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Multiplicative perturbations of local C-semigroups

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Abstract. In this paper, we establish some left and right multiplicative perturbation theorems concerning local C-semigroups when the generator A of a perturbed local C-semigroup $S(\cdot)$ may not be densely defined and the perturbation operator B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$, which can be applied to obtain some additive perturbation theorems for local C-semigroups in which B is a bounded linear operator from $[D(A)]$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$. We also show that the perturbations of a (local) C-semigroup $S(\cdot)$ are exponentially bounded (resp., norm continuous, locally Lipschitz continuous, or exponentially Lipschitz continuous) if $S(\cdot)$ is.

Keywords. Local C-semigroup; generator; abstract Cauchy problem; perturbation.

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1. Introduction

Throughout this paper, we always assume that X is a Banach space with norm $\|\cdot\|$, and $L(X)$ denotes the set of all bounded linear operators on X. Let $0 < T_0 \leq \infty$, and $C \in L(X)$ be injective. A family $S(\cdot)(= \{S(t)\,|\,0 \leq t < T_0\})$ in $L(X)$ is called a local C-semigroup on X if it is strongly continuous, $S(0) = C$ on X and satisfies

(1.1) $S(t)S(s) = S(t+s)C$ for all $0 \leq t, s, t+s < T_0$ (see [8]). In this case, the generator of $S(\cdot)$ is a linear operator A in X defined by $D(A) = \{x \in X \mid \lim_{h\to 0^+}(S(h)x - Cx)/h \in R(C)\}$ and $Ax = C^{-1}\lim_{h\to 0^+}(S(h)x - Cx)/h$ for $x \in D(A)$;
(1.2) We say that $S(\cdot)$ is locally Lipschitz continuous, if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that $\|S(t+h) - S(t)\| \leq K_{t_0}h$ for all $0 \leq t, h, t+h \leq t_0$;
(1.3) Exponentially bounded, if $T_0 = \infty$ and there exist $K, \omega \geq 0$ such that $\|S(t)\| \leq K\mathrm{e}^{\omega t}$ for all $t \geq 0$;
(1.4) Exponentially Lipschitz continuous, if $T_0 = \infty$ and there exist $K, \omega \geq 0$ such that $\|S(t+h) - S(t)\| \leq Kh\mathrm{e}^{\omega(t+h)}$ for all $t, h \geq 0$.

Perturbation of local C-semigroups is one of the subjects in the theory of semigroup which has been extensively studied by many authors for multiplicative and additive perturbations of c_0-semigroups and C-semigroups appearing in [1, 2, 6, 7, 10, 12, 13].

Liang *et al.* [5] have also obtained some right-multiplicative perturbation theorems for local C-semigroups in which the operator C may not commute with the bounded perturbation operator B on X and satisfies an estimation that is similar to the condition (2.6) below. In this case, $C^{-1}A(I+B)C$ generates a local C-semigroup on X when $CA(I+B) \subset A(I+B)C$. Along this line, they also establish some left-multiplicative perturbation theorems for local C-semigroups on X with densely defined generators. In this case, $(I+B)A$ generates a local C-semigroup on X when $C^{-1}(I+B)AC = (I+B)A$ (see [14]); this can be applied to obtain an extension of the classical additive perturbation for c_0-semigroups as in [6]. Unfortunately, the growth conditions of additive and multiplicative perturbations of $S(\cdot)$ are not considered in [5, 14]. Recently, the author obtained some additive perturbation theorems in [3] for local C-semigroups with or without the exponential boundedness (resp., norm continuity, local Lipschitz continuity, or exponential Lipschitz continuity) in which the perturbation operator B may not be a bounded linear operator in X. Some results concerning this topic are also investigated in [4, 8, 9] except for growth conditions. The purpose of this paper is to establish some left and right multiplicative perturbation theorems for local C-semigroups with or without exponential boundedness (resp., norm continuity, local Lipschitz continuity, or exponential Lipschitz continuity) as in [5, 14] except for growth conditions. In [14], the assumption of the generator A of a perturbed local C-semigroup $S(\cdot)$ is densely defined and can be deleted here. The perturbation operator B is only a bounded linear operator on $\overline{D(A)}$, and the assumption of $C^{-1}(I+B)AC = (I+B)A$ is not necessary (see Theorems 2.5 and 2.9 below), which can be applied to obtain some new left and right multiplicative perturbation theorems for locally Lipschitz continuous local C-semigroups (see Theorems 2.6 and 2.10 below) and to extend some Miyadera type additive perturbation results in [14] for local C-semigroups with or without the exponential boundedness (resp., norm continuity, local Lipschitz continuity, or exponential Lipschitz continuity) (see Theorems 2.13–2.14 and 2.17–2.18 below). An illustrative example concerning these results is also presented in this paper.

2. Perturbation theorems

In this section, we first note some basic properties of a local C-semigroup and known results about connections between the generator of a local C-semigroup and strong solutions of the following abstract Cauchy problem:

$$ACP(A, f, x)\begin{cases} u'(t) = Au(t) + f(t) & \text{for } t \in (0, T_0),\\ u(0) = x, \end{cases}$$

where $x \in X$ and f is an X-valued function defined on a subset of $[0, T_0)$.

