Proc. Indian Acad. Sci. (Math. Sci.) Vol. 125, No. 1, February 2015, pp. 103–111.

Generalization of Samuelson's inequality and location of eigenvalues

R SHARMA* and R SAINI

Department of Mathematics, H. P. University, Shimla 171 005, India
*Corresponding author.
E-mail: rajesh_hpu_math@yahoo.co.in

MS received 7 September 2013; revised 29 October 2013

Abstract. We prove a generalization of Samuelson's inequality for higher order central moments. Bounds for the eigenvalues are obtained when a given complex $n \times n$ matrix has real eigenvalues. Likewise, we discuss bounds for the roots of polynomial equations.

Keywords. Maximum deviation; central moments; Hermitian matrix; eigenvalues; condition number; polynomial; roots.

2010 Mathematics Subject Classification. 60E15, 15A42, 12D10.

1. Introduction

Let $x_1, x_2, \dots, x_n$ denote n real numbers. Their arithmetic mean is the number

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{1.1}$$

and the r-th central moment is

$$m_r = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^r . \tag{1.2}$$

Then, $s = \sqrt{m_2}$ is the standard deviation of x_i $(i = 1, 2, \dots, n)$. The Samuelson inequality [10] says that

$$s^2 \geq \frac{1}{n-1}\left(x_j - \bar{x}\right)^2, \tag{1.3}$$

or equivalently

$$\bar{x} - \sqrt{n-1}\, s \leq x_j \leq \bar{x} + \sqrt{n-1} s. \tag{1.4}$$

The Samuelson inequality also provides an upper bound for the maximum deviation d from the mean

$$d \leq \sqrt{n-1}\, s, \tag{1.5}$$

where

$$d = \max_i |x_i - \bar{x}|. \tag{1.6}$$

Refinements, extensions and applications of such inequalities have been studied extensively in the literature (see [2, 4, 8, 11, 14]). One refinement of the Samuelson inequality (1.5) gives an upper bound for the maximum deviation in a given range $r = M - m$, where $m \leq x_i \leq M$, $i = 1, 2, \ldots, n$. Thus if $ns \geq \sqrt{n-1}r$, then

$$d \leq \frac{r}{2} + \sqrt{\left(\frac{r}{2}\right)^2 - s^2}$$

and if $ns \leq \sqrt{n-1}r$, then

$$d \leq \frac{r}{2} + \sqrt{\frac{n-2}{2}\left(s^2 - \frac{r^2}{2n}\right)}.$$

(see [11]). Such inequalities are useful in various contexts. Wolkowicz and Styan [14] have observed that if the eigenvalues of an $n \times n$ complex matrix are real, as in the case of Hermitian matrices, the inequalities (1.4) provide bounds for the extreme eigenvalues. Let λ_i be eigenvalues of A, $i = 1, 2, \ldots, n$. Let $B = A - \frac{\mathrm{tr}A}{n} I$, where $\mathrm{tr}A$ denotes the trace of A. Then,

$$\frac{\mathrm{tr}\,A}{n} - \sqrt{\frac{n-1}{n}\mathrm{tr}\,B^2} \leq \lambda_i \leq \frac{\mathrm{tr}\,A}{n} + \sqrt{\frac{n-1}{n}\mathrm{tr}\,B^2}. \tag{1.7}$$

In some different context and notations, inequalities related to (1.4) also appeared in the work of Laguerre [6]. In particular, let $x_1, x_2, \ldots, x_n$ denote the roots, all of which we assume to be real, of the n-th degree monic polynomial equation with $n \geq 2$,

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0. \tag{1.8}$$

Then [6],

$$\begin{aligned} -\frac{a_1}{n} - \sqrt{\frac{(n-1)^2 a_1^2}{n^2} - \frac{2(n-1)a_2}{n}} &\leq x_i \\ \leq -\frac{a_1}{n} + \sqrt{\frac{(n-1)^2 a_1^2}{n^2} - \frac{2(n-1)a_2}{n}} & \end{aligned} \tag{1.9}$$

for all x_i.

It is natural to consider the genralization of Samuelson's inequality for higher order moments and look for related extensions and applications. We obtain a generalization of the Samuelson inequality that gives a lower bound for the central moment m_{2r} and bounds for all x_i in terms of $\bar{x}$ and m_{2r} (Theorem 2.1 and Corollary 2.2, below). This also provides an upper bound for the maximum deviation in terms of central moment m_{2r} (Corollary 2.3). As an application we find bounds for the eigenvalues of a matrix when all its eigenvalues are real (Theorem 3.1). An upper bound for the condition number is given in Corollary 3.2. We give examples and compare our bounds with those given by Wolkowicz and Styan [14]. Likewise, we give bounds for the largest and smallest roots of a polynomial equation when all its roots are real (Theorem 4.1).

