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h-p Spectral element methods for three dimensional elliptic problems on non-smooth domains, Part-I: Regularity estimates and stability theorem

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Abstract. This is the first of a series of papers devoted to the study of h-p spectral element methods for solving three dimensional elliptic boundary value problems on non-smooth domains using parallel computers. In three dimensions there are three different types of singularities namely; the vertex, the edge and the vertex-edge singularities. In addition, the solution is anisotropic in the neighbourhoods of the edges and vertex-edges. To overcome the singularities which arise in the neighbourhoods of vertices, vertex-edges and edges, we use local systems of coordinates. These local coordinates are modified versions of spherical and cylindrical coordinate systems in their respective neighbourhoods. Away from these neighbourhoods standard Cartesian coordinates are used. In each of these neighbourhoods we use a geometrical mesh which becomes finer near the corners and edges. The geometrical mesh becomes a quasi-uniform mesh in the new system of coordinates. We then derive differentiability estimates in these new set of variables and state our main stability estimate theorem using a non-conforming h-p spectral element method whose proof is given in a separate paper.

Keywords. Spectral element method; non-smooth domains; geometric mesh; vertex singularity; edge singularity; vertex-edge singularity; differentiability estimates; stability estimates; exponential accuracy.

Mathematics Subject Classification. 65N35, 35J25.

1. Introduction

Finite element methods (FEM) are one of the most widely used techniques for solving problems in structural mechanics. There are three versions of the FEM namely; the h

version, the p version and the h-p version. The h version uses polynomials of a fixed degree and the mesh size h is reduced to increase accuracy. In the p version, a fixed mesh is used and polynomial degree p is raised to increase accuracy. The h-p version combines the approaches of the h and p versions. It simultaneously refines the mesh and increases the polynomial degree to solve problems on non-smooth domains and achieve optimal convergence. The h-p version of Spectral element method (SEM) employ global polynomials of higher degree in order to recover the so-called *spectral/exponential convergence*.

A method for obtaining a numerical solution to exponential accuracy for elliptic problems on non-smooth domains in $\mathbb{R}^2$ was first proposed by Babuška and Guo [7, 8] in the frame work of the finite element method. In [18, 19], an exponentially accurate h-p spectral element method was proposed for two dimensional elliptic problems on non-smooth domains with analytic coefficients posed on curvilinear polygons with piecewise analytic boundary. The method is able to resolve the singularities which arise at the corners using a geometrical mesh as proposed by Babuška and Guo.

In contrast to the two dimensional case, the character of the singularities in three dimensions is much more complex not only because of higher dimension but also due to the varied nature of the singularities which are the vertex singularity, the edge singularity and the vertex-edge singularity. Thus, we have to distinguish between the behaviour of the solution in the neighbourhoods of the vertices, edges and vertex-edges. Unlike the two dimensional case where weighted isotropic spaces are used, in three dimensions we have to utilize weighted anisotropic spaces because the solution is smooth along the edges but singular in the direction perpendicular to the edges [4]. Behaviour of the solution is even more complex at the vertices where the edges are joined together and the solution is not smooth along the edges too. Guo [24] introduced the relevant anisotropic weighted spaces to study elliptic problems on non-smooth polyhedral domains. Since then the proof that the regularity of solutions of elliptic boundary value problems on non-smooth domains is described by these sapces remained an open problem for a long time.

To prove the analytic regularity for these problems, Babuška and Guo [2, 3] started the study of analytic regularity of elliptic problems on non-smooth domains in $\mathbb{R}^3$ in the frame work of weighted Sobolev spaces with Cauchy type control of all derivatives in the so-called *countably normed spaces* in the neighbourhoods of vertices, edges and vertex-edges in spherical, cylindrical and Cartesian coordinates. However, proving these regularity results is quite technical and sometimes difficult to follow as can be seen in the papers by Babuška and Guo [2–4, 24]. We remark that these regularity estimates on polyhedral domains were assumed to be true in the error analysis of h-p FEM in [24, 29] and in the error analysis of h-p SEM in [1].

Recently, Costabel and coworkers settled the proof of the analytic regularity estimates in [11] by filling the gap which was left over by Babuška and Guo. They combined *a priori* basic regularity results of low order for elliptic problems on polyhedral domains [14, 27] with the regularity shift estimates of Cauchy type to complete the proof using a nested open set technique and dyadic partition technique near corners [11–14]. The techniques employed in [11–13] extends to the general elliptic problems having lower order terms and variable coefficients examined in this paper (see [11] and references therein). It follows from [11, 12] that the solution to the problem under consideration in this paper belongs to an analytic class which is defined using anisotropic weighted Sobolev spaces introduced in [2, 3].

The h-p version of the finite element method for solving three dimensional elliptic problems on non-smooth domains with exponential accuracy was proposed by Guo in [21, 24]. To overcome the singularities which arise along vertices and edges they used geometric meshes which are defined in neighbourhoods of vertices, edges and vertex-edges. We refer to [4–6, 9, 10, 22–24] for a detailed discussion of the h-p FEM and [23] on the use of auxiliary mappings for the finite element solutions of three dimensional elliptic problems on non-smooth domains.

An efficient and exponentially accurate h-p spectral element method to solve general elliptic problems on non-smooth domains in three dimensions is now described. We also use a geometric mesh in the neighbourhoods of vertices, edges and vertex-edges and in each of these neighbourhoods we switch to a modified system of local coordinates using auxiliary mappings and this enables us to obtain the solution with exponential accuracy. These local coordinates are modified versions of the standard spherical and cylindrical coordinate systems in vertex and edge neighbourhoods respectively and a hybrid combination of spherical and cylindrical coordinates in vertex-edge neighbourhoods. The geometric mesh becomes geometrically fine in these neighbourhoods and in the new set of variables in these neighbourhoods the geometric mesh is mapped to a quasi uniform mesh. Hence Sobolev's embedding theorems and the trace theorems apply for spectral element functions defined on mesh elements in the new system of variables with a uniform constant. This new system of coordinates in two dimensions was first proposed by Kondratiev [25] and we shall refer to them as *modified system of coordinates*. Away from the neighbourhoods of vertices, edges and vertex-edges, we retain the standard Cartesian coordinate system (x_1, x_2, x_3) in the regular region of the polyhedron.

We now seek a solution to elliptic BVP's as in [18, 19, 30–32] which minimizes the sum of a weighted squared norm of the residuals in the partial differential equation and a fractional Sobolev norm of the residuals in the boundary conditions and enforce continuity across inter element boundaries by adding a term which measures the sum of the squares of the jump in the function and its derivatives at inter element boundaries in appropriate Sobolev norms to the functional being minimized. Here we examine the non-conforming version of the method. The case when the spectral element functions are conforming will be examined in future work.

This series of papers is devoted to the study of regularity theory and implementation of the h-p spectral element method for three dimensional elliptic problems on non-smooth domains with analytic coefficients and piecewise analytic boundary data using parallel computers. The first paper deals with the regularity of the solution in the neighbourhoods of vertices, edges and vertex-edges and the stability theorem. The second paper is more technical in nature and addresses proof of the stability theorem [16]. The numerical scheme, error estimates, construction of preconditioners on regular as well as singular regions of the polyhedron and the solution techniques are discussed in the third paper. Theoretical results have been validated by computational experiments independently in [17]. We mention here that these papers are based on the thesis work of one of the authors and we refer to [1] for complete details.

The organization of this paper is as follows. In §2, we introduce the problem under consideration. We define various neighbourhoods of vertices, edges and vertex-edges and recall the function spaces in these neighbourhoods, as defined in [2], which will be needed in the sequel. In §3 we drive differentiability (regularity) estimates in various neighbourhoods. In §4, we impose a geometrical mesh on each of these neighbourhoods, define the spectral element functions on these elements and give construction of the stability estimate (without proof) which is the main result of the paper. Section 5 gives concluding remarks.

2. Preliminaries and notations

Let Ω denote a polyhedron in R^3, as shown in figure 1. We shall denote the boundary of Ω by $\partial\Omega$. Let Γ_i, $i \in \mathcal{I} = \{1, 2, \ldots, I\}$ be the faces of the polyhedron. Let $\mathcal{D}$ be a subset of $\mathcal{I}$ and $\mathcal{N} = \mathcal{I} \setminus \mathcal{D}$. We impose Dirichlet boundary conditions on the faces Γ_i, $i \in \mathcal{D}$ and Neumann boundary conditions on the faces Γ_j, $j \in \mathcal{N}$. Further, let $\partial\Omega = \Gamma^{[0]} \cup \Gamma^{[1]}$, $\Gamma^{[0]} = \bigcup_{i\in\mathcal{D}} \bar{\Gamma}_i$ and $\Gamma^{[1]} = \bigcup_{i\in\mathcal{N}} \bar{\Gamma}_i$.

We consider an elliptic boundary value problem posed on Ω with mixed Neumann and Dirichlet boundary conditions:

$$\begin{aligned} Lw &= F \quad \text{in } \Omega, \\ w &= g^{[0]} \quad \text{for } x \in \Gamma^{[0]}, \\ \left(\frac{\partial w}{\partial n}\right)_A &= g^{[1]} \quad \text{for } x \in \Gamma^{[1]}, \end{aligned} \tag{2.1}$$

where n denotes the outward normal and $\left(\frac{\partial w}{\partial n}\right)_A$ is the usual conormal derivative.

It is assumed that the differential operator

$$Lw(x) = \sum_{i,j=1}^{3} -(a_{ij} w_{x_j})_{x_i} + \sum_{i=1}^{3} b_i w_{x_i} + cw \tag{2.2}$$

is a strongly elliptic differential operator which satisfies the Lax–Milgram conditions. Moreover, $A = a_{ij} = a_{ji}$ for all i, j and the coefficients of the differential operator are analytic. The data F, $g^{[0]}$ and $g^{[1]}$ are analytic on each open face and $g^{[0]}$ is continuous on $\bigcup_{i\in\mathcal{D}} \bar{\Gamma}_i$.

2.1 *The neighbourhoods of vertices, edges and vertex-edges*

Let Γ_i, $i \in \mathcal{I} = \{1, 2, \ldots, I\}$ be the faces (open), S_j, $j \in \mathcal{J} = \{1, 2, \ldots, J\}$ be the edges and A_k, $k \in \mathcal{K} = \{1, 2, \ldots, K\}$ be the vertices of the polyhedron. We shall also denote an edge by e, where $e \in \mathcal{E} = \{S_1, S_2, \ldots, S_J\}$, the set of edges, and a vertex by v where

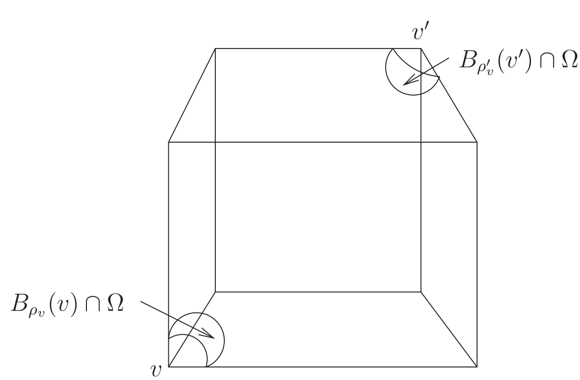

Figure 1. Polyhedral domain Ω.

$v \in \mathcal{V} = \{A_1, A_2, \ldots, A_K\}$, the set of vertices. Let $B_{\rho_v}(v) = \{x : \text{dist}\,(x, v) < \rho_v\}$. For every vertex v, ρ_v is chosen so small that $B_{\rho_v}(v) \cap B_{\rho_{v'}}(v') = \emptyset$ if the vertices v and v' are distinct.

Now consider a vertex v which has n_v edges passing through it (figure 2). We shall let x_3 axis denote one of these edges. Consider first the edge e which coincides with the x_3 axis. Let ϕ denote the angle which $x = (x_1, x_2, x_3)$ makes with the x_3 axis.

Let

$$\mathcal{V}_{\rho_v,\phi_v}(v, e) = \{x \in \Omega :\ 0 < \text{dist}(x, v) < \rho_v,\ 0 < \phi < \phi_v\},$$

where ϕ_v is a constant. Let us choose ϕ_v sufficiently small so that

$$\mathcal{V}_{\rho_v,\phi_v}(v, e') \bigcap \mathcal{V}_{\rho_v,\phi_v}(v, e'') = \emptyset,$$

where e' and e'' are distinct edges which have v as a common vertex. Now we define Ω^v, the vertex neighbourhood of the vertex v. Let $\mathcal{E}^v$ denote the subset of $\mathcal{E}$, the set of edges, such that $\mathcal{E}^v = \{e \in \mathcal{E} : v \text{ is a vertex of } e\}$. Then

$$\Omega^v = \left(B_{\rho_v}(v) \Big\backslash \bigcup_{e \in \mathcal{E}^v} \overline{\mathcal{V}_{\rho_v,\phi_v}(v, e)} \right) \bigcap \Omega\,.$$

Here, ρ_v and ϕ_v are chosen so that $\rho_v \sin(\phi_v) = Z$, a constant for all $v \in V$, the set of vertices. Let

$$\begin{aligned}
\rho &= \sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2},\\
\phi &= \cos^{-1}(x_3/\rho),\\
\theta &= \tan^{-1}(x_2/x_1),
\end{aligned}$$

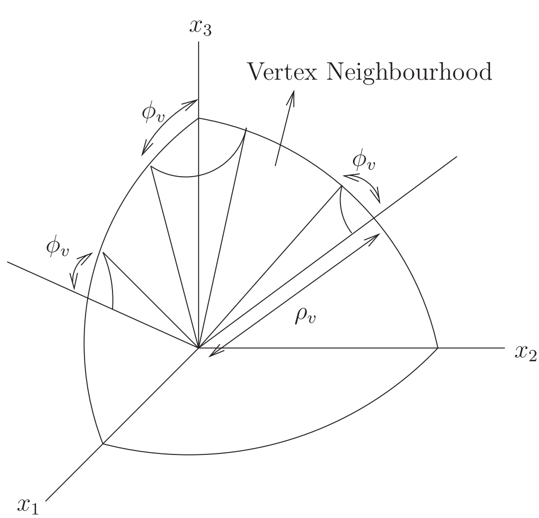

Figure 2. Vertex neighbourhood Ω^v.

denote the usual spherical coordinates in Ω^v. We now introduce a set of modified coordinates in the vertex neighbourhood Ω^v by

$$\begin{aligned} x_1^v &= \phi, \\ x_2^v &= \theta, \\ x_3^v &= \chi = \ln\rho\,. \end{aligned} \tag{2.3}$$

Let e denote an edge, which for convenience we assume to coincide with the x_3 axis as before, whose end points are the vertices v and v' as shown in figure 3.

