

Revisiting the Zassenhaus conjecture on torsion units for the integral group rings of small groups

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Abstract. In recent years several new restrictions on integral partial augmentations for torsion units of $\mathbb{Z}G$ have been introduced, which have improved the effectiveness of the Luthar–Passi method for checking the Zassenhaus conjecture for specific groups G . In this note, we report that the Luthar–Passi method with the new restrictions are sufficient to verify the Zassenhaus conjecture with a computer for all groups of order less than 96, except for one group of order 48 – the non-split covering group of S_4 , and one of order 72 of isomorphism type $(C_3 \times C_3) \rtimes D_8$. To verify the Zassenhaus conjecture for this group we give a new construction of normalized torsion units of $\mathbb{Q}G$ that are not conjugate to elements of $\mathbb{Z}G$.

Keywords. Integral group rings; torsion units; Zassenhaus conjectures.

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1. Introduction

The Zassenhaus conjecture $ZC1$ for torsion units of the integral group ring states that for all finite groups G , every unit u of $\mathbb{Z}G$ having augmentation 1 is conjugate in $\mathbb{Q}G$ to an element of G . A general reference for known results on $ZC1$ is [4]. In [4] it is explained why $ZC1$ holds for all groups of order ≤ 71 . There the authors used a computer implementation of the Luthar–Passi method of [5] to check $ZC1$ for those small groups for which $ZC1$ did not follow by a general theorem. Since that time new restrictions on partial augmentations have emerged. This motivated the authors to examine the situation for groups of slightly larger order.

2. Partial augmentations of units

If x and g are elements of a group G , we write $x \sim g$ when x and g are conjugate in G . When $u = \sum_{g \in G} u(g)g \in \mathbb{Z}G$, $\varepsilon(u) = \sum_{g \in G} u(g)$ denotes the augmentation of u , and $\varepsilon_x(u) = \sum_{x \sim g} u(g)$ denotes the partial augmentation of u with respect to $x \in G$. Let $\text{Irr}(G)$ be the set of complex irreducible characters of the finite group G .

If R is a ring with identity, then $U(R)$ denotes the group of units of R . The subgroup of $U(\mathbb{Z}G)$ consisting of units with augmentation 1 is denoted by $V(\mathbb{Z}G)$. $V(\mathbb{Z}G)^{\text{tor}}$ denotes the subset of $V(\mathbb{Z}G)$ consisting of torsion units; i.e. units of finite order.

Our first theorem examines the extent to which rational conjugacy of units in $\mathbb{Q}G$ is determined by their partial augmentations.

Theorem 1. *Let G be a finite group.*

- (i) *If $u, v \in \mathbb{Q}G$, and $u \sim v$ in $\mathbb{Q}G$, then $\varepsilon_x(u) = \varepsilon_x(v)$, for all $x \in G$.*
- (ii) *Suppose u and v are torsion units in $\mathbb{Q}G$ with the same order $k > 1$, that $u^d \sim v^d$ for all divisors d of k with $1 < d \leq k$, and that $\varepsilon_x(u) = \varepsilon_x(v)$ for all $x \in G$. Then $u \sim v$ in $\mathbb{Q}G$.*

Proof.

- (i) Suppose $u \sim v$ in $\mathbb{Q}G$. Then $\chi(u) = \chi(v)$ for all $\chi \in \text{Irr}(G)$. Let $K(G)$ be a complete set of representatives for the conjugacy classes of G . Then, for all $\chi \in \text{Irr}(G)$, we have that

$$\chi(u) = \chi(v) \implies \sum_{x \in K(G)} \varepsilon_x(u) \chi(x) = \sum_{x \in K(G)} \varepsilon_x(v) \chi(x).$$

Since the character table of a finite group G is an invertible matrix, it follows that $\varepsilon_x(u) = \varepsilon_x(v)$, for all $x \in G$.

- (ii) Let u_1 be a torsion unit of $\mathbb{Q}G$ with order k . Then u_1 produces a specific list $(\mu) = (\mu_\ell(u_1, \chi))_{\ell, \chi}$ of nonnegative integers that are associated with the collection of all the Luthar–Passi equations

$$\mu_\ell(u_1, \chi) = \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(\zeta_k^d)/\mathbb{Q}}(\chi(u_1^d) \xi^{-\ell d})$$

as ℓ runs through 0 to $k - 1$ and χ runs over $\text{Irr}(G)$ [5]. Here ζ_k denotes a primitive k -th root of unity. Our assumptions that $\chi(u^d) = \chi(v^d)$ for $d > 1$ dividing k and that u and v have the same partial augmentations mean that the right-hand side of the collection of LP-equations for the unit u matches the right-hand side of the LP-equations for the unit v . Therefore, the left hand sides also match, so $\mu_\ell(u, \chi) = \mu_\ell(v, \chi)$ for all ℓ and χ .