2. Main results

Theorem 2.1. *If m_{2r} is the central moment of n real numbers $x_1, x_2, \ldots, x_n$, then*

$$m_{2r} \geq \frac{1 + (n-1)^{2r-1}}{n\,(n-1)^{2r-1}} \left(x_j - \bar{x}\right)^{2r} \tag{2.1}$$

for all $j = 1, 2, \ldots, n$ and $r = 1, 2, \ldots$.

Proof. From eq. (1.2), we write

$$m_{2r} = \frac{\left(x_j - \bar{x}\right)^{2r}}{n} + \frac{n-1}{n} \frac{1}{n-1} \sum_{i=1, i \neq j}^{n} (x_i - \bar{x})^{2r} . \tag{2.2}$$

For m positive real numbers y_i, $i = 1, 2, \ldots, m$,

$$\frac{1}{m} \sum_{i=1}^{m} y_i^k \geq \left(\frac{1}{m} \sum_{i=1}^{m} y_i \right)^k , \; k = 1, 2, \cdots . \tag{2.3}$$

Applying (2.3) to $n-1$ positive real numbers $(x_i - \bar{x})^2$, $i = 1, 2, \ldots, n$ and $i \neq j$, we get

$$\frac{1}{n-1} \sum_{i=1, i \neq j}^{n} \left((x_i - \bar{x})^2\right)^r \geq \left(\frac{1}{n-1} \sum_{i=1, i \neq j}^{n} (x_i - \bar{x})^2 \right)^r . \tag{2.4}$$

Further, the Cauchy–Schwarz inequality implies that the inequality (2.3) holds good for all real numbers y_i when $k = 2$. Therefore

$$\frac{1}{n-1} \sum_{i=1, i \neq j}^{n} (x_i - \bar{x})^2 \geq \left(\frac{1}{n-1} \sum_{i=1, i \neq j}^{n} (x_i - \bar{x}) \right)^2 . \tag{2.5}$$

On the other hand, the sum of all the deviations from the mean is zero, therefore

$$\sum_{i=1}^{n} (x_i - \bar{x}) = 0,$$

and we get that

$$\sum_{i=1, i \neq j}^{n} (x_i - \bar{x}) = \bar{x} - x_j . \tag{2.6}$$

Combining (2.4)–(2.6), we find that

$$\frac{1}{n-1} \sum_{i=1, i \neq j}^{n} (x_i - \bar{x})^{2r} \geq \left(\frac{\bar{x} - x_j}{n-1} \right)^{2r} . \tag{2.7}$$

Inserting (2.7) in (2.2), and by computing leads to (2.1). □

It is clear from the derivation that inequality (2.1) is strict if x_j is different from m or M. Equality holds in (2.1) when $x_j = m$ and $x_i = M$ or $x_j = M$ and $x_i = m$, $i = 1, 2, \ldots, n$ and $i \neq j$. We now find bounds for x_j in terms of mean $(\bar{x})$ and central moment (m_{2r}) in the following corollary.

COROLLARY 2.2

With notations as above,

$$\bar{x} - \left(\frac{n\,(n-1)^{2r-1}}{1+(n-1)^{2r-1}} m_{2r}\right)^{\frac{1}{2r}} \leq x_j \leq \bar{x} + \left(\frac{n\,(n-1)^{2r-1}}{1+(n-1)^{2r-1}} m_{2r}\right)^{\frac{1}{2r}} \tag{2.8}$$

for all $j = 1, 2, \ldots, n$.

Proof. It follows from inequality (2.1) that

$$\left(x_j - \bar{x}\right)^2 \leq \left(\frac{n\,(n-1)^{2r-1}}{1+(n-1)^{2r-1}} m_{2r}\right)^{\frac{1}{r}}. \tag{2.9}$$

Since, $y^2 \leq a^2$ if and only if $-a \leq y \leq a$, the inequalities (2.8) follow from (2.9). □

Equality holds on the left(right) of (2.8) if and only if $n-1$ largest(smallest) x_i are all equal. The upper bound for the maximum deviation d from the mean is given in the following corollary.