Assume that the vertex v coincides with the origin. Let the length of the edge e be l_e, $\delta_v = \rho_v \cos(\phi_v)$ and $\delta_{v'} = \rho_{v'} \cos(\phi_{v'})$. Let (r, θ, x_3) denote the usual cylindrical coordinates

$$\begin{aligned} r &= \sqrt{{x_1}^2 + {x_2}^2}, \\ \theta &= \tan^{-1}(x_2/x_1) \end{aligned}$$

and Ω^e the edge neighbourhood (figure 3)

$$\Omega^e = \{x \in \Omega :\ \delta_v < x_3 < l_e - \delta_{v'},\ 0 < r < Z\}\,.$$

We introduce a set of modified system of coordinates in the edge neighbourhood Ω^e by

$$\begin{aligned} x_1^e &= \tau = \ln r, \\ x_2^e &= \theta, \\ x_3^e &= x_3. \end{aligned} \tag{2.4}$$

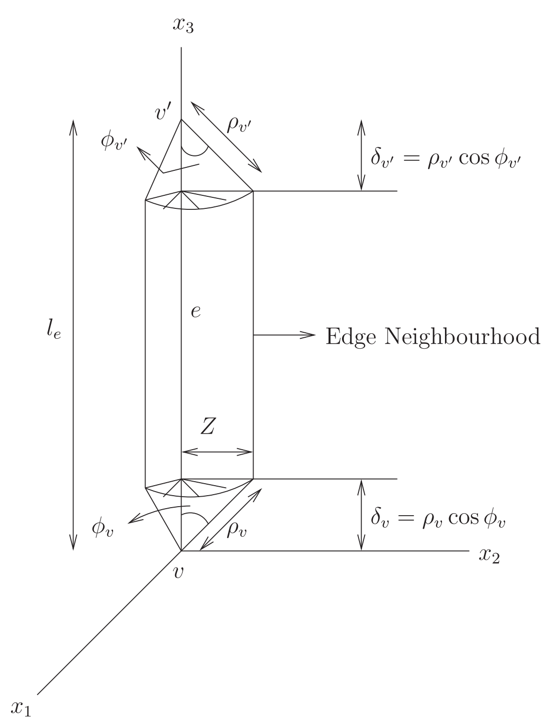

Figure 3. Edge neighbourhood Ω^e.

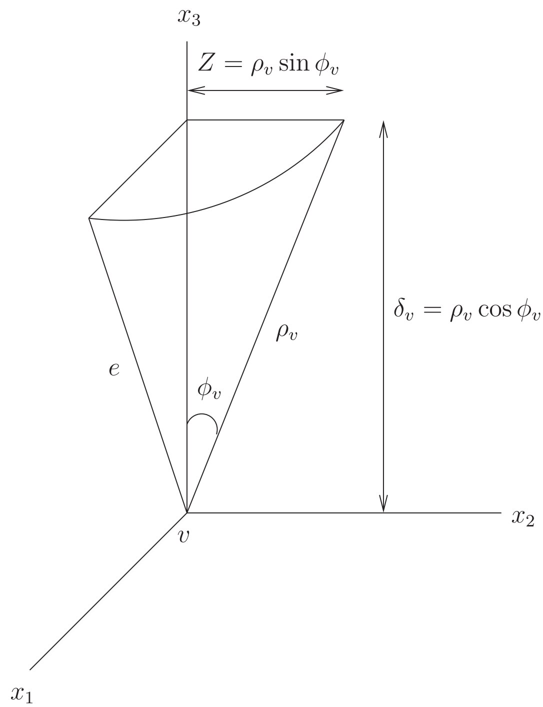

Figure 4. Vertex-edge neighbourhood Ω^{v-e}.

Next, we define the vertex-edge neighbourhood Ω^{v-e}, shown in figure 4, as follows

$$\Omega^{v-e} = \{x \in \Omega : \ 0 < \phi < \phi_v, \ 0 < x_3 < \delta_v = \rho_v \cos\phi_v\}.$$

We thus obtain a set of vertex-edge neighbourhoods Ω^{v-e}, where $v - e \in \mathcal{V} - \mathcal{E}$, the set of vertex-edge neighbourhoods.

Let us introduce a set of modified coordinates in the vertex-edge neighbourhood Ω^{v-e},

$$\begin{aligned} x_1^{v-e} &= \psi = \ln(\tan\phi), \\ x_2^{v-e} &= \theta, \\ x_3^{v-e} &= \zeta = \ln x_3. \end{aligned} \tag{2.5}$$

2.2 *Function spaces and dynamical weights associated with various neighbourhoods*

We need to review a set of function spaces and weight functions as described in [2, 21]. By $H^m(\Omega)$, we denote the usual Sobolev space of integer order $m \geq 0$ furnished with the norm

$$||w||^2_{H^m(\Omega)} = \sum_{|\alpha| \leq m} ||D^\alpha w||^2_{L^2(\Omega)},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $D^\alpha w = D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} D^{\alpha_3}_{x_3} w = w_{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}}$ is the distributional (weak) derivative of w. As usual $H^0(\Omega) = L^2(\Omega)$, $H^1_0(\Omega) =$

$\{w \in L^2(\Omega) : Dw \in L^2(\Omega), w = 0 \text{ on } \partial\Omega\}$. A seminorm on $H^m(\Omega)$ is given by

$$|w|^2_{H^m(\Omega)} = \sum_{|\alpha|=m} ||D^\alpha w||^2_{L^2(\Omega)}.$$

Let $\rho = \rho(x) = \text{dist}(x, v)$ for $x \in \Omega^v$ and $\beta_v \in (0, 1/2)$. We introduce a weight function

$$\mathbf{\Phi}^{\alpha,2}_{\beta_v}(x) = \begin{cases} \rho^{\beta_v+|\alpha|-2}, & \text{for } |\alpha| \geq 2 \\ \quad 1, & \text{for } |\alpha| < 2 \end{cases}$$

in the neighbourhood Ω^v as in (2.3) of [2]. Let

$$\mathbf{H}^{k,2}_{\beta_v}(\Omega^v) = \left\{ w |\ \|w\|^2_{\mathbf{H}^{k,2}_{\beta_v}(\Omega^v)} = \sum_{|\alpha|\leq k} \|\mathbf{\Phi}^{\alpha,2}_{\beta_v} D^\alpha w\|^2_{L^2(\Omega^v)} < \infty \right\}$$

and

$$\mathbf{B}^2_{\beta_v}(\Omega^v) = \{w | w \in \mathbf{H}^{k,2}_{\beta_v}(\Omega^v) \text{ for all } k \geq 2 \text{ and } \|\mathbf{\Phi}^{\alpha,2}_{\beta_v} D^\alpha w\|_{L^2(\Omega^v)} \leq C\, d^\alpha\, \alpha!\}$$

respectively, denote the weighted Sobolev space and countably normed space defined on Ω^v as in [2].

Let $r = r(x) = \text{dist}(x, e)$ for $x \in \Omega^e$ and $\beta_e \in (0, 1)$. Define a weight function

$$\mathbf{\Phi}^{\alpha,2}_{\beta_e}(x) = \begin{cases} r^{\beta_e+|\alpha'|-2}, & \text{for } |\alpha'| = \alpha_1 + \alpha_2 \geq 2 \\ 1, & \text{for } |\alpha| < 2 \end{cases}$$

in the neighbourhood Ω^e as in (2.1) of [2]. Let

$$\mathbf{H}^{k,2}_{\beta_e}(\Omega^e) = \left\{ w |\ \|w\|^2_{\mathbf{H}^{k,2}_{\beta_e}(\Omega^e)} = \sum_{|\alpha|\leq k} \|\mathbf{\Phi}^{\alpha,2}_{\beta_e} D^\alpha w\|^2_{L^2(\Omega^e)} < \infty \right\}$$

and

$$\mathbf{B}^2_{\beta_e}(\Omega^e) = \{w | w \in \mathbf{H}^{k,2}_{\beta_e}(\Omega^e) \text{ for all } k \geq 2 \text{ and} \|\mathbf{\Phi}^{\alpha,2}_{\beta_e} D^\alpha w\|_{L^2(\Omega^e)} \leq C\, d^\alpha\, \alpha!\}$$

respectively, denote the weighted Sobolev space and countably normed space defined on Ω^e as in [2].

Let $\rho = \rho(x)$ and $\phi = \phi(x)$ for $x \in \Omega^{v-e}$ and $\beta_{v-e} = (\beta_v, \beta_e)$, $\beta_v \in (0, 1/2)$, $\beta_e \in (0, 1)$. We introduce a weight function

$$\mathbf{\Phi}^{\alpha,2}_{\beta_{v-e}}(x) = \begin{cases} \rho^{\beta_v+|\alpha|-2}(\sin(\phi))^{\beta_e+|\alpha'|-2}, & \text{for } |\alpha'| = \alpha_1 + \alpha_2 \geq 2 \\ \rho^{\beta_v+|\alpha|-2}, & \text{for } |\alpha'| < 2 \leq |\alpha| \\ 1, & \text{for } |\alpha| < 2 \end{cases}$$

in the neighbourhood Ω^{v-e} as in (2.2) of [2]. Let

$$\mathbf{H}^{k,2}_{\beta_{v-e}}(\Omega^{v-e}) = \left\{ w |\ \|w\|^2_{\mathbf{H}^{k,2}_{\beta_{v-e}}(\Omega^{v-e})} = \sum_{|\alpha|\leq k} \|\mathbf{\Phi}^{\alpha,2}_{\beta_{v-e}} D^\alpha w\|^2_{L^2(\Omega^{v-e})} < \infty \right\}$$

and

$$\mathbf{B}^2_{\beta_{v-e}}(\Omega^{v-e}) = \Big\{ w \,|\, w \in \mathbf{H}^{k,2}_{\beta_e}(\Omega^{v-e}) \text{ for all } k \geq 2 \text{ and } \| \mathbf{\Phi}^{\alpha,2}_{\beta_{v-e}} D^\alpha w \|_{L^2(\Omega^{v-e})} \leq C\, d^\alpha\, \alpha! \Big\}$$

respectively, denote the weighted Sobolev space and countably normed space defined on Ω^{v-e} as in [2].

Let us recall that by $\mathbf{C}^2_{\beta_v}(\Omega^v)$ we denote a countably normed space as described in [2, 21] which is the set of functions $w(x) \in \mathbf{C}^0(\bar{\Omega}^v)$ such that for all α, $|\alpha| \geq 0$,

$$|D^\alpha_x(w(x) - w(v))| \leq C\, d^\alpha\, \alpha!\, \rho^{-(\beta_v + |\alpha| - 1/2)}(x). \tag{2.6}$$

Here, $\bar{\Omega}^v$ denotes the closure of Ω^v and $w(v)$ denotes the value of w at the vertex v.

Next, we recall that by $\mathbf{C}^2_{\beta_e}(\Omega^e)$, $\beta_e \in (0, 1)$ is denoted a countably normed space as described in [2, 21] which is the set of functions $w \in \mathbf{C}^0(\bar{\Omega}^e)$ such that for $|\alpha| \geq 0$,

$$\| r^{\beta_e + \alpha_1 + \alpha_2 - 1} D^\alpha_x\, (w(x) - w(0, 0, x_3)) \|_{\mathbf{C}^0(\bar{\Omega}^e)} \leq C\, d^\alpha\, \alpha! \tag{2.7}$$

and for $k \geq 0$

$$\left\| \frac{d^k}{(dx_3)^k} w(0, 0, x_3) \right\|_{\mathbf{C}^0(\bar{\Omega}^e \cap \{x:\, x_1 = x_2 = 0\})} \leq C\, d^k\, k!, \tag{2.8}$$

where $\bar{\Omega}^e$ denotes the closure of Ω^e.

Finally, by $\mathbf{C}^2_{\beta_{v-e}}(\Omega^{v-e})$ is denoted a countably normed space which is the set of functions $w(x) \in \mathbf{C}^0(\bar{\Omega}^{v-e})$ such that

$$\| \rho^{\beta_v + |\alpha| - 1/2}\, (\sin\phi)^{\beta_e + \alpha_1 + \alpha_2 - 1}\, D^\alpha_x\, (w(x) - w(0, 0, x_3)) \|_{\mathbf{C}^0(\bar{\Omega}^{v-e})} \leq C\, d^\alpha \alpha! \tag{2.9}$$

and

$$\left| |x_3|^{\beta_v + k - 1/2} \frac{d^k}{dx_3^k}\, (w(0, 0, x_3) - w(v)) \right|_{\mathbf{C}^0(\bar{\Omega}^{v-e} \cap \{x: x_1 = x_2 = 0\})} \leq C\, d^k\, k! \tag{2.10}$$

as described in [2, 21]. Here $\bar{\Omega}^{v-e}$ denotes the closure of Ω^{v-e}.