The k nonnegative integers $\mu_\ell(u_1, \chi)$, $\ell = 0, \dots, k - 1$ for a fixed χ each represent the multiplicity of ζ_k^ℓ as an eigenvalue of $\mathcal{X}(v)$, for any irreducible representation \mathcal{X} with character χ . Observe that the full spectrum of $\mathcal{X}(u_1)$ is determined by these multiplicities. When u_1 has finite order, $\mathcal{X}(u_1)$ is a matrix of finite order, and is therefore diagonalizable. If χ has degree n , any matrix in $GL(n, \mathbb{C})$ with this same spectrum will be conjugate to $\mathcal{X}(u_1)$ in $GL(n, \mathbb{C})$. In particular, since $\mu_\ell(u, \chi) = \mu_\ell(v, \chi)$ for all ℓ , $\mathcal{X}(u)$ is conjugate to $\mathcal{X}(v)$ in $GL(n, \mathbb{C})$. Since χ is an absolutely irreducible character, $\mathcal{X}(\mathbb{C}G) = M_n(\mathbb{C})$, so in fact we have that $\mathcal{X}(u)$ and $\mathcal{X}(v)$ are conjugate in $\mathcal{X}(\mathbb{C}G)$.

Since this is the case for all irreducible characters χ of G , if we let \mathcal{R} be the regular representation of G , then it must be the case that $\mathcal{R}(u)$ and $\mathcal{R}(v)$ are conjugate in $\mathcal{R}(\mathbb{C}G)$. Since \mathcal{R} is a faithful representation of G , this implies that u and v are conjugate in $\mathbb{C}G$.

Since u and v are torsion units of $\mathbb{Q}G$, it follows from Lemma 37.5 of [6] that u and v are in fact conjugate in $\mathbb{Q}G$. □

Some recent observations by Hertweck are particularly useful in improving performance of computer verifications of the Zassenhaus conjecture for small groups using

the Luthar–Passi method. We use two of these in particular. The first of these is that if $u \in V(\mathbb{Z}G)$ has prime power order p^n for some prime p and positive integer n , then $\varepsilon_x(u) = 0$ for any element x of G for which $o(x)_p > p^n$ (see Remark 2.4 of [2]). From this it follows that $\varepsilon_x(u)$ can be nonzero only when $o(x)$ divides $o(u)$. The second is that if G is a solvable group and $u \in V(\mathbb{Z}G)^{\text{tor}}$, then G has an element x for which $o(x) = o(u)$ and $\varepsilon_x(u) \neq 0$ [3].

By G Higman and S Berman (see Proposition 1.4 of [6]), if $u \in V(\mathbb{Z}G)^{\text{tor}}$, then $u \neq 1 \implies u(1) = 0$. This ensures that the coefficient of the identity in any nontrivial torsion unit of $\mathbb{Z}G$ is always 0. Another property that is useful for dealing with some nontrivial solutions to the Luthar–Passi equations is a special case of Theorem 4.1 of [1].

PROPOSITION 2

Let $u \in V(\mathbb{Z}G)^{\text{tor}}$, $o(u) = p^n$. Then

$$0 \equiv \sum_{o(g)=p^m} u(g) \pmod{p}, \text{ for } 1 \leq m < n,$$

and

$$1 \equiv \sum_{o(g)=p^n} u(g) \pmod{p}.$$

For further restrictions on partial augmentations, see [2].

3. Construction of partially central torsion units of $\mathbb{Q}G$

A torsion unit u of $\mathbb{Q}G$ has *trivial* partial augmentations if there is a unique $x \in K(G)$ for which $\varepsilon_x(u) = 1$ and $\varepsilon_g(u) = 0$ for all other $g \in K(G)$. The idea of the Luthar–Passi method for verifying ZC1 for a group G is based on showing that for all $k > 1$ dividing $|G|$, the trivial partial augmentations are the only possible integral solutions of the Luthar–Passi equations for units of order k .