COROLLARY 2.3

If d is a maximum deviation from the mean $(\bar{x})$ of n real numbers $x_1, x_2, \ldots, x_n$, then

$$d \leq \left(\frac{n\,(n-1)^{2r-1}}{1+(n-1)^{2r-1}} m_{2r}\right)^{\frac{1}{2r}}. \tag{2.10}$$

Proof. The inequality (2.9) is evidently equivalent to (2.10). □

Note that Samuelson's inequalities (1.3), (1.4) and (1.5) are respectively the special cases of inequalities (2.1), (2.8) and (2.10), $r = 1$. One of the interests in the special case $r = 2$ is that the sample kurtosis is defined in terms of the second and fourth central moment as

$$\beta_2 = \frac{m_4}{m_2^2}.$$

For $r = 2$, the inequality (2.8) is tighter than (1.4). This follows from the fact that [3]

$$\beta_2 \leq \frac{n^2 - 3n + 3}{n-1}.$$

For $r = 2$ and $n = 3$, the inequalities (1.4) and (2.8) give equal estimates. It also follows that if the value of the fourth central moment (m_4) is prescribed, then (2.10) gives a better bound for maximum deviation than the corresponding bound given by Samuelson's inequality (1.5).

3. Bounds for eigenvalues using traces

Let A be an $n \times n$ complex matrix with real eigenvalues as in case of a Hermitian matrix. Several inequalities for the eigenvalues in terms of traces of A, A^2 and A^3 are given in the literature (see [1, 12–14]). It is costly to calculate the traces of higher powers of A. However, it is always of interest to know if better estimates can be obtained at the cost of more calculations. We obtain here a generalization of inequality (1.7) that involves a trace of B^{2r}, $r = 1, 2, \ldots$ and we mention a number of examples to illustrate the utility of these bounds. The condition number of a positive definite matrix is the ratio of the largest to the smallest eigenvalue. The bounds for this ratio in terms of the traces are given in [14]. We show by means of examples that our bounds are also useful in estimating the upper limit of the condition number.

Theorem 3.1. *Let A be a complex $n \times n$ matrix with real eigenvalues λ_i, $i = 1, 2, \ldots, n$ and let $B = A - \frac{\operatorname{tr} A}{n} I$. Then,*

$$\frac{\operatorname{tr} A}{n} - \left(\frac{(n-1)^{2r-1}}{1+(n-1)^{2r-1}} \operatorname{tr} B^{2r}\right)^{\frac{1}{2r}} \le \lambda_i \le \frac{\operatorname{tr} A}{n} + \left(\frac{(n-1)^{2r-1}}{1+(n-1)^{2r-1}} \operatorname{tr} B^{2r}\right)^{\frac{1}{2r}} \tag{3.1}$$

for all $i = 1, 2, \ldots, n$, and $r = 1, 2, 3, \ldots$.

Proof. The arithmetic mean of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i = \frac{\operatorname{tr} A}{n}. \tag{3.2}$$

The eigenvalues of the matrix B are $\lambda_i - \frac{\operatorname{tr} A}{n}$. Therefore

$$m_{2r} = \frac{1}{n} \sum_{i=1}^{n} \left(\lambda_i - \frac{\operatorname{tr} A}{n}\right)^{2r} = \frac{1}{n} \operatorname{tr} B^{2r}. \tag{3.3}$$

The inequality (3.1) follows from (2.8), and substitute the values of $\bar{x}$ and m_{2r} from (3.2) and (3.3), respectively. □

Let λ_i be the eigenvalues of a Hermitian matrix A, and suppose $m \le \lambda_i \le M$, $i = 1, 2, \ldots, n$. Wolkowicz and Styan [14] have shown that if $\operatorname{tr} A > 0$ and

$$p = \frac{(\operatorname{tr} A)^2}{\operatorname{tr} A^2} - (n-1) > 0\,, \tag{3.4}$$

then A is positive definite and

$$\frac{M}{m} \le \frac{1 + \left(1 - p^2\right)^{\frac{1}{2}}}{p}. \tag{3.5}$$

The inequality (3.5) gives an upper bound for the condition number $\left(\frac{M}{m}\right)$. But, for example, for the matrix

$$A = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 6 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 8 \end{bmatrix}, \tag{3.6}$$

the value of $p < 0$ and therefore (3.5) is not applicable. The following corollary is stronger than (3.5).