Unless otherwise stated, as in Babuška and Guo [2, 3], we let $w(x^v)$, $w(x^{v-e})$, $w(x^e)$ denote $w(x(x^v))$, $w(x(x^{v-e}))$, $w(x(x^e))$ respectively. The same notation is used for the spectral element functions $u(x^v)$, $u(x^{v-e})$, $u(x^e)$ etc. in the ensuing sections.

3. Differentiability estimates in modified coordinates

3.1 *Differentiability estimates in modified coordinates in vertex neighbourhoods*

Let $w_v = w(v)$, denote the value of w at the vertex v and let $\tilde{\Omega}^v$ denote the image of Ω^v in x^v coordinates. We can now state the differentiability estimates in these modified coordinates in vertex neighbourhoods. The proof is based on the regularity results proved by Babuška and Guo in [2]. These estimates are obtained when the differential operator is the Laplacian. However they are valid for the more general situation examined in this paper.

PROPOSITION 3.1

There exists a constant $\beta_v \in (0, 1/2)$ *such that for all* $0 < \nu \leq \rho_v$ *the estimate*

$$\int\limits_{\tilde{\Omega}^v \cap \{x^v:\, x_3^v \leq \ln(\nu)\}} \sum_{|\alpha| \leq m} e^{x_3^v} |D_{x_v}^{\alpha} (w(x^v) - w_v)|^2 \, dx^v \leq C (d^m \, m!)^2 \, \nu^{(1-2\beta_v)} \tag{3.1}$$

holds for all integers $m \geq 1$. *Here,* C *and* d *denote positive constants and* dx^v *denotes a volume element in* x^v *coordinates.*

Proof. By Theorem 3.19 of [2], for $\beta_v \in (0, 1/2)$, $\mathbf{H}_{\beta_v}^{2,2}(\Omega^v)$ is embedded in $\mathbf{C}^0(\bar{\Omega}^v)$ and

$$\|w\|_{\mathbf{C}^0(\bar{\Omega}^v)} \leq C \|w\|_{\mathbf{H}_{\beta_v}^{2,2}(\Omega^v)} .$$

Here, C denotes the positive constant and $\bar{\Omega}^v$ the closure of Ω^v. Hence, we can define $w_v = w(v)$, the value of w at the vertex v and

$$|w_v| \leq C \|w\|_{\mathbf{H}_{\beta_v}^{2,2}(\Omega^v)}.$$

Let $\mathcal{D}^\alpha w = w_{\phi^{\alpha_1} \theta^{\alpha_2} \rho^{\alpha_3}}$. Here, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha' = (\alpha_1, \alpha_2)$. Let

$$\mathcal{H}_{\beta_v}^{k,2}(\Omega^v) = \left\{ w | \ \|w\|^2_{\mathbf{H}_{\beta_v}^{k,2}(\Omega^v)} = \sum_{|\alpha| \leq k} \| \mathbf{\Phi}_{\beta_v}^{\alpha,2} \, \rho^{-|\alpha'|} \mathcal{D}^\alpha w \, \|^2_{L^2(\tilde{\Omega}^v)} < \infty \right\}$$

and

$$\mathcal{B}_{\beta_v}^2(\Omega^v) = \{ w | w \in \mathcal{H}_{\beta_v}^{k,2}(\Omega^v) \text{ for all } k \geq 2, \| \mathbf{\Phi}_{\beta_v}^{\alpha,2} \rho^{-|\alpha'|} \mathcal{D}^\alpha w \|_{L^2(\tilde{\Omega}^v)} \leq C d^\alpha \alpha! \}.$$

Then from Theorem 4.13 of [2], we have that $w \in \mathcal{B}_{\beta_v}^2(\Omega^v)$ if and only if $w \in \mathbf{B}_{\beta_v}^2(\Omega^v)$.

Hence,

$$\sum_{2 \leq |\alpha| \leq m} \int\limits_{\tilde{\Omega}^v} |\rho^{\beta_v - 2} \rho^{\alpha_3} w_{\phi^{\alpha_1} \theta^{\alpha_2} \rho^{\alpha_3}}|^2 \rho^2 \, \sin\phi \, d\rho \, d\phi \, d\theta \leq (C d^m m!)^2. \tag{3.2}$$

Define $\chi = \ln \rho$. Then

$$\frac{\partial}{\partial \chi} = \rho \frac{\partial}{\partial \rho} \quad \text{and} \quad \frac{d\rho}{\rho} = d\chi \, .$$

Now using (3.2) it can be shown that

$$\sum_{2 \leq |\alpha| \leq m} \int\limits_{\tilde{\Omega}^v} \mathrm{e}^{(2\beta_v - 1)\chi} \left| w_{\phi^{\alpha_1} \theta^{\alpha_2} \chi^{\alpha_3}} \right|^2 \, d\chi \, d\phi \, d\theta \leq (C d^m m!)^2.$$

Here, C and d denote generic constants. Hence,

$$\sum_{2\leq|\alpha|\leq m} \int\limits_{\tilde{\Omega}^v\cap\{x^v:\,\chi\leq\ln\nu\}} |D^{\alpha}_{x^v} w|^2 dx^v \leq (Cd^m m!)^2\, \nu^{1-2\beta_v}. \tag{3.3}$$

We now obtain estimates for $0 \leq |\alpha| \leq 1$. By Lemma 5.5 of [2], since $w \in \mathbf{H}^{2,2}_{\beta_v}(\Omega^v)$, the estimate

$$\int\limits_{\Omega^v} \rho^{2(\beta_v-2)}\, |w - w(v)|^2 dx \leq C\|u\|^2_{\mathbf{H}^{2,2}_{\beta_v}(\Omega^v)}$$

holds. Hence,

$$\int\limits_{\tilde{\Omega}^v} \mathrm{e}^{(2\beta_v-1)\chi}\, |w - w_v|^2 d\chi\, d\phi\, d\theta \leq (Cd^m m!)^2.$$

Thus, we conclude that

$$\int\limits_{\tilde{\Omega}^v\cap\{x^v:\,\chi\leq\ln\nu\}} |w - w_v|^2 dx^v \leq C(d^m m!)^2 \nu^{1-2\beta_v}. \tag{3.4}$$

We know by Theorem 5.6 of [2] that $\mathbf{B}^2_{\beta_v}(\Omega^v) \subseteq \mathbf{C}^2_{\beta_v}(\Omega^v)$. Now,

$$\rho\nabla_x w = Q^v \nabla_{x_v} w, \quad \text{where } Q^v = O^v P^v. \tag{3.5}$$

Here, O^v is the orthogonal matrix

$$O^v = \begin{bmatrix} \cos\phi\cos\theta & -\sin\theta & \sin\phi\cos\theta \\ \cos\phi\sin\theta & \cos\theta & \sin\phi\sin\theta \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

and

$$P^v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sin\phi} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, in Ω^v

$$\phi^v < \phi < \pi - \phi^v.$$

Hence, from (2.6) and (3.5) we can conclude that

$$|\nabla_{x^v} w| \leq C\rho^{-\beta_v+1/2}. \tag{3.6}$$

Using (3.6), the estimate

$$\int\limits_{\tilde{\Omega}^v\cap\{x^v:\,\chi\leq\ln\nu\}} \sum_{|\alpha|=1} |D^{\alpha}_{x^v} w|^2 dx^v \leq C\int_{-\infty}^{\ln\nu} \mathrm{e}^{(-2\beta_v+1)\chi} d\chi \leq C\nu^{1-2\beta_v} \tag{3.7}$$

follows. Combining (3.3), (3.4) and (3.7) we obtain the result. □

3.2 *Differentiability estimates in modified coordinates in edge neighbourhoods*

Let $\tilde{\Omega}^e$ denote the image of Ω^e in x^e coordinates. The differentiability estimates for the solution w in edge neighbourhoods in these modified coordinates can now be stated.

PROPOSITION 3.2

Let $s(x_3) = w\,(x_1, x_2, x_3)\,|_{(x_1=0, x_2=0)}$. *Then*

$$\int_{\delta_v}^{l_e-\delta_{v'}} \sum_{k\le m} |D^k_{x^e_3}\, s(x^e_3)|^2 dx^e_3 \le C(d^m m!)^2 \tag{3.8}$$

for all integers $m \ge 1$.

Moreover, there exists a constant $\beta_e \in (0,1)$ *such that for* $\mu \le Z$,

$$\int\limits_{\tilde{\Omega}^e \cap \{x^e :\, x^e_1 < \ln \mu\}} \sum_{|\alpha|\le m} |D^\alpha_{x^e}(w(x^e) - s(x^e_3))|^2 dx^e \le\ C(d^m m!)^2 \mu^{2(1-\beta_e)} \tag{3.9}$$

for all integers $m \ge 1$. *Here, C and d denote positive constants and* dx^e *denotes a volume element in* x^e *coordinates.*

Proof. By Theorem 5.3 of [2], $\mathbf{B}^2_{\beta_e}(\Omega^e) \subseteq \mathbf{C}^2_{\beta_e}(\Omega^e)$. Define $s(x_3) = w(x_1, x_2, x_3)|_{(x_1=0, x_2=0)}$. Then (3.8) follows immediately from (2.8) since $w \in \mathbf{B}^2_{\beta_e}(\Omega^e)$ and hence $w \in \mathbf{C}^2_{\beta_e}(\Omega^e)$. Let

$$p(x) = w(x) - s(x_3).$$

Then from (2.7), we have that

$$\|r^{\beta_e+\alpha_1+\alpha_2-1} D^\alpha_x\, p(x)\|_{\mathbf{C}^0(\bar{\Omega}^e)} \le C d^\alpha \alpha!\,. \tag{3.10}$$

Now we can show just as in Theorem 4.1 of [2] that

$$\|r^{\beta_e-1} D^\alpha_{x^e}\, p(x^e)\|_{\mathbf{C}^0(\tilde{\Omega}^e)} \le C d^\alpha \alpha! \tag{3.11}$$

using the estimate (3.10). Hence,

$$\left|D^\alpha_{x^e}\, p(x^e)\right| \le C d^\alpha\, \alpha!\, e^{(1-\beta_e)x^e_1}$$

for $x^e \in \tilde{\Omega}^e$. Using the above we conclude that

$$\int\limits_{\tilde{\Omega}^e \cap \{x^e : x^e_1 \le \ln \mu\}} \sum_{|\alpha|\le m} \left|D^\alpha_{x^e}\, p(x^e)\right|^2 dx^e \le C\,(d^m\, m!)^2 \int_{-\infty}^{\ln \mu} \mathrm{e}^{2(1-\beta_e)\tau}\, d\tau \qquad \square$$

and this gives the required estimate (3.9).

3.3 *Differentiability estimates in modified coordinates in vertex-edge neighbourhoods*

Let $\tilde{\Omega}^{v-e}$ denote the image of Ω^{v-e} in x^{v-e} coordinates. We can now state the differentiability estimates in modified coordinates in vertex-edge neighbourhoods.

PROPOSITION 3.3

Let $w_v = w(v)$, the value of w evaluated at the vertex v, and $s(x_3) = w(x_1, x_2, x_3)|_{(x_1=0,x_2=0)}$. Then there exists a constant $\beta_v \in (0, 1/2)$ such that for any $0 < \nu \leq \delta_v$,

$$\int_{-\infty}^{\ln \nu} \mathrm{e}^{x_3^{v-e}} \sum_{k\leq m} |D^k_{x_3^{v-e}}(s(x_3^{v-e})-w_v)|^2 dx_3^{v-e} \leq C(d^m\, m!)^2\, \nu^{(1-2\beta_v)}. \tag{3.12}$$

Moreover, there exists a constant $\beta_e \in (0, 1)$ such that for any $0 < \alpha \leq \tan\phi_v$ and $0 < \nu \leq \delta_v$,

$$\int_{\tilde{\Omega}^{v-e}\cap\{x^{v-e}:\, x_1^{v-e}<\ln\alpha,\ x_3^{v-e}<\ln\nu\}} \mathrm{e}^{x_3^{v-e}} \sum_{|\gamma|\leq m} \left|D^\gamma_{x^{v-e}}(w(x^{v-e})-s(x_3^{v-e}))\right|^2 dx^{v-e}$$
$$\leq C(d^m\, m!)^2\, \alpha^{2(1-\beta_e)}\, \nu^{(1-2\beta_v)} \tag{3.13}$$

for all integers $m \geq 1$. Here, C and d denote positive constants and dx^{v-e} denotes a volume element in x^{v-e} coordinates.

