach space, and got the following theorem.

**Theorem B.** *Let  $\{X, X_n; n \geq 1\}$  be a sequence of i.i.d.  $B$ -valued random variables satisfying*

$$EX = 0, \quad E \left( \frac{\|X\|^2}{\log \log \|X\|} \right) < \infty, \quad Ef^2(X) < \infty, \quad \forall f \in B^*$$

and

$$\frac{S_n}{\sqrt{2n \log \log n}} \xrightarrow{p} 0.$$

Then we have

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{\|S_k\|}{\sqrt{k}} = \sigma(X) \quad a.s., \quad (1.1)$$

where  $\sigma^2(X) = \sup_{f \in B_1^*} Ef^2(X)$  and  $B_1^*$  is the unit ball of  $B^*$ .

In this paper, we intend to establish a limit LIL for the  $B$ -valued trimmed sums. For this end, we denote by  $L_{p,q}$  the space of all real random variables  $\xi$  such that

$$\int_0^\infty (t^p \mathbf{P}(|\xi| > t))^{q/p} \frac{dt}{t} < \infty.$$

It is not difficult to verify

$$L_{p,q_1} \subset L_{p,q_2} \quad \text{if } q_1 \leq q_2$$

and

$$\lim_{t \rightarrow \infty} (t^p \mathbf{P}(|\xi| > t))^{q/p} = 0 \quad \text{if } \xi \in L_{p,q}.$$

Now it is in a position to state our main result.

**Theorem 1.1.** *Let  $\{X, X_n; n \geq 1\}$  be a sequence of i.i.d.  $B$ -valued random variables satisfying*

$$\|X\|^2 / \log \log \|X\| \in L_{1,r+1},$$

where  $r \geq 0$ . If

$$Ef(X) = 0, \quad Ef^2(X) < \infty, \quad \forall f \in B^* \quad (1.2)$$

and

$$\frac{S_n}{\sqrt{2n \log \log n}} \xrightarrow{p} 0,$$

then we have

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{\|^{(r)}S_k\|}{\sqrt{k}} = \sigma(X) \quad a.s.$$

As a direct application of Theorem 1.1, we can obtain the following result for trimmed sums in real spaces.

#### COROLLARY 1.1

Let  $\{X, X_n; n \geq 1\}$  be i.i.d. real-valued random variables with  $EX = 0$  and  $EX^2 = \sigma^2$ , and  $r \geq 0$ . Then we have

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{|^{(r)}S_k|}{\sqrt{k}} = \sigma \quad a.s.$$

*Remark 1.1.* The limit LIL of the trimmed sums is obtained under the same conditions of Zhang [11] and Fu and Zhang [2] for the classical limsup LIL of the trimmed sums, which means that our result is derived under minimal conditions.

## 2. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. Before stating the proof, we first introduce two useful lemmas.

*Lemma 2.1.* Let  $\{a_n; n \geq 1\}$  be a nondecreasing sequence of positive real numbers such that

$$\lim_{n \rightarrow \infty} a_n = \infty.$$

Then for any sequence  $\{b_n; n \geq 1\}$  of real numbers, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n} \max_{1 \leq k \leq n} b_k = 0 \vee \gamma,$$

where  $\gamma = \limsup_{n \rightarrow \infty} \frac{b_n}{a_n}$ .

*Proof.* See Lemma 2.1 of [7]. □

*Lemma 2.2.* Let  $\{X, X_n; n \geq 1\}$  be a sequence of i.i.d.  $B$ -valued random variables such that (1.2) is satisfied. Then for any  $1 \leq r \leq n$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{|f(X_k^{(r)})|}{\sqrt{k}} = 0. \quad (2.1)$$

*Proof.* Note that for any  $\varepsilon > 0$  and  $f \in B^*$ ,  $E f^2(X) < \infty$  if and only if

$$\sum_{n=1}^{\infty} P(|f(X_n)| \geq \varepsilon \sqrt{n}) = \sum_{n=1}^{\infty} P(|f(X)| \geq \varepsilon \sqrt{n}) < \infty.$$

Thus an application of the Borel–Cantelli lemma and (1.2), along with the arbitrariness of  $\varepsilon$ , yields that for any  $f \in B^*$ ,

$$\lim_{n \rightarrow \infty} \frac{|f(X_n)|}{\sqrt{n}} = 0 \quad \text{a.s.}$$

Then, by applying Lemma 2.1, it gives

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} |f(X_k)| = 0 \quad \text{a.s.},$$

which leads to

$$\lim_{n \rightarrow \infty} \frac{\max_{1 \leq k \leq n} |f(X_k)| / \sqrt{n}}{\sqrt{2 \log \log n}} = 0 \quad \text{a.s.}$$

Hence, by applying Lemma 2.1 again, we have

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \left( \frac{\max_{1 \leq m \leq k} |f(X_m)|}{\sqrt{k}} \right) = 0 \quad \text{a.s.},$$

and this ensures (2.1) holds, since  $|f(X_k^{(r)})| \leq \max_{1 \leq m \leq k} |f(X_m)|$ .  $\square$

*Proof of Theorem 1.1.* We first prove the upper bound. Recalling the limsup LIL for the trimmed sums (see [2]), we have that

$$\limsup_{n \rightarrow \infty} \frac{\|^{(r)}S_n\|}{\sqrt{2n \log \log n}} = \sigma(X) \quad \text{a.s.}$$

By taking

$$a_n = \sqrt{2 \log \log n} \quad \text{and} \quad b_k = \frac{\|^{(r)}S_k\|}{\sqrt{k}}$$

into Lemma 2.1, it implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{\|^{(r)}S_k\|}{\sqrt{k}} \leq 0 \vee \sigma(X) = \sigma(X) \quad \text{a.s.},$$

and thus the upper bound follows, as desired.

Now we prove the lower bound. By virtue of (1.2), it follows from Lemma 2.2 and Theorem A that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{\|^{(r)}S_k\|}{\sqrt{k}} &\geq \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{|f(^{(r)}S_k)|}{\sqrt{k}} \\ &= \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{|f(S_k)|}{\sqrt{k}} \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{f(S_k)}{\sqrt{k}} \\ &= E f^2(X) \quad \text{a.s.,} \quad \forall f \in B^*, \end{aligned}$$

and this gives

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{\|^{(r)} S_k\|}{\sqrt{k}} \geq \sigma(X) \quad \text{a.s.}$$

which ends the proof of Theorem 1.1.

### Acknowledgements

The authors would like to thank the referees for pointing out some errors in a previous version, as well as for several comments that led to the improvement of the paper. This project supported by the National Natural Science Foundation of China (Nos 11201422, 11301481 and 11371321), Zhejiang Provincial Natural Science Foundation of China (Nos Y6110639 & LQ12A01017) and Foundation for Young Talents of ZJGSU (No. 1020XJ1314019).

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