COROLLARY 3.2

Let A be an $n \times n$ Hermitian matrix. If $\operatorname{tr} A > 0$ *and*

$$\frac{\operatorname{tr} A}{n} \geq \left(\frac{(n-1)^{2r-1}}{1 + (n-1)^{2r-1}} \operatorname{tr} B^{2r} \right)^{\frac{1}{2r}} \tag{3.7}$$

for some $r = 1, 2, 3, \ldots$, then A is positive definite and

$$\frac{M}{m} \leq \frac{\beta}{\alpha}, \tag{3.8}$$

where α and β are respectively the lower and upper bounds in (3.1).

Proof. The assertions of the corollary follow immediately from Theorem 3.1. If (3.7) holds, the first inequality (3.1) shows that all the eigenvalues of A are positive. □

The matrix A in (3.6) satisfies the condition of Corollary 3.2 for $r = 3$, therefore $\frac{M}{m} \leq 55.181$. This also shows that (3.7) is more useful than (3.4).

We now mention several examples to illustrate the effectiveness of the bounds in Theorem 3.1.

Example 1. For the 3×3 matrix,

$$A = \begin{bmatrix} 5 & 1 & 2-i \\ 1 & 1 & 1+2i \\ 2+i & 1-2i & 3 \end{bmatrix},$$

the estimates of Wolkowicz and Styan [14] and (3.1) with $r = 2$ both give $-1.4721 \leq \lambda_i \leq 7.4721$. For $r = 3$, we have from (3.1), $-1.325 \leq \lambda_i \leq 7.325$. The eigenvalues of A are $3, 3 \pm \sqrt{15}$.

We now borrow examples of Wolkowicz and Styan [14] to compare the bounds (3.5) and (3.8) for the condition number of a positive definite matrix.

Example 2. Let

$$A = \begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{bmatrix}.$$

The estimate (3.5) gives $\frac{M}{m} \leq 13.928$ (see [14]), while from (3.8), $\frac{M}{m} \leq 12.600$, $r = 2$.

Example 3. Let

$$A = \begin{bmatrix} 4 & 1 & 1 & 2 & 2 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 \\ 2 & 1 & 1 & 7 & 1 \\ 2 & 1 & 1 & 1 & 8 \end{bmatrix}.$$

From (3.5), $\frac{M}{m} \leq 36.973$ (see [14]), while from (3.8), $\frac{M}{m} \leq 18.918$, $r = 2$.

4. Bounds on roots of polynomials

In the theory of polynomial equations, the study of polynomials with real roots is of special interest (see [7, 9]). It is also of interest to find bounds on the roots in terms of the coefficients (see [6, 12]). One such bound is the Laguerre inequality (1.9) which gives an estimate for the roots of the polynomial eq. (1.8) in terms of its first two coefficients a_1 and a_2. It is sufficient to consider the polynomial equation in which the coefficient of x^{n-1} is zero,

$$f(x) = x^n + b_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_{n-1} x + b_n = 0. \tag{4.1}$$

The eq. (4.1) is obtained on diminishing the roots of (1.8) by $-\frac{a_1}{n}$. Then the Laguerre inequality says that all the roots of (4.1) lie in the interval $[-D_1, D_1]$, where $D_1 = \sqrt{-2\frac{(n-1)b_2}{n}}$. We show that better bounds can be given if we involve other coefficients of the eq. (4.1).

Let $x_1, x_2, \ldots, x_n$ be the roots of (4.1). On using the well-known Newton's identity

$$\alpha_k + b_1 \alpha_{k-1} + b_2 \alpha_{k-2} + \cdots + b_{k-1} \alpha_1 + k b_k = 0,$$

where $\alpha_k = \sum_{i=1}^n x_i^k$ and $k = 1, 2, \ldots, n$. We have

$$m_1 = 0,\ m_2 = \frac{1}{n} \sum_{i=1}^n x_i^2 = -\frac{2}{n} b_2,\ m_4 = \frac{1}{n} \sum_{i=1}^n x_i^4 = \frac{2}{n}(b_2^2 - 2b_4) \tag{4.2}$$

and

$$m_6 = \frac{1}{n} \sum_{i=1}^n x_i^6 = \frac{-2b_2^3 + 3b_3^2 + 6b_2 b_4 - 6b_6}{n}. \tag{4.3}$$

Using such relations in inequality (2.8) we can obtain better estimates for the roots of eq. (4.1). As an example, we mention here an improvement of Laguerre's inequality in the following theorem.