Proof. By Theorem 5.9 of [2], $\mathbf{B}^2_{\beta_{v-e}}(\Omega^{v-e}) \subseteq \mathbf{C}^2_{\beta_{v-e}}(\Omega^{v-e})$. Since $w \in \mathbf{B}^2_{\beta_{v-e}}(\Omega^{v-e})$, we conclude that $w \in \mathbf{C}^2_{\beta_{v-e}}(\Omega^{v-e})$. Let $s(x_3) = w(0, 0, x_3)$ and $w_v = w(v)$. Then

$$\left| |x_3|^{\beta_v+k-1/2} \frac{d^k}{dx_3^k}(s(x_3) - w_v) \right| \leq Cd^k k!.$$

Now $x_3^{v-e} = \ln x_3$. Hence, it can be shown as before that

$$\int_{-\infty}^{\ln\nu} \sum_{k\leq m} |D^k_{x_3^{v-e}}(s(x_3^{v-e}) - w_v)|^2\, \mathrm{d}x_3^{v-e} \leq C(d^m\, m!)^2\, \nu^{(1-2\beta_v)}.$$

Let $p(x) = w(x) - s(x_3)$. Then by (2.9) we have that

$$\|\rho^{\beta_v+|\alpha|-1/2}(\sin\phi)^{\beta_e+\alpha_1+\alpha_2-1} D^\alpha_x p(x)\|_{\mathbf{C}^0(\bar{\Omega}^{v-e})} \leq Cd^\alpha\, \alpha!\,.$$

It can be shown as in Theorem 4.8 of [2] that

$$\|(\rho^{\beta_v-1/2}(\sin\phi)^{\beta_e-1})\, \rho^{\alpha_3}(\sin\phi)^{\alpha_1}\, p_{\phi^{\alpha_1}\theta^{\alpha_2}\rho^{\alpha_3}}\|_{\mathbf{C}^0(\breve{\Omega}^{v-e})} \leq Cd^\alpha \alpha!\,.$$

Here, $\breve{\Omega}^{v-e}$ is the image of $\bar{\Omega}^{v-e}$ in (ϕ, θ, ρ) coordinates. From the above, the estimate

$$\|\mathrm{e}^{(\beta_v-1/2)\chi}\, (\sin\phi)^{\beta_e-1}\, (\sin\phi)^{\alpha_1}\, p_{\phi^{\alpha_1}\theta^{\alpha_2}\chi^{\alpha_3}}\|_{\mathbf{C}^0(\widehat{\Omega}^{v-e})} \leq Cd^\alpha \alpha! \tag{3.14}$$

follows. Here, $\widehat{\Omega}^{v-e}$ is the image of $\bar{\Omega}^{v-e}$ in x^v coordinates, $x^v = (\phi, \theta, \chi)$ and $\chi = \ln\rho$. Now

$$\nabla_{x^v} w = J^{v-e}\nabla_{x^{v-e}} w,$$

where

$$J^{v-e} = \begin{bmatrix} \sec^2\phi\cot\phi & 0 & -\tan\phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore,

$$\nabla_{x^{v-e}} u = (J^{v-e})^{-1} \nabla_{x^v} u.$$

Here,

$$(J^{v-e})^{-1} = \begin{bmatrix} \cos\phi \sin\phi & 0 & \sin^2\phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence,

$$\frac{\partial u}{\partial \psi} = \cos\phi \sin\phi \frac{\partial u}{\partial \phi} + \sin^2\phi \frac{\partial u}{\partial \chi}, \quad \frac{\partial u}{\partial \zeta} = \frac{\partial u}{\partial \chi}.$$

From the above, we obtain

$$\frac{\partial^m u}{\partial \psi^m} = \sum_{k=1}^{m} \sum_{\alpha_1+\alpha_2=k} \left(\sum_{j_1+j_2=2m-\alpha_1} a^m_{\alpha_1,\alpha_2,j_1,j_2} (\cos\phi)^{j_1} (\sin\phi)^{j_2} ((\sin\phi)^{\alpha_1}) u_{\phi^{\alpha_1}\chi^{\alpha_2}} \right). \quad (3.15)$$

It can be shown that the coefficients $a^m_{\alpha_1,\alpha_2,j_1,j_2}$ satisfy the recurrence relation

$$\begin{aligned} a^{m+1}_{\alpha_1,\alpha_2,j_1,j_2} &= a^m_{\alpha_1-1,\alpha_2,j_1-1,j_2} + (\alpha_1 + j_2) a^m_{\alpha_1,\alpha_2,j_1-2,j_2} \\ &\quad - j_1 a^m_{\alpha_1,\alpha_2,j_1,j_2-2} + a^m_{\alpha_1,\alpha_2-1,j_1,j_2-2} \end{aligned} \quad (3.16)$$

for $|\alpha| \le m$. For $|\alpha| = m+1$,

$$a^{m+1}_{\alpha_1,\alpha_2,j_1,j_2} = \begin{cases} 1, & \text{if } j_1 = \alpha_1,\ j_2 = 2m+2-\alpha_1 \\ 0, & \text{otherwise}. \end{cases} \quad (3.17)$$

Since $0 \le \phi \le \phi_v$, where $\phi_v < \pi/2$, we can conclude from (3.14) that

$$\| e^{(\beta_v - 1/2)\zeta}\ e^{(\beta_e - 1)\psi}\ (\sin\phi)^{\alpha_1}\ p_{\phi^{\alpha_1}\theta^{\alpha_2}\chi^{\alpha_3}} \|_{\mathbf{C}^0_{(\tilde{\Omega}^{v-e})}} \le C\, d^\alpha\, \alpha!. \quad (3.18)$$

Here, $\tilde{\Omega}^{v-e}$ denotes the image of $\bar{\Omega}^{v-e}$ in x^{v-e} coordinates. From (3.18), the estimate

$$\| e^{(\beta_v - 1/2)x_3^{v-e}}\ e^{(\beta_e - 1)x_1^{v-e}}\ D^\alpha_{x^{v-e}}\ p \|_{\mathbf{C}^0_{(\tilde{\Omega}^{v-e})}} \le C\, d^\alpha\, \alpha! \quad (3.19)$$

follows.

As in [2], we show (3.19) for the cases $\alpha = (m, 0, 0)$, $\alpha = (0, m, 0)$ and $\alpha = (0, 0, m)$ since the general case can be shown in the same way. It is enough to prove (3.19) for $\alpha = (m, 0, 0)$ since the other two cases are trivial. Let

$$A^m_k = \sum_{\alpha_1+\alpha_2=k} \sum_{j_1+j_2=2m-\alpha_1} |a^m_{\alpha_1,\alpha_2,j_1,j_2}|. \quad (3.20)$$

Then

$$A^m_m \le 4^m. \quad (3.21)$$

Moreover, for $k < m$,

$$A_k^m \leq 4^m \frac{m!}{k!}. \tag{3.22}$$

The proof is by induction. Using the recurrence relation (3.16), we obtain

$$\begin{aligned} A_k^{m+1} &\leq 2mA_k^m + 2A_{k-1}^m \\ &\leq 2m\left(\frac{4^m m!}{k!}\right) + 2\left(\frac{4^m m!}{(k-1)!}\right) \\ &\leq \frac{4^{m+1}}{k!}(m+1)!. \end{aligned} \tag{3.23}$$

Now using (3.14), (3.15), (3.21) and (3.22) it can be shown that

$$\begin{aligned} &\left\| e^{(\beta_v - 1/2)x_3^{v-e}} e^{(\beta_e - 1)x_1^{v-e}} \frac{\partial^m p}{\partial \psi^m} \right\|_{\mathbf{C}^0(\tilde{\Omega}^{v-e})} \\ &\quad \leq \sum_{k=1}^m \sum_{\alpha_1+\alpha_2=k} \sum_{j_1+j_2=2m-\alpha_1} |a^m_{\alpha_1,\alpha_2,j_1,j_2}| \, Cd^{\alpha_1+\alpha_2} \, \alpha_1!\alpha_2! \\ &\quad \leq \sum_{k=1}^m A_k^m (Cd^k \, k!) \\ &\quad \leq Cd^m m!, \end{aligned}$$

here C and d denote generic constants. The inequality (3.19) is obtained in the same way. Now the estimate (3.13) follows immediately from (3.19). □

3.4 *Differentiability estimates in standard coordinates in the regular region of the polyhedron*

Let Ω^r denote the portion of the polyhedron Ω obtained after the closure of the vertex-neighbourhoods, edge neighbourhoods and vertex-edge neighbourhoods that have been removed from it. Thus, let

$$\Delta = \left\{ \bigcup_{v \in \mathcal{V}} \bar{\Omega}^v \right\} \bigcup \left\{ \bigcup_{e \in \mathcal{E}} \bar{\Omega}^e \right\} \bigcup \left\{ \bigcup_{v-e \in \mathcal{V}-\mathcal{E}} \bar{\Omega}^{v-e} \right\}.$$

Then

$$\Omega^r = \Omega \setminus \Delta.$$

The solution w is analytic in Ω^r and we denote it as the regular region of the polyhedron. In Ω^r, the standard coordinate system $x = (x_1, x_2, x_3)$ is retained. The differentiability estimates in these coordinates in the regular region of the polyhedron are now stated.

PROPOSITION 3.4

The estimate

$$\int_{\Omega^r} \sum_{|\alpha|\leq m} |D_x^{\alpha} w(x)|^2 dx \leq C(d^m\, m!)^2 \tag{3.24}$$

holds for all integers $m \geq 1$. Here, C and d denote positive constants and dx denotes a volume element in x coordinates.

Proof. Now $w(x)$ is analytic in an open neighbourhood of $\bar{\Omega}^r$. Hence, (3.24) follows. □

4. The stability theorem

In §2, we had partitioned the domain Ω into a regular region Ω^r, a set of vertex neighbourhoods Ω^v, where $v \in \mathcal{V}$, a set of edge neighbourhoods Ω^e, where $e \in \mathcal{E}$ and a set of vertex-edge neighbourhoods Ω^{v-e}, where $v - e \in \mathcal{V} - \mathcal{E}$. In the regular region Ω^r standard coordinates $x = (x_1, x_2, x_3)$ are used and in the remaining regions modified coordinates are used (which are introduced in Section 2). We now divide Ω^r into a set of curvilinear hexahedrons, tetrahedrons and prisms. We impose a geometrically graded mesh in the remaining regions consisting of hexahedrons and prisms which is described in this section. We remark that a tetrahedron (figure 5) can always be divided into four hexahedrons (see [28, 29]), in the same way that a triangle can be divided into three rectangles by joining the centre of the triangle to the midpoints of the sides. Moreover a prism can be divided into three hexahedral elements. Hence we can choose all our elements to be hexahedrons.

A set of spectral element functions are defined on the elements. In edge neighbourhoods and vertex-edge neighbourhoods these spectral element functions are a sum of

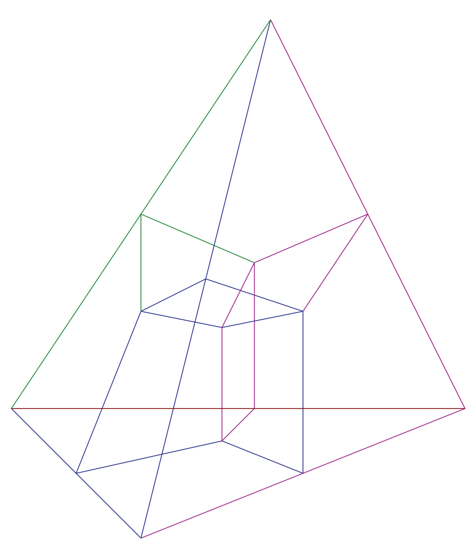

Figure 5. Decomposition of a tetrahedron into four hexahedrons.

tensor products of polynomials in the modified coordinates. Let $\{\mathcal{F}_u\}$ denote the spectral element representation of the function u. We shall examine two cases. The first case is when the spectral element functions are nonconforming. The second case is when the spectral element functions are conforming on the wirebasket WB of the elements, i.e. the union of the edges and vertices of the elements. In both these cases the spectral element functions are nonconforming on the faces (open) of the elements.

To state the stability theorem we need to define some quadratic forms. Let N denote the number of refinements in the geometrical mesh and W denote an upper bound on the degree of the polynomial representation of the spectral element functions. We shall define two quadratic forms $\mathcal{V}^{N,W}(\{\mathcal{F}_u\})$ and $\mathcal{U}^{N,W}(\{\mathcal{F}_u\})$.

Now

$$\begin{aligned}\mathcal{V}^{N,W}(\{\mathcal{F}_u\}) \;=\; & \mathcal{V}^{N,W}_{\text{regular}}(\{\mathcal{F}_u\}) + \mathcal{V}^{N,W}_{\text{vertices}}(\{\mathcal{F}_u\}) + \mathcal{V}^{N,W}_{\text{vertex-edges}}(\{\mathcal{F}_u\}) \\ & + \mathcal{V}^{N,W}_{\text{edges}}(\{\mathcal{F}_u\}). \end{aligned} \tag{4.1}$$

In the same way,

$$\begin{aligned}\mathcal{U}^{N,W}(\{\mathcal{F}_u\}) \;=\; & \mathcal{U}^{N,W}_{\text{regular}}(\{\mathcal{F}_u\}) + \mathcal{U}^{N,W}_{\text{vertices}}(\{\mathcal{F}_u\}) + \mathcal{U}^{N,W}_{\text{vertex-edges}}(\{\mathcal{F}_u\}) \\ & + \mathcal{U}^{N,W}_{\text{edges}}(\{\mathcal{F}_u\}). \end{aligned} \tag{4.2}$$

Let us first consider the regular region Ω^r of Ω and define the two quadratic forms $\mathcal{V}^{N,W}_{\text{regular}}(\{\mathcal{F}_u\})$ and $\mathcal{U}^{N,W}_{\text{regular}}(\{\mathcal{F}_u\})$. The regular region Ω^r is divided into N_r curvilinear hexahedrons, tetrahedrons and prisms. Let Ω^r_l be one of the elements into which Ω^r is divided, which we shall assume is a curvilinear hexahedron to keep the exposition simple. Let Q denote the standard cube $Q = (-1, 1)^3$. Then there is an analytic map M^r_l from Q to Ω^r_l which has an analytic inverse. Let Ω^r_l be as shown in figure 6 and let $\{\Gamma^r_{l,i}\}_{1\le i\le n^r_l}$ denote its faces.