We have used a computer to apply the Luthar–Passi method with the new restrictions on partial augmentations to all groups of order less than 96. The calculations show all of these groups except two satisfy ZC1. The smallest exception is $G = \text{SmallGroup}(48, 30)$, the non-split covering group of S_4 , for which the LP-equations for elements of order 4 (and no other order) have integral solutions corresponding to units with nontrivial partial augmentations. In the next example we show how to construct units of $\mathbb{Q}G$ that produce these partial augmentations. The units we construct turn out to have a ‘partially central’ property that prevents them from being rationally conjugate to elements of $\mathbb{Z}G$. The next exception is identified as $\text{SmallGroup}(72, 40)$ in GAP. It produces solutions to the LP equations for units of order 6 for which ZC1 is not verified by method of this article. Our computer has now started through the list of groups of order 96, and has found more of these interesting cases, but so far the ones we have found can be dealt with using our constructive method.

Example. Let G be the group identified by $\text{SmallGroup}(48, 30)$ in the GAP library. We will only consider the LP-equations for units of order 4, for other orders the solutions to the LP-equations turn out to be trivial.

The group G is a non-split central extension of the form $C_2 : S_4$. Our notation for the character table and generators of G was produced using GAP [7]. The generators of G in its polycyclic presentation in GAP are: f_1 (of order 4), f_2 (a central element of order 2),

f_3 (of order 3), f_4 and f_5 (both of order 2). There are 8 columns of GAP's character table of G corresponding to elements of order dividing 4:

	1a	2a	2b	2c	4a	4b	4c	4d
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1
χ_3	1	-1	1	-1	-i	i	-i	i
χ_4	1	-1	1	-1	i	-i	i	-i
χ_5	2	-2	2	-2	0	0	0	0
χ_6	2	2	2	2	0	0	0	0
χ_7	3	3	-1	-1	-1	-1	1	1
χ_8	3	3	-1	-1	1	1	-1	-1
χ_9	3	-3	-1	1	-i	i	i	-i
χ_{10}	3	-3	-1	1	i	-i	-i	i

The nontrivial integer solutions to the Luthar–Passi equations for order $k = 4$ have $\varepsilon_{2a} = \varepsilon_{2b} = \varepsilon_{2c} = 0$. All of them arise after choosing u^2 to lie in the class $2a$. The solutions for the partial augmentations on classes of order 4 come in two patterns:

$$(\varepsilon_{4a}, \varepsilon_{4b}, \varepsilon_{4c}, \varepsilon_{4d}) \in \{(2, 0, -1, 0), (0, 2, 0, -1), (-1, 0, 2, 0), (0, -1, 0, 2), (1, -1, 0, 1), (-1, 1, 0, 1), (1, 0, 1, -1), (0, 1, 1, -1), (0, -1, 1, 1), (-1, 0, 1, 1), (1, 1, 0, -1), (1, 1, -1, 0)\}.$$

The lists of multiplicities corresponding to these solutions produce the following lists of eigenvalues for our candidates for $\mathcal{X}_\chi(u)$:

χ	$\text{spec}(\mathcal{X}(u))$	u_i
χ_1	(1)	
χ_2	(-1)	
χ_3	(i), (-i)	$\leftarrow u_3 = f_1, f_1 f_2$
χ_4	(-i), (i)	
χ_5	(i, -i)	
χ_6	(1, -1)	
χ_7	(-1, -1, 1), (-1, 1, 1), (-1, -1, -1), (1, 1, 1)	$\leftarrow u_7 = f_1, -f_1, v, -v$
χ_8	(-1, 1, 1), (-1, -1, 1), (1, 1, 1), (-1, -1, -1)	$\leftarrow u_8 = u_7$
χ_9	(-i, -i, -i), (i, i, i), (-i, -i, i), (-i, i, i)	$\leftarrow u_9 = -v, v, -f_1, f_1$
χ_{10}	(i, i, i), (-i, -i, -i), (-i, i, i), (-i, -i, i)	$\leftarrow u_9 = u_{10}$