Theorem 4.1. *All the roots of the polynomial equation* (4.1) *with* $n \geq 5$ *lie in the interval* $[-D_2, D_2]$, *where*

$$D_2 = \left(\frac{2(n-1)^3}{1 + (n-1)^3} (b_2^2 - 2b_4) \right)^{\frac{1}{4}}.$$

Proof. From (2.8), for $m_1 = \bar{x} = 0$ and $r = 2$, we have

$$-\left(\frac{n\,(n-1)^3}{1+(n-1)^3}m_4\right)^{\frac{1}{4}} \leq x_j \leq \left(\frac{n\,(n-1)^3}{1+(n-1)^3}m_4\right)^{\frac{1}{4}}. \tag{4.4}$$

Substituting the value of m_4 from (4.2) in (4.4), we find that $\left|x_j\right| \leq D_2$. □

Example 4. Let

$$f(x) = x^5 + 25x^4 + 112x^3 + 96x^2 + 14x + \frac{1}{4} = 0. \tag{4.5}$$

If all the coefficients a_i's in the polynomial eq. (1.8) are positive and

$$a_{n-i}^2 - 4a_{n-i+1}a_{n-i-1} > 0, \quad i = 1, 2, \ldots, n-1,$$

then all its roots are real (see [5]). So, all the roots x_i of (4.5) are real, $i = 1, 2, \ldots, 5$. Let $y_i = x_i - 5$ be the roots of the diminished equation

$$f(y) = y^5 - 138y^3 + 916y^2 - 1921y + \frac{3321}{4} = 0.$$

The Laguerre inequality (1.9) gives $|y_i| \leq 14.859$ while from Theorem 4.1, $|y_i| \leq 14.57$.

Example 5. Let

$$f(x) = x^5 + 80x^4 + 1500x^3 + 5000x^2 + 3750x + \frac{1}{5} = 0. \tag{4.6}$$

The roots of $f\,(x)$ are real. Let $y = x - 16$, then (4.6) gives

$$f(y) = y^5 - 1060y^3 + 14920y^2 + 12710y - \frac{3648479}{5} = 0.$$

From (1.9), $|y_i| \leq 41.183$ while from Theorem 4.1, $|y_i| \leq 38.348$.

Acknowledgements

The authors are grateful to Prof. Rajendra Bhatia for useful discussions and suggestions, and to ISI, Delhi for a visit in January 2013 when this work had begun. The support of the UGC-SAP is acknowledged.

References

[1] Bhatia R, Matrix analysis (2000) (New York: Springer Verlag)
[2] Bhatia R and Davis C, A better bound on the variance, *Amer. Math. Monthly* **107** (2000) 353–357
[3] Dalen J, Algebraic bounds on standardized sample moments, *Statist. Prob. Lett.* **5** (1987) 329–331
[4] Jensen S T and Styan G P H, Some comments and a bibliography on the Laguerre–Samuelson inequality with extensions and applications in statistics and matrix theory, in: Analytic and Geometric Inequalities and applications (eds) H M Srivastava and T M Rassias (1999) (Kluwer) pp. 151–181

[5] Kurtz D C, A sufficient condition for all the roots of a polynomial to be real, *Amer. Math. Monthly* **99** (1992) 259–263

[6] Laguerre E N, Sur une methode pour obtenir par approximation les racines d'une equation algebrique qui a toutes ses raciness reelles, *Nouv. Ann. de Math.* **19** (1880) 161–171 & 193–202

[7] Marden M, Geometry of polynomials (2005) (American Mathematical Society)

[8] Olkin I, A matrix formulation on how deviant an observation can be, *The Amer. Statist.* **46** (1992) 205–209

[9] Rahman Q I and Schmeisser G, Analytic theory of polynomials (2002) (Oxford University Press)

[10] Samuelson P A, How deviant can you be?, *J. Amer. Statist. Assoc.* **63** (1968) 1522–1525

[11] Sharma R, Kaura A, Gupta M and Ram S, Some bounds on sample parameters with refinements of Samuelson and Brunk inequalities, *J. Math. Inequal.* **3** (2009) 99–106

[12] Sharma R, Gupta M and Kapoor G, Some better bounds on the variance with applications, *J. Math. Inequal.* **4** (2010) 355–363

[13] Sharma R, Bhandari R and Gupta M, Inequalities related to the Cauchy–Schwarz inequality, *Sankhya* **74-A** (2012) 101–111

[14] Wolkowicz H and Styan G P H, Bounds for eigenvalues using traces, *Linear Algebra Appl.* **29** (1980) 471–506

COMMUNICATING EDITOR: Parameswaran Sankaran