Now the map M^r_l is of the form

$$x = M^r_l(\lambda_1, \lambda_2, \lambda_3),$$

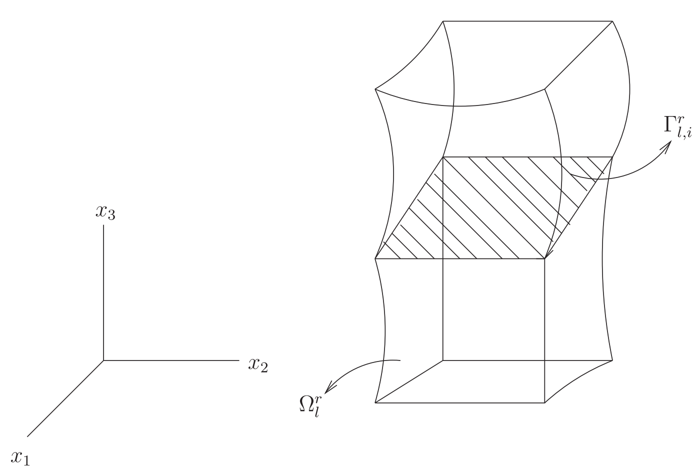

Figure 6. Elements in Ω^r.

where $(\lambda_1, \lambda_2, \lambda_3) \in Q$, the master cube. Define the spectral element function u_l^r on Ω_l^r by

$$u_l^r(\lambda) = \sum_{i=0}^{W}\sum_{j=0}^{W}\sum_{k=0}^{W} \alpha_{i,j,k}\, \lambda_1^i \lambda_2^j \lambda_3^k.$$

Now the spectral element functions are nonconforming in the general case. Let $[u]|_{\Gamma_{l,i}^r}$ denote the jump in u across the face $\Gamma_{l,i}^r$. Let the face $\Gamma_{l,i}^r = \Gamma_{m,j}^r$ where $\Gamma_{m,j}^r$ is a face of the element Ω_m^r. We may assume that the face $\Gamma_{l,i}^r$ corresponds to $\lambda_3 = 1$ and $\Gamma_{m,j}^r$ corresponds to $\lambda_3 = -1$. Then $[u]|_{\Gamma_{l,i}^r}$ is a function of only λ_1 and λ_2.

We now define

$$\begin{aligned} \mathcal{V}_{\text{regular}}^{N,W}(\{\mathcal{F}_u\}) \;=\; & \sum_{l=1}^{N_r} \int_{\Omega_l^r} |Lu_l^r(x)|^2\, dx \\ & + \sum_{\Gamma_{l,i}^r \subseteq \bar{\Omega}^r \setminus \partial\Omega} \left(\|[u]\|_{0,\Gamma_{l,i}^r}^2 + \sum_{k=1}^{3} \|[u_{x_k}]\|_{1/2,\Gamma_{l,i}^r}^2 \right) \\ & + \sum_{\Gamma_{l,i}^r \subseteq \Gamma^{[0]}} \|u_l^r\|_{3/2,\Gamma_{l,i}^r}^2 + \sum_{\Gamma_{l,i}^r \subseteq \Gamma^{[1]}} \left\| \left(\frac{\partial u_l^r}{\partial \nu} \right)_A \right\|_{1/2,\Gamma_{l,i}^r}^2 . \end{aligned} \tag{4.3}$$

The fractional Sobolev norms used above are as defined in [20].

Since $\Gamma_{l,i}^r$, corresponding to $\lambda_3 = 1$, is the image of $S = (-1,1)^2$, or T the master triangle, in λ_1, λ_2 coordinates,

$$\|w\|_{\sigma,\Gamma_{l,i}^r}^2 = \|w\|_{0,E}^2 + \int_E \int_E \frac{\left(w(\lambda_1,\lambda_2) - w(\lambda_1',\lambda_2')\right)^2}{\left((\lambda_1-\lambda_1')^2 + (\lambda_2-\lambda_2')^2\right)^{1+\sigma}} d\lambda_1\, d\lambda_2\, d\lambda_1'\, d\lambda_2' \tag{4.4a}$$

for $0 < \sigma < 1$. Here, E denote either S or T. However, if E is S, then we prefer to use the equivalent norm [26]

$$\begin{aligned} \|w\|_{\sigma,\Gamma_{l,i}^r}^2 \;=\; & \|w\|_{0,E}^2 + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{(w(\lambda_1,\lambda_2) - w(\lambda_1',\lambda_2))^2}{(\lambda_1-\lambda_1')^{1+2\sigma}}\, d\lambda_1 d\lambda_1' d\lambda_2 \\ & + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{(w(\lambda_1,\lambda_2) - w(\lambda_1,\lambda_2'))^2}{(\lambda_2-\lambda_2')^{1+2\sigma}}\, d\lambda_2 d\lambda_2' d\lambda_1. \end{aligned} \tag{4.4b}$$

Moreover,

$$\|w\|_{1+\sigma,\Gamma_{l,i}^r}^2 = \|w\|_{0,E}^2 + \sum_{i=1}^{2} \left\| \frac{\partial w}{\partial \lambda_i} \right\|_{\sigma,E}^2 . \tag{4.5}$$

Next, we define

$$\mathcal{U}_{\text{regular}}^{N,W}(\{\mathcal{F}_u\}) = \sum_{l=1}^{N_r} \int_{Q=(M_l^r)^{-1}(\Omega_l^r)} \sum_{|\alpha|\le 2} \left| D_\lambda^\alpha u_l^r \right|^2 d\lambda. \tag{4.6}$$

Let v be one of the vertices of Ω. In figure 7, the vertex neighbourhood, described in §2, is shown. Let S^v denote the intersection of the surface of the sphere $B_{\rho_v}(v)$ with $\bar{\Omega}^v$, i.e.

$$S^v = \left\{x \in \bar{\Omega}^v : \operatorname{dist}(x, v) = \rho_v\right\}.$$

We divide the surface S^v into a set of triangular and quadrilateral elements. Let S_j^v denote these elements, where $1 \leq j \leq I_v$. Here, I_v denotes a fixed constant. Let μ_v be a positive constant less than one which shall be used to define a geometric mesh (as in figure 8) in the vertex neighbourhood Ω^v of the vertex v. We now divide Ω^v into $N_v = I_v(N+1)$ curvilinear hexahedrons and prisms $\{\Omega_l^v\}_{1 \leq l \leq N_v}$, where Ω_l^v is of the form

$$\Omega_l^v = \{x : (\phi, \theta) \in S_j^v,\ \rho_k^v < \rho < \rho_{k+1}^v\}$$

for $1 \leq j \leq I_v$ and $0 \leq k \leq N$. Here, $\rho_k^v = \rho_v(\mu_v)^{N+1-k}$ and $0 < \mu_v < 1$ for $1 \leq k \leq N+1$. Moreover, $\rho_0^v = 0$.

Let $\tilde{\Omega}^v$ denote the image of Ω^v in x^v coordinates (introduced in §2) and $\tilde{\Omega}_l^v$ denote the image of the element Ω_l^v. Then the geometric mesh $\{\Omega_l^v\}_{1 \leq l \leq N_v}$ which has been defined on Ω^v, is mapped to a quasi-uniform mesh $\{\tilde{\Omega}_l^v\}_{1 \leq l \leq N_v}$ on $\tilde{\Omega}^v$, except that the corner elements

$$\Omega_l^v = \{x : (\phi, \theta) \in S_j^v,\ 0 < \rho < \rho_1^v\}$$

are mapped to semi-infinite elements

$$\tilde{\Omega}_l^v = \{x^v : (\phi, \theta) \in S_j^v,\ -\infty < \chi < \ln \rho_1^v\}.$$

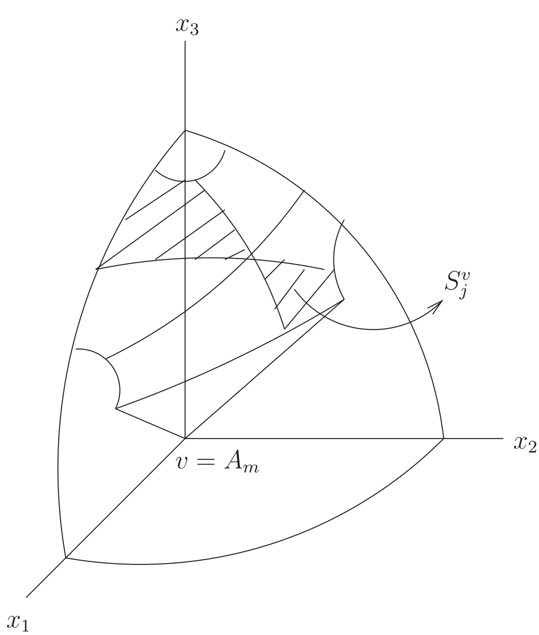

Figure 7. Mesh imposed on the spherical boundary S^v.

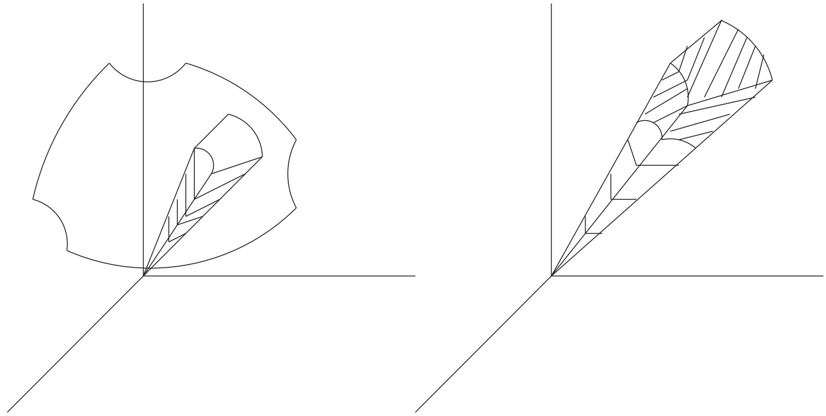

Figure 8. Geometrical mesh imposed on Ω^v.

We now specify the form of the spectral element functions $u_l^v(x^v)$ on the elements. Consider first the case when $\tilde{\Omega}_l^v$ is a corner element of the form

$$\tilde{\Omega}_l^v = \{x^v : (\phi, \theta) \in S_j^v, \ -\infty < \chi < \ln \rho_1^v\}.$$

In this case, we define $u_l^v(x^v) = h^v$, where h^v is a constant. Thus, at all corner elements the spectral element functions assume the same constant value for that corner.

Now there is an analytic map M_l^v from Q, the master cube to $\tilde{\Omega}_l^v$, which has an analytic inverse. Here, the map M_l^v is of the form

$$x^v = M_l^v(\lambda_1, \lambda_2, \lambda_3).$$

We define the spectral element function u_l^v on $\tilde{\Omega}_l^v$ by

$$u_l^v(\lambda) = \sum_{t=0}^{W_l} \sum_{s=0}^{W_l} \sum_{r=0}^{W_l} \beta_{r,s,t}\, \lambda_1^r \lambda_2^s \lambda_3^t.$$

Here, $1 \leq W_l \leq W$. Moreover, as in [21], $W_l = [\mu_1 i]$ for $1 \leq i \leq N$, where $\mu_1 > 0$ is a degree factor. Hereafter $[a]$ denotes the greatest positive integer $\leq a$.

Let $v \in \mathcal{V}$ denote the vertices of Ω. Define

$$\mathcal{V}^{N,W}_{\text{vertices}}(\{\mathcal{F}_u\}) = \sum_{v \in \mathcal{V}} \mathcal{V}_v^{N,W}(\{\mathcal{F}_u\}) \tag{4.7}$$

and

$$\mathcal{U}^{N,W}_{\text{vertices}}(\{\mathcal{F}_u\}) = \sum_{v \in \mathcal{V}} \mathcal{U}_v^{N,W}(\{\mathcal{F}_u\}). \tag{4.8}$$

We now fix a vertex v and define the quadratic forms $\mathcal{V}_v^{N,W}(\{\mathcal{F}_u\})$ and $\mathcal{U}_v^{N,W}(\{\mathcal{F}_u\})$. Consider the vertex neighbourhood Ω^v and let Ω_l^v be one of the elements into which it is divided. Now Ω_l^v has n_l^v faces $\{\Gamma_{l,i}^v\}_{1 \leq i \leq n_l^v}$. Let $\tilde{\Omega}_l^v$ be the image of Ω_l^v and $\tilde{\Gamma}_{l,i}^v$ be the image of $\Gamma_{l,i}^v$ in x^v coordinates.

Define $L^v u(x^v)$ so that

$$\int_{\tilde{\Omega}_l^v} |L^v u(x^v)|^2 dx^v = \int_{\Omega_l^v} \rho^2 \, |Lu(x)|^2 dx. \tag{4.9}$$

Here, dx^v denote a volume element in x^v coordinates and dx a volume element in x coordinates. In Chapter 3 of [1], it is shown that

$$L^v u(x^v) = -\mathrm{div}_{x^v}(\mathrm{e}^{\chi/2}\sqrt{\sin\phi}A^v \nabla_{x^v} u) + \sum_{i=1}^{3} \hat{b}_i^v u_{x_i^v} + \hat{c}^v u. \tag{4.10}$$

In the above, A^v is a symmetric, positive definite matrix.