The element $v = -\frac{1}{2}\widehat{C_{4a}}$ is an appropriate scalar multiple of a class sum that will be mapped onto the desired scalar multiples of the identity matrix by an irreducible representation \mathcal{X}_i affording the character χ_i . Let e_{χ_i} be the centrally primitive idempotent of $\mathbb{C}G$ corresponding to χ_i . The idea is that choosing an appropriate $u_i \in \mathbb{Q}G$ for $i = 1, \dots, 10$ will result in a unit $u = \sum_{i=1}^{10} u_i e_{\chi_i}$ that has the desired partial augmentations. It turns out that the choice of u_3 will be an appropriate choice for u_1 through u_6 , the choice for u_7 is appropriate for u_8 , and the choice for u_9 will work for u_{10} . We have given the above 2 choices for u_3 , 4 choices for u_7 , and 4 choices for u_9 . These have been produced using the irreducible representations produced by the GAP command `IrreducibleRepresentations(G)`. (One must be careful while using it since the

ordering of the representations it gives may not match the ordering in the character table.) With these we can produce 32 units of $\mathbb{Q}G$, 16 of which give integral partial augmentations, 4 of these corresponding to cases of trivial partial augmentations, and the other 12 are the nontrivial ones listed above.

The torsion units u we construct from u_i will have the form $b_1e_1 + b_2e_2 + b_3e_3$, where

$$\begin{aligned} e_1 &= e_{\chi_1} + e_{\chi_2} + e_{\chi_3} + e_{\chi_4} + e_{\chi_5} + e_{\chi_6}, \\ e_2 &= e_{\chi_7} + e_{\chi_8}, \\ e_3 &= e_{\chi_9} + e_{\chi_{10}} \end{aligned}$$

are nontrivial orthogonal central idempotents of $\mathbb{Q}G$ that sum to 1, and b_1, b_2 , and b_3 vary among the selections listed above for the u_i for $i = 3, 7$, and 9 . The 12 torsion units of order 4 that we get having nontrivial partial augmentations are

u	$(\varepsilon_{4a}, \varepsilon_{4b}, \varepsilon_{4c}, \varepsilon_{4d})$
$f_1e_1 - ve_2 - ve_3$	$(2, 0, -1, 0)$
$f_1e_1 + ve_2 + ve_3$	$(-1, 0, 2, 0)$
$f_1f_2e_1 - ve_2 + ve_3$	$(0, 2, 0, -1)$
$f_1f_2e_1 + ve_2 - ve_3$	$(0, -1, 0, 2)$
$f_1e_1 + f_1e_2 + ve_3$	$(0, 1, 1, -1)$
$f_1f_2e_1 + f_1e_2 - ve_3$	$(1, 0, -1, 1)$
$f_1e_1 - f_1e_2 - ve_3$	$(1, -1, 0, 1)$
$f_1f_2e_1 - f_1e_2 + ve_3$	$(-1, 1, 1, 0)$
$f_1e_1 - ve_2 - f_1e_3$	$(1, 1, 0, -1)$
$f_1f_2e_1 - ve_2 + f_1e_3$	$(1, 1, -1, 0)$
$f_1e_1 + ve_2 + f_1e_3$	$(0, -1, 1, 1)$
$f_1f_2e_1 + ve_2 - f_1e_3$	$(-1, 0, 1, 1)$

A key observation concerning these units is that the ve 's for $e \in \{e_2, e_3\}$ do not lie in $\mathbb{Z}Ge$. We have checked this with GAP by showing that ve does not lie in the integral span of Ge . This allows us to make use of the following lemma.

Lemma 3. Let e be a centrally primitive idempotent of $\mathbb{Q}G$ with $e \neq 0, 1$. Suppose $v \in \mathbb{Q}G$ with $ve \in Z(\mathbb{Q}G)$ and $ve \notin \mathbb{Z}Ge$. Then for all $t \in \mathbb{Q}G$, $ve + t(1 - e)$ is not conjugate in $\mathbb{Q}G$ to an element of $\mathbb{Z}G$.

Proof. Let $t \in \mathbb{Q}G$ and let $u = ve + t(1 - e)$. If w is a unit of $\mathbb{Q}G$, then $u^w = ve + t^w(1 - e)$. Since $t^w(1 - e)$ always lies in $\mathbb{Q}G(1 - e)$ and $\mathbb{Q}Ge \cap \mathbb{Q}G(1 - e) = \{0\}$, it cannot make up the difference between ve and any element of $\mathbb{Z}Ge$. This means that u^w cannot be an element of $\mathbb{Z}G$, since multiplying it by e does not result in an element of $\mathbb{Z}Ge$. \square

Now that $ZC1$ holds for units of $\mathbb{Z}G$ with order 4 follows from the lemma and Theorem 1.

Remark 4. We note that the above construction of ‘partially central’ torsion units of $\mathbb{Q}G$ can be carried out whenever $\chi(g)$ is a rational multiple of a root of unity for some $\chi \in \text{Irr}(G)$ and noncentral $g \in G$.

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