Let $\Gamma_{l,i}^v$ be one of the faces of Ω_l^v and $\tilde{\Gamma}_{l,i}^v$ denote its image in x^v coordinates. Let $\tilde{P}$ be a point belonging to $\tilde{\Gamma}_{l,i}^v$ and $\boldsymbol{\nu}^v$ be the unit normal to $\tilde{\Gamma}_{l,i}^v$ at the point $\tilde{P}$. Then define

$$\left(\frac{\partial u}{\partial \boldsymbol{\nu}^v}\right)_{A^v}(\tilde{P}) = (\boldsymbol{\nu}^v)^T A^v \nabla_{x^v} u. \tag{4.11}$$

Here, the matrix A^v is as in (4.10). Let

$$R_{l,i}^v = \sup_{x^v \in \tilde{\Gamma}_{l,i}^v} (\mathrm{e}^{x_3^v}).$$

We now define

$$\begin{aligned}
\mathcal{V}_v^{N,W}(\{\mathcal{F}_u\}) \;=\; & \sum_{l=1,\mu(\tilde{\Omega}_l^v)<\infty}^{N_v} \int_{\tilde{\Omega}_l^v} |L^v u_l^v(x^v)|^2 \, dx^v \\
& + \sum_{\Gamma_{l,i}^v \subseteq \bar{\Omega}^v \setminus \partial\Omega, \; \mu(\tilde{\Gamma}_{l,i}^v)<\infty} \left(\left\| \sqrt{R_{l,i}^v}[u] \right\|_{0,\tilde{\Gamma}_{l,i}^v}^2 \right. \\
& \left. + \sum_{k=1}^{3} \left\| \sqrt{R_{l,i}^v}[u_{x_k^v}] \right\|_{1/2,\tilde{\Gamma}_{l,i}^v}^2 \right) \\
& + \sum_{\Gamma_{l,i}^v \subseteq \Gamma^{[0]}, \; \mu(\tilde{\Gamma}_{l,i}^v)<\infty} \left\| \sqrt{R_{l,i}^v} u_l^v \right\|_{3/2,\tilde{\Gamma}_{l,i}^v}^2 \\
& + \sum_{\Gamma_{l,i}^v \subseteq \Gamma^{[1]}, \; \mu(\tilde{\Gamma}_{l,i}^v)<\infty} \left\| \sqrt{R_{l,i}^v} \left(\frac{\partial u_l^v}{\partial \boldsymbol{\nu}^v}\right)_{A^v} \right\|_{1/2,\tilde{\Gamma}_{l,i}^v}^2 .
\end{aligned} \tag{4.12}$$

The fractional Sobolev norms used above are as in (4.4) and (4.5). Moreover, μ denotes measure. Finally, the quadratic form $\mathcal{U}_v^{N,W}(\{\mathcal{F}_u\})$ is given by

$$\mathcal{U}_v^{N,W}(\{\mathcal{F}_u\}) = \sum_{l=1}^{N_v} \int_{\tilde{\Omega}_l^v} \mathrm{e}^{x_3^v} \sum_{|\alpha|\le 2} |D_{x^v}^{\alpha} u_l^v(x^v)|^2 \, dx^v. \tag{4.13}$$

We now define $\mathcal{V}_{\text{vertex-edges}}^{N,W}(\{\mathcal{F}_u\})$ and $\mathcal{U}_{\text{vertex-edges}}^{N,W}(\{\mathcal{F}_u\})$. Let $v-e$ denote one of the vertex-edges of Ω. Here, $v-e \in \mathcal{V}-\mathcal{E}$, the set of vertex-edges of Ω. Let Ω^{v-e} denote the vertex-edge neighbourhood corresponding to the vertex-edge $v-e$. We divide Ω^{v-e} into N_{v-e} elements $\Omega_l^{v-e}, l = 1, 2, \ldots, N_{v-e}$, using a geometric mesh.

Figure 9 shows the vertex-edge neighbourhood Ω^{v-e} of the vertex v and the edge e. Now

$$\Omega^{v-e} = \{x \in \Omega : \; 0 < x_3 < \delta_v, \; 0 < \phi < \phi_v \}.$$

Here, $\delta_v = \rho_v \cos\phi_v$. We impose a geometrical mesh on Ω^{v-e} as shown in figure 9 by defining

$$(x_3)_0 = 0 \quad \text{and} \quad (x_3)_i = \delta_v(\mu_v)^{N+1-i}$$

for $1 \le i \le N+1$. Let

$$\zeta_i^{v-e} = \ln((x_3)_i)$$

for $0 \le i \le N+1$.

Let us introduce points $\phi_0^{v-e}, \ldots, \phi_{N+1}^{v-e}$ such that $\phi_0^{v-e} = 0$ and $\tan\phi_i^{v-e} = \mu_e^{N+1-i}\tan(\phi_v)$, for $1 \le i \le N+1$. Here μ_e is a positive constant less than one. Thus, we impose a geometrical mesh on ϕ with mesh ratio μ_e. Finally, $\theta_l^{v-e} < \theta < \theta_u^{v-e}$. A quasi-uniform mesh

$$\theta_l^{v-e} = \theta_0^{v-e} < \theta_1^{v-e} < \cdots < \theta_{I_{v-e}}^{v-e} = \theta_u^{v-e}$$

is imposed in θ. Let $\tilde{\Omega}^{v-e}$ be the image of Ω^{v-e} in x^{v-e} coordinates (introduced in §2). Thus, $\tilde{\Omega}^{v-e}$ is divided into $N_{v-e} = I_{v-e}(N+1)^2$ hexahedrons $\{\tilde{\Omega}_n^{v-e}\}_{n=1,\ldots,N_{v-e}}$, where

$$\tilde{\Omega}_n^{v-e}\{x^{v-e} : \; \psi_i^{v-e} < \psi < \psi_{i+1}^{v-e}, \; \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e}, \; \zeta_k^{v-e} < \zeta < \zeta_{k+1}^{v-e}\}.$$

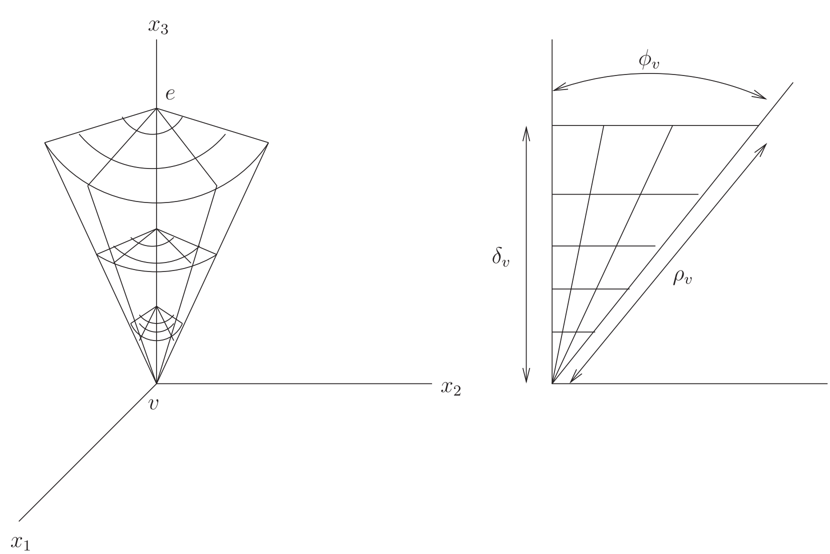

Figure 9. Geometrical mesh imposed on Ω^{v-e}.

We now define the spectral element functions on the elements in $\tilde{\Omega}^{v-e}$. Consider an element

$$\tilde{\Omega}_n^{v-e} = \{x^{v-e} : \ \psi_i^{v-e} < \psi < \psi_{i+1}^{v-e}, \ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e}, -\infty < \zeta < \zeta_1^{v-e}\}.$$

Then on $\tilde{\Omega}_n^{v-e}$, we define

$$u_n^{v-e} = h^{v-e} = h^v,$$

where h^v is the same constant as for the spectral element function u_n^v defined on the corner element

$$\tilde{\Omega}_n^v = \{x^v : \ (\phi, \theta) \in S_j^v, \ -\infty < \chi < \ln(\rho_1^v)\}.$$

Next, we consider the element

$$\tilde{\Omega}_p^{v-e} = \{x^{v-e} : -\infty < \psi < \psi_1^{v-e}, \ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e}, \ \zeta_k^{v-e} < \zeta < \zeta_{k+1}^{v-e}\}.$$

Here, $k \geq 1$. Then on $\tilde{\Omega}_p^{v-e}$, we define

$$u_p^{v-e}(x^{v-e}) = \sum_{l=0}^{W_p} \beta_l \, \zeta^l.$$

Here, $1 \leq W_p \leq W$. Moreover, $W_p = [\mu_2 k]$ for $1 \leq k \leq N$, where $\mu_2 > 0$ is a degree factor [21]. Now consider

$$\tilde{\Omega}_q^{v-e} = \left\{x^{v-e} : \ \psi_i^{v-e} < \psi < \psi_{i+1}^{v-e}, \ \theta_j^{v-e} < \theta < \theta_{j+1}^{v-e}, \ \zeta_k^{v-e} < \zeta < \zeta_{k+1}^{v-e}\right\}$$

for $1 \leq i \leq N$, $1 \leq k \leq N$. Then on $\tilde{\Omega}_q^{v-e}$, we define

$$u_q^{v-e}(x^{v-e}) = \sum_{r=0}^{W_q} \sum_{s=0}^{W_q} \sum_{t=0}^{V_q} \gamma_{r,s,t} \, \psi^r \theta^s \zeta^t.$$

Here, $1 \leq W_q \leq W$ and $1 \leq V_q \leq W$. Moreover, $W_q = [\mu_1 i], V_q = [\mu_2 k]$ for $1 \leq i, k \leq N$, where $\mu_1, \mu_2 > 0$ are degree factors [21].

Let $\tilde{\Gamma}_{n,i}^{v-e}$ be one of the faces of $\tilde{\Omega}_n^{v-e}$ such that $\mu(\tilde{\Gamma}_{n,i}^{v-e}) < \infty$, where μ denotes measure. We introduce a norm $|||\, u \,|||^2_{\tilde{\Gamma}_{n,i}^{v-e}}$ as follows:

Let $E_{n,i}^{v-e} = \sup_{x^{v-e} \in \tilde{\Gamma}_{n,i}^{v-e}} (\sin\phi)$ and $F_{n,i}^{v-e} = \sup_{x^{v-e} \in \tilde{\Gamma}_{n,i}^{v-e}} (e^{x_3^{v-e}})$. We also define $G_{n,i}^{v-e}$ which is used in (4.16).

(1) If $\tilde{\Gamma}_{n,i}^{v-e} = \{x^{v-e} : \ \alpha_0 < x_1^{v-e} < \alpha_1, \ \beta_0 < x_2^{v-e} < \beta_1, \ x_3^{v-e} = \gamma_0\}$, then define $G_{n,i}^{v-e} = E_{n,i}^{v-e}$ and

$$|||\, u \,|||^2_{\tilde{\Gamma}_{n,i}^{v-e}} = E_{n,i}^{v-e} F_{n,i}^{v-e} \Bigg(\int_{\beta_0}^{\beta_1} \int_{\alpha_0}^{\alpha_1} u^2(\psi, \theta, \gamma_0) \, d\psi \, d\theta$$

$$+ \int_{\beta_0}^{\beta_1} d\theta \int_{\alpha_0}^{\alpha_1} \int_{\alpha_0}^{\alpha_1} \frac{\big(u(\psi, \theta, \gamma_0) - u(\psi', \theta, \gamma_0)\big)^2}{(\psi - \psi')^2} \, d\psi \, d\psi'$$

$$+\int_{\alpha_0}^{\alpha_1} d\psi \int_{\beta_0}^{\beta_1}\int_{\beta_0}^{\beta_1} \frac{(u(\psi,\theta,\gamma_0)-u(\psi,\theta',\gamma_0))^2}{(\theta-\theta')^2}\,d\theta\,d\theta'\Bigg). \tag{4.14a}$$

(2) If $\tilde{\Gamma}_{n,i}^{v-e}=\{x^{v-e}:\ x_1^{v-e}=\alpha_0,\ \beta_0<x_2^{v-e}<\beta_1,\ \gamma_0<x_3^{v-e}<\gamma_1\}$, then define $G_{n,i}^{v-e}=1$ and

$$\begin{aligned}|||\,u\,|||^2_{\tilde{\Gamma}_{n,i}^{v-e}} = F_{n,i}^{v-e}\Bigg(&\int_{\gamma_0}^{\gamma_1}\int_{\beta_0}^{\beta_1} u^2(\alpha_0,\theta,\zeta)\,d\theta\,d\zeta\\ &+\int_{\gamma_0}^{\gamma_1} d\zeta\int_{\beta_0}^{\beta_1}\int_{\beta_0}^{\beta_1}\frac{(u(\alpha_0,\theta,\zeta)-u(\alpha_0,\theta',\zeta))^2}{(\theta-\theta')^2}\,d\theta\,d\theta'\\ &+E_{n,i}^{v-e}\int_{\beta_0}^{\beta_1} d\theta\int_{\gamma_0}^{\gamma_1}\int_{\gamma_0}^{\gamma_1}\frac{(u(\alpha_0,\theta,\zeta)-u(\alpha_0,\theta,\zeta'))^2}{(\zeta-\zeta')^2}\,d\zeta\,d\zeta'\Bigg).\end{aligned} \tag{4.14b}$$

(3) If $\tilde{\Gamma}_{n,i}^{v-e}=\{x^{v-e}:\ \alpha_0<x_1^{v-e}<\alpha_1,\ x_2^{v-e}=\beta_0,\ \gamma_0<x_3^{v-e}<\gamma_1\}$, then define $G_{n,i}^{v-e}=1$ and

$$\begin{aligned}|||\,u\,|||^2_{\tilde{\Gamma}_{n,i}^{v-e}} = F_{n,i}^{v-e}\Bigg(&\int_{\gamma_0}^{\gamma_1}\int_{\alpha_0}^{\alpha_1} u^2(\psi,\beta_0,\zeta)\,d\psi\,d\zeta\\ &+\int_{\gamma_0}^{\gamma_1} d\zeta\int_{\alpha_0}^{\alpha_1}\int_{\alpha_0}^{\alpha_1}\frac{(u(\psi,\beta_0,\zeta)-u(\psi',\beta_0,\zeta))^2}{(\psi-\psi')^2}\,d\psi\,d\psi'\\ &+E_{n,i}^{v-e}\int_{\alpha_0}^{\alpha_1} d\psi\int_{\gamma_0}^{\gamma_1}\int_{\gamma_0}^{\gamma_1}\frac{(u(\psi,\beta_0,\zeta)-u(\psi,\beta_0,\zeta'))^2}{(\zeta-\zeta')^2}\,d\zeta\,d\zeta'\Bigg).\end{aligned} \tag{4.14c}$$

Let L^{v-e} be a differential operator such that

$$\int_{\tilde{\Omega}_n^{v-e}} \left|L^{v-e}u(x^{v-e})\right|^2\,dx^{v-e} = \int_{\Omega_n^v} \rho^2\sin^2\phi\,|Lu(x)|^2\,dx.$$

Here, dx^{v-e} denotes a volume element in x^{v-e} coordinates and dx a volume element in x coordinates. In Chapter 3 of [1], it is shown that

$$L^{v-e}u(x^{v-e}) = -\mathrm{div}_{x^{v-e}}(\mathrm{e}^{\zeta/2}A^{v-\mathrm{e}}\nabla_{x^{v-\mathrm{e}}}u)+\sum_{i=1}^{3}\hat{b}_i^{v-\mathrm{e}}u_{x_i^{v-\mathrm{e}}}+\hat{c}^{v-\mathrm{e}}u. \tag{4.15}$$

Here, A^{v-e} is a symmetric, positive definite matrix. We now define the quadratic form

$$\begin{aligned}\mathcal{V}_{v-e}^{N,W}(\{\mathcal{F}_u\}) = &\sum_{l=1,\mu(\tilde{\Omega}_l^{v-e})<\infty}^{N_{v-e}}\int_{\tilde{\Omega}_l^{v-e}}\left|L^{v-e}u_l^{v-e}(x^{v-e})\right|^2\,dx^{v-e}\\ &+\sum_{\Gamma_{n,i}^{v-e}\subseteq\bar{\Omega}^{v-e}\setminus\partial\Omega,\ \mu(\tilde{\Gamma}_{n,i}^{v-e})<\infty}\Bigg(\left\|\sqrt{F_{n,i}^{v-e}G_{n,i}^{v-e}}\,[u]\right\|^2_{0,\tilde{\Gamma}_{n,i}^{v-e}}\\ &+|||\,[u_{x_1^{v-e}}]\,|||^2_{\tilde{\Gamma}_{n,i}^{v-e}}\end{aligned}$$

$$+||| [u_{x_2^{v-e}}] |||^2_{\tilde{\Gamma}^{v-e}_{n,i}} + ||| E^{v-e}_{n,i} [u_{x_3^{v-e}}] |||^2_{\tilde{\Gamma}^{v-e}_{n,i}} \Bigg)$$

$$+ \sum_{\Gamma^{v-e}_{n,i} \subseteq \Gamma^{[0]},\, \mu(\tilde{\Gamma}^{v-e}_{n,i}) < \infty} \left(\left\| \sqrt{F^{v-e}_{n,i}}\, u^{v-e}_n \right\|^2_{0,\tilde{\Gamma}^{v-e}_{n,i}} \right.$$

$$\left. +||| u_{x_1^{v-e}} |||^2_{\tilde{\Gamma}^{v-e}_{n,i}} + ||| E^{v-e}_{n,i}\, u_{x_3^{v-e}} |||^2_{\tilde{\Gamma}^{v-e}_{n,i}} \right)$$

$$+ \sum_{\Gamma^{v-e}_{n,i} \subseteq \Gamma^{[1]},\, \mu(\tilde{\Gamma}^{v-e}_{n,i}) < \infty} \left||| \left(\frac{\partial u}{\partial \boldsymbol{\nu}}^{v-e} \right)_{A^{v-e}} \right|||^2_{\tilde{\Gamma}^{v-e}_{n,i}}. \tag{4.16}$$

Here, μ denotes measure and the term $\left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}}\right)_{A^{v-e}}$ is defined as follows: Let $\tilde{\Gamma}^{v-e}_{n,i}$ be a face of $\tilde{\Omega}^{v-e}_{n,i}$, $\tilde{P}$ be a point belonging to $\tilde{\Gamma}^{v-e}_{n,i}$ and $\boldsymbol{\nu}^{v-e}$ denote the unit normal to $\tilde{\Gamma}^{v-e}_{n,i}$ at the point $\tilde{P}$. Then

$$\left(\frac{\partial u}{\partial \boldsymbol{\nu}^{v-e}} \right)_{A^{v-e}} (\tilde{P}) = (\boldsymbol{\nu}^{v-e})^T A^{v-e} \nabla_{x^{v-e}} u. \tag{4.17}$$

Now the quadratic form $\mathcal{V}^{N,W}_{\text{vertex-edges}}(\{\mathcal{F}_u\})$ is given by

$$\mathcal{V}^{N,W}_{\text{vertex-edges}}(\{\mathcal{F}_u\}) = \sum_{v-e \in \mathcal{V}-\mathcal{E}} \mathcal{V}^{N,W}_{v-e}(\{\mathcal{F}_u\}). \tag{4.18}$$

Next, we define the quadratic form $\mathcal{U}^{N,W}_{v-e}(\{\mathcal{F}_u\})$. Let $w^{v-e}(x_1^{v-e})$ be a weight function such that

$$w^{v-e}(x_1^{v-e}) = 1 \quad \text{for } x_1^{v-e} \geq \zeta_1^{v-e} = \ln(\tan \phi_1^{v-e})$$

and which satisfies

$$\int_{-\infty}^{\zeta_1^{v-e}} w^{v-e}(x_1^{v-e})\, dx_1^{v-e} = 1.$$

We shall choose

$$w^{v-e}(x_1^{v-e}) = 1 \quad \text{for } x_1^{v-e} \geq \zeta_1^{v-e} - 1$$

and

$$w^{v-e}(x_1^{v-e}) = 0 \quad \text{for } x_1^{v-e} < \zeta_1^{v-e} - 1\,.$$

Then

$$\begin{aligned}\mathcal{U}^{N,W}_{v-e}(\{\mathcal{F}_u\}) \;=\; & \sum_{l=1,\mu(\tilde{\Omega}^{v-e}_l)<\infty}^{N_{v-e}} \int_{\tilde{\Omega}^{v-e}_l} \mathrm{e}^{x_3^{v-e}} \left(\sum_{i,j=1,2} \left(\frac{\partial^2 u_l^{v-e}}{\partial x_i^{v-e} \partial x_j^{v-e}} \right)^2 \right. \\ & + \sum_{i=1}^{2} \sin^2\phi \left(\frac{\partial^2 u_l^{v-e}}{\partial x_i^{v-e} \partial x_3^{v-e}} \right)^2 + \sin^4\phi \left(\frac{\partial^2 u_l^{v-e}}{(\partial x_3^{v-e})^2} \right)^2 \\ & \left. + \sum_{i=1}^{2} \left(\frac{\partial u_l^{v-e}}{\partial x_i^{v-e}} \right)^2 + \sin^2\phi \left(\frac{\partial u_l^{v-e}}{\partial x_3^{v-e}} \right)^2 + (u_l^{v-e})^2 \right) dx^{v-e}\end{aligned}$$

$$+\sum_{l=1,\ \mu(\tilde{\Omega}_l^{v-e})=\infty}^{N_{v-e}} \int_{\tilde{\Omega}_l^{v-e}} (u_l^{v-e})^2 e^{x_3^{v-e}}\ w^{v-e}(x_1^{v-e})\, dx^{v-e}. \tag{4.19}$$

The quadratic form $\mathcal{U}^{N,W}_{\text{vertex-edges}}(\{\mathcal{F}_u\})$ is then given by

$$\mathcal{U}^{N,W}_{\text{vertex-edges}}(\{\mathcal{F}_u\}) = \sum_{v-e\in\mathcal{V}-\mathcal{E}} \mathcal{U}^{N,W}_{v-e}(\{\mathcal{F}_u\}). \tag{4.20}$$

Finally, we define the quadratic forms $\mathcal{V}^{N,W}_{\text{edges}}(\{\mathcal{F}_u\})$ and $\mathcal{U}^{N,W}_{\text{edges}}(\{\mathcal{F}_u\})$. Consider the edge e whose end points are v and v'. The edge e coincides with the x_3 axis and the vertex v with the origin. Let the length of the edge e be l_e. Now the edge neighbourhood Ω^e is defined as

$$\Omega^e = \left\{x \in \Omega:\ 0 < r < Z = \rho_v \sin\phi_v,\ \theta_l^e < \theta < \theta_u^e,\ \delta_v < x_3 < l_e - \delta_v'\right\}.$$

Here, (r, θ, x_3) denote cylindrical coordinates with origin at v, $\delta_v = \rho_v \cos\phi_v$ and $\delta_v' = \rho_v' \cos\phi_v'$ are as shown in figure 10.

A geometrical mesh is imposed in r by defining $r_0^e = 0$ and $r_j^e = Z(\mu_e)^{N+1-j}$ for $j = 1, 2, \ldots, N+1$. We impose the same quasi-uniform mesh on θ as we did in the vertex-edge neighbourhood, viz.

$$\theta_l^e = \theta_0^e < \theta_1^e < \cdots < \theta_{I^e}^e = \theta_u^e.$$

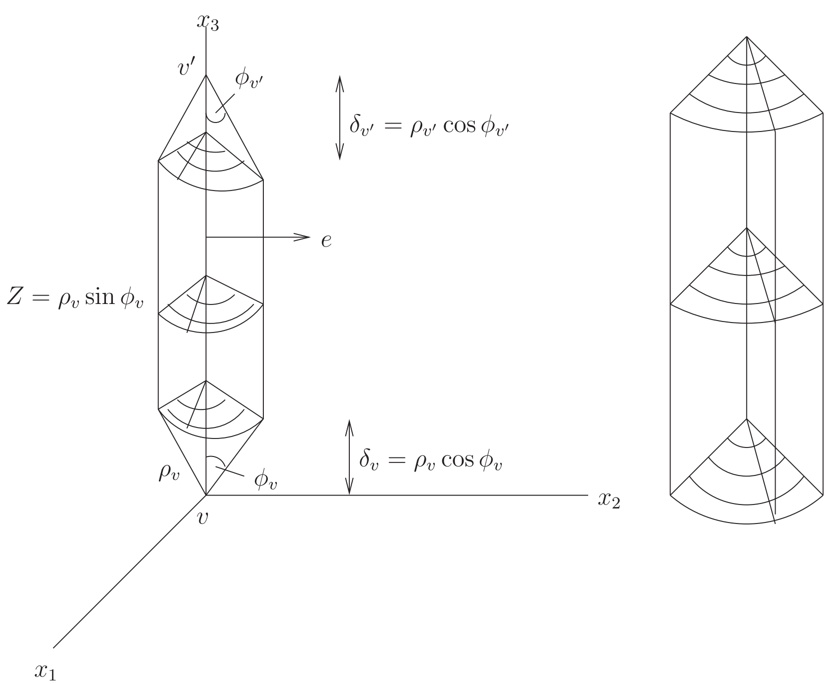

Figure 10. Geometrical mesh imposed on Ω^e.

Here, $I_e = I_{v-e}$ and $\theta_k^e = \theta_k^{v-e}$ for $0 \le k \le I_e$. A quasi-uniform mesh is defined in x_3, by choosing

$$\delta_v = Z_0^e < Z_1^e < \cdots < Z_{J_e}^e = l_e - \delta_v'.$$

Thus, Ω^e is divided into $N_e = I_e\, J_e\,(N+1)$ elements. Let $\tilde{\Omega}^e$ be the image of Ω^e in x^e coordinates (introduced in §2). Thus, $\tilde{\Omega}^e$ is divided into N_e hexahedrons $\{\tilde{\Omega}_m^e\}_{m=1,\ldots,N_e}$, where

$$\tilde{\Omega}_m^e = \{x^e :\ \ln(r_i^e) < x_1^e < \ln(r_{i+1}^e),\ \theta_j^e < x_2^e < \theta_{j+1}^e,\ Z_k^e < x_3^e < Z_{k+1}^e\}.$$

We now define the spectral element functions on the elements in $\tilde{\Omega}^e$. Consider an element

$$\tilde{\Omega}_p^e = \{x^e :\ -\infty < x_1^e < \ln(r_1^e),\ \theta_j^e < x_2^e < \theta_{j+1}^e,\ Z_n^e < x_3^e < Z_{n+1}^e\}.$$

Then, we define

$$u_p^e(x^e) = \sum_{t=1}^{W} \alpha_r\,(x_3^e)^t.$$

This representation is valid for all j for fixed n. Next, consider the element

$$\tilde{\Omega}_q^e = \{x^e :\ \ln(r_i^e) < x_1^e < \ln(r_{i+1}^e),\ \theta_j^e < x_2^e < \theta_{j+1}^e,\ Z_n^e < x_3^e < Z_{n+1}^e\}$$

for $1 \le i \le N$. Then, we define

$$u_q^e(x^e) = \sum_{r=1}^{W_q}\sum_{s=1}^{W_q}\sum_{t=1}^{W} \alpha_{r,s,t}\ (x_1^e)^r (x_2^e)^s (x_3^e)^t.$$

Here, $1 \le W_q \le W$. Moreover, as in [21], $W_q = [\mu_1 i]$ for all $1 \le i \le N$, where $\mu_1 > 0$ is a degree factor.

Let $\tilde{\Gamma}_{m,i}^e$ be one of the faces of $\tilde{\Omega}_m^e$ such that $\mu(\tilde{\Gamma}_{m,i}^e) < \infty$, where μ denotes measure. We define a norm $|||\, u\, |||^2_{\tilde{\Gamma}_{m,i}^e}$ as follows:

Let $G_{m,i}^e = \sup_{x^e \in \tilde{\Gamma}_{m,i}^e} (e^\tau)$. We also define $H_{m,i}^e$ which is needed in (4.25).

(1) If $\tilde{\Gamma}_{m,i}^e = \{x^e :\ \alpha_0 < x_1^e < \alpha_1,\ \beta_0 < x_2^e < \beta_1,\ x_3^e = \gamma_0\}$, then define $H_{m,i}^e = G_{m,i}^e$ and

$$\begin{aligned} |||\, u\, |||^2_{\tilde{\Gamma}_{m,i}^e} \;=\; & G_{m,i}^e \left(\int_{\beta_0}^{\beta_1}\int_{\alpha_0}^{\alpha_1} u^2(\tau,\theta,\gamma_0)\, d\tau d\theta \right. \\ & + \int_{\beta_0}^{\beta_1} d\theta \int_{\alpha_0}^{\alpha_1}\int_{\alpha_0}^{\alpha_1} \frac{(u(\tau,\theta,\gamma_0) - u(\tau',\theta,\gamma_0))^2}{(\tau-\tau')^2}\, d\tau d\tau' \\ & \left. + \int_{\alpha_0}^{\alpha_1} d\tau \int_{\beta_0}^{\beta_1}\int_{\beta_0}^{\beta_1} \frac{(u(\tau,\theta,\gamma_0) - u(\tau,\theta',\gamma_0))^2}{(\theta-\theta')^2}\, d\theta d\theta' \right). \end{aligned} \tag{4.21a}$$

(2) If $\tilde{\Gamma}^e_{m,i} = \{x^e : x^e_1 = \alpha_0,\ \beta_0 < x^e_2 < \beta_1,\ \gamma_0 < x^e_3 < \gamma_1\}$, then define $H^e_{m,i} = 1$ and

$$\begin{aligned}|||\,u\,|||^2_{\tilde{\Gamma}^e_{m,i}} \;=\; & \left(\int_{\gamma_0}^{\gamma_1}\int_{\beta_0}^{\beta_1} u^2(\alpha_0,\theta,x_3)\,d\theta dx_3\right.\\ & + \int_{\gamma_0}^{\gamma_1} dx_3 \int_{\beta_0}^{\beta_1}\int_{\beta_0}^{\beta_1} \frac{\big(u(\alpha_0,\theta,x_3)-u(\alpha_0,\theta',x_3)\big)^2}{(\theta-\theta')^2}\,d\theta d\theta'\\ & \left. + G^e_{m,i}\int_{\beta_0}^{\beta_1} d\theta \int_{\gamma_0}^{\gamma_1}\int_{\gamma_0}^{\gamma_1} \frac{(u(\alpha_0,\theta,x_3)-u(\alpha_0,\theta,x_3'))^2}{(x_3-x_3')^2}\,dx_3\,dx_3'\right).\end{aligned} \tag{4.21b}$$

(3) If $\tilde{\Gamma}^e_{m,i} = \{x^e : \alpha_0 < x^e_1 < \alpha_1,\ x^e_2 = \beta_0,\ \gamma_0 < x^e_3 < \gamma_1\}$, then define $H^e_{m,i} = 1$ and

$$\begin{aligned}|||\,u\,|||^2_{\tilde{\Gamma}^e_{m,i}} \;=\; & \left(\int_{\gamma_0}^{\gamma_1}\int_{\alpha_0}^{\alpha_1} u^2(\tau,\beta_0,x_3)\,d\tau dx_3\right.\\ & + \int_{\gamma_0}^{\gamma_1} dx_3 \int_{\alpha_0}^{\alpha_1}\int_{\alpha_0}^{\alpha_1} \frac{(u(\tau,\beta_0,x_3)-u(\tau',\beta_0,x_3))^2}{(\tau-\tau')^2}\,d\tau d\tau'\\ & \left. + G^e_{m,i}\int_{\alpha_0}^{\alpha_1} d\tau \int_{\gamma_0}^{\gamma_1}\int_{\gamma_0}^{\gamma_1} \frac{(u(\tau,\beta_0,x_3)-u(\tau,\beta_0,x_3'))^2}{(x_3-x_3')^2}\,dx_3 dx_3'\right).\end{aligned} \tag{4.21c}$$

Let L^e be a differential operator such that

$$\int_{\tilde{\Omega}^e_m} |L^e u(x^e)|^2 dx^e = \int_{\Omega^e_m} r^2 |Lu(x)|^2 dx. \tag{4.22}$$

Here, dx^e denotes a volume element in x^e coordinates and dx a volume element in x coordinates. In Chapter 3 of [1], it is shown that

$$L^e u(x^e) = -\mathrm{div}_{x^e}\left(A^e \nabla_{x^e} u\right) + \sum_{i=1}^{3} \hat{b}^e_i u_{x^e_i} + \hat{c}^e u, \tag{4.23}$$

where A^e is a symmetric, positive definite matrix. Let $\tilde{\Gamma}^e_{m,i}$ be one of the sides of $\tilde{\Omega}^e_m$ and $\tilde{P}$ a point belonging to $\tilde{\Gamma}^e_{m,i}$. Let $\boldsymbol{\nu}^e$ be the normal to $\tilde{\Gamma}^e_{m,i}$ at $\tilde{P}$. Then

$$\left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e}(\tilde{P}) = (\nu^e)^T A^e \nabla_{x^e} u(\tilde{P})\,. \tag{4.24}$$

We now define the quadratic form

$$\begin{aligned}\mathcal{V}^{N,W}_e(\{\mathcal{F}_u\}) = & \sum_{l=1,\,\mu(\tilde{\Omega}^e_l)<\infty}^{N_e} \int_{\tilde{\Omega}^e_l} |L^e u^e_l(x^e)|^2\, dx^e\\ & + \sum_{\Gamma^e_{l,i}\subseteq\bar{\Omega}^e\setminus\partial\Omega,\ \mu(\tilde{\Gamma}^e_{l,i})<\infty} \left(\left\| \sqrt{H^e_{l,i}}\,[u]\right\|^2_{0,\tilde{\Gamma}^e_{l,i}} + |||\,[u_{x^e_1}]\,|||^2_{\tilde{\Gamma}^e_{l,i}}\right.\end{aligned}$$

$$
\begin{aligned}
&+|||\,[u_{x_2^e}]\,|||^2_{\tilde{\Gamma}^e_{l,i}} + |||\,G^e_{l,i}[u_{x_3^e}]\,|||^2_{\tilde{\Gamma}^e_{l,i}}\Bigg) \\
&+ \sum_{\Gamma^e_{l,i}\subseteq\Gamma^{[0]},\,\mu(\tilde{\Gamma}^e_{l,i})<\infty} (\|u^e_l\|^2_{0,\tilde{\Gamma}^e_{l,i}} + |||u_{x_1^e}|||^2_{\tilde{\Gamma}^e_{l,i}} + \Big|\Big|\Big|G^e_{l,i}\,u_{x_3^e}|||^2_{\tilde{\Gamma}^e_{l,i}}) \\
&+ \sum_{\Gamma^e_{l,i}\subseteq\Gamma^{[1]},\,\mu(\tilde{\Gamma}^e_{l,i})<\infty} \left|\left|\left|\left(\frac{\partial u}{\partial \boldsymbol{\nu}^e}\right)_{A^e}\right|\right|\right|^2_{\tilde{\Gamma}^e_{l,i}}. \qquad (4.25)
\end{aligned}
$$

The quadratic form $\mathcal{V}^{N,W}_{\text{edges}}(\{\mathcal{F}_u\})$ is given by

$$
\mathcal{V}^{N,W}_{\text{edges}}(\{\mathcal{F}_u\}) = \sum_{e\in\mathcal{E}} \mathcal{V}^{N,W}_e(\{\mathcal{F}_u\}). \qquad (4.26)
$$

Next, let us define the quadratic form $\mathcal{U}^{N,W}_e(\{\mathcal{F}_u\})$. Let $w^e(x_1^e)$ be a weight function such that

$$
w^e(x_1^e) = 1 \quad \text{for } x_1^e \geq \ln(r_1^e)
$$

and

$$
\int_{-\infty}^{\ln(r_1^e)} w^e(x_1^e)\, dx_1^e = 1.
$$

We shall choose

$$
w^e(x_1^e) = 1 \quad \text{for } x_1^e \geq \ln(r_1^e) - 1
$$

and

$$
w^e(x_1^e) = 0 \quad \text{for } x_1^e < \ln(r_1^e) - 1.
$$

Then

$$
\begin{aligned}
\mathcal{U}^{N,W}_e(\{\mathcal{F}_u\}) = &\sum_{l=1,\mu(\tilde{\Omega}^e_l)<\infty}^{N_e} \int_{\tilde{\Omega}^e_l} \Bigg(\sum_{i,j=1,2} \left(\frac{\partial^2 u^e_l}{\partial x_i^e \partial x_j^e}\right)^2 + e^{2\tau} \sum_{i=1}^{2} \left(\frac{\partial^2 u^e_l}{\partial x_i^e \partial x_3^e}\right)^2 \\
&+ e^{4\tau}\left(\frac{\partial^2 u^e_l}{(\partial x_3^e)^2}\right)^2 + \sum_{i=1}^{2}\left(\frac{\partial u^e_l}{\partial x_i^e}\right)^2 + e^{2\tau}\left(\frac{\partial u^e_l}{\partial x_3^e}\right)^2 + (u^e_l)^2\Bigg) dx^e \\
&+ \sum_{l=1,\mu(\tilde{\Omega}^e_l)=\infty}^{N_e} \int_{\tilde{\Omega}^e_l} (u^e_l)^2 w^e(x_1^e) dx^e. \qquad (4.27)
\end{aligned}
$$

Here, μ denotes measure. The quadratic form $\mathcal{U}^{N,W}_{\text{edges}}(\{\mathcal{F}_u\})$ is then given by

$$
\mathcal{U}^{N,W}_{\text{edges}}(\{\mathcal{F}_u\}) = \sum_{e\in\mathcal{E}} \mathcal{U}^{N,W}_e(\{\mathcal{F}_u\}). \qquad (4.28)
$$

Finally, using (4.1) and (4.2) we can define the quadratic forms $\mathcal{V}^{N,W}(\{\mathcal{F}_u\})$ and $\mathcal{U}^{N,W}(\{\mathcal{F}_u\})$.

We now state the stability results whose proof can be found in [16]. It is assumed that N is proportional to W.

Theorem 4.1. *Consider the elliptic boundary value problem* (2.1). *Suppose the boundary conditions are Dirichlet. Then*

$$\mathcal{U}^{N,W}(\{\mathcal{F}_u\}) \leq C(\ln W)^2 \mathcal{V}^{N,W}(\{\mathcal{F}_u\}) \tag{4.29}$$

provided $W = O(\mathrm{e}^{N^\alpha})$ *for* $\alpha < 1/2$. *Since we choose N proportional to W this condition is satisfied.*

Next, we state the corresponding result for general boundary conditions.

Theorem 4.2. *If the boundary conditions for the elliptic boundary value problem* (2.1) *are mixed*, *then*

$$\mathcal{U}^{N,W}(\{\mathcal{F}_u\}) \leq CN^4 \mathcal{V}^{N,W}(\{\mathcal{F}_u\}) \tag{4.30}$$

provided $W = O(\mathrm{e}^{N^\alpha})$ *for* $\alpha < 1/2$.

The rapid growth of the factor CN^4 with N creates difficulties in parallelizing the numerical scheme. To overcome this problem, we state a version of Theorem 4.2 when the spectral element functions vanish on the wirebasket. Let *WB* denote the wirebasket along which the spectral element functions that need to be conforming. Here the wirebasket denotes the union of the vertices and edges of the elements.

Theorem 4.3. *If the boundary conditions are mixed and the spectral element functions* $(\{\mathcal{F}_u\})$ *are conforming on the wirebasket WB and vanish on WB*, *then*

$$\mathcal{U}^{N,W}(\{\mathcal{F}_u\}) \leq C(\ln W)^2 \mathcal{V}^{N,W}(\{\mathcal{F}_u\}) \tag{4.31}$$

provided $W = O(\mathrm{e}^{N^\alpha})$ *for* $\alpha < 1/2$.

5. Conclusions

We will use the stability theorems 4.1 and 4.2 in the forthcoming papers to formulate a numerical scheme and a parallel preconditioner (similar to that described in [15] for two dimensional problems) to obtain an exponentially accurate solution to the elliptic boundary value problem on non-smooth domains considered in this paper.

Another version of the method can be defined by choosing the spectral element functions to be conforming on the wirebasket of the elements (Theorem 4.3). An efficient preconditioner can be obtained for the Schur complement of the common boundary values. We intend to examine this in future work.

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