PROPOSITION 2.1 (see [8, 11])

Let A be the generator of a local C-semigroup $S(\cdot)$ on X. Then

(2.1) $S(t)S(s) = S(s)S(t)$ *for* $0 \le t, s, t+s < T_0$;
(2.2) A *is closed and* $C^{-1}AC = A$;
(2.3) $S(t)x \in D(A)$ *and* $S(t)Ax = AS(t)x$ *for* $x \in D(A)$ *and* $0 \le t < T_0$;
(2.4) $\int_0^t S(r)x\mathrm{d}r \in D(A)$ *and* $A\int_0^t S(r)x\mathrm{d}r = S(t)x - Cx$ *for* $x \in X$ *and* $0 \le t < T_0$;
(2.5) $R(S(t)) \subset \overline{D(A)}$ *for* $0 \le t < T_0$.

DEFINITION 2.2

Let $A : D(A) \subset X \to X$ be a closed linear operator in a Banach space X with domain $D(A)$ and range $R(A)$. A function $u : [0, T_0) \to X$ is called a (strong) solution of $ACP(A, f, x)$ if $u \in C^1((0, T_0), X) \cap C([0, T_0), X) \cap C((0, T_0), [D(A)])$ and satisfies $ACP(A, f, x)$. Here $[D(A)]$ denotes the Banach space $D(A)$ with norm $|\cdot|$ defined by $|x| = \|x\| + \|Ax\|$ for all $x \in D(A)$.

PROPOSITION 2.3 (see [8])

Let $A : D(A) \subset X \to X$ be a closed linear operator in X such that $AC = CA$ on $D(A)$. Assume that for each $x \in X$ $ACP(A, Cx, 0)$ has a unique (strong) solution u in $C^1([0, T_0), X)$. Then $C^{-1}AC$ generates a local C-semigroup $S(\cdot)$ on X. In this case, $S(\cdot)x = u'(\cdot)$ on $[0, T_0)$.

PROPOSITION 2.4 (see [8])

Let A be the generator of a local C-semigroup $S(\cdot)$ on X, $x \in X$ and $f \in L^1_{\mathrm{loc}}([0, T_0), X) \cap C((0, T_0), X)$. Then $ACP(A, Cf, Cx)$ has a (strong) solution u in $C^1([0, T_0), X)$ if and only if $v(\cdot) = S(\cdot)x + S * f(\cdot) \in C^1([0, T_0), X)$. In this case, $u = v$ on $[0, T_0)$. Here $S * f(\cdot) = \int_0^{\cdot} S(\cdot - s) f(s)\mathrm{d}s$.

The next result is a right-multiplicative perturbation theorem for local C-semigroups which is an extension of Desch–Schappacher type perturbation results for c_0-semigroups as given in [6, 7, 12], and has been established by Liang *et al.* in [5] by another method for which $C^{-1}A(I + B)C$ generates a local C-semigroup on X when B is a bounded linear operator from X into $R(C)$, $CA(I + B) \subset A(I + B)C$ and the generator A of a perturbed local C-semigroup is also not densely defined.

Theorem 2.5. *Let $S(\cdot)$ be a local C-semigroup on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$, and for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that $(S * C^{-1}Bf)(t) \in D(A)$ and*

$$\|A(S * C^{-1}Bf)(t)\| \le M_{t_0} \int_0^t \|f(s)\|\mathrm{d}s \tag{2.6}$$

for all $f \in C([0, t_0], \overline{D(A)})$ and $0 \le t \le t_0$. Then $A(I + C^{-1}BC)$ generates a local C-semigroup $T(\cdot)$ on X satisfying

$$T(\cdot)x = S(\cdot)x + A(S * C^{-1}BT)(\cdot)x \quad \text{on } [0, T_0) \tag{2.7}$$

for all $x \in X$. Here I denotes the identity operator on X.

Proof. Indeed, fix $x \in X$ and $0 < t_0 < T_0$. We define $U : C([0, t_0], \overline{D(A)}) \to C([0, t_0], \overline{D(A)})$ by $U(f)(\cdot) = S(\cdot)x + A(S * C^{-1}Bf)(\cdot)$ on $[0, t_0]$ for all $f \in C([0, t_0], \overline{D(A)})$, and so U is well-defined. By induction, we obtain from (2.6) that

$$\begin{aligned}\|U^n f(t) - U^n g(t)\| &= \|A(S * C^{-1}B(U^{n-1}f - U^{n-1}g)(t))\| \\ &\le M_{t_0}^n j_n(t_0)\|f - g\|\end{aligned}$$

for all $f, g \in C([0, t_0], \overline{D(A)})$, $0 \le t \le t_0$ and $n \in \mathbb{N}$. Here $j_k(t) = t^k/k!$ for $t \ge 0$ and $k \in \mathbb{N} \cup \{0\}$ and $\|f - g\| = \max_{0 \le s \le t_0} \|f(s) - g(s)\|$. It follows from the contraction mapping theorem that there exists a unique function w_{x,t_0} in $C([0, t_0], \overline{D(A)})$ such that $w_{x,t_0}(\cdot) = S(\cdot)x + AS * C^{-1}Bw_{x,t_0}(\cdot)$ on $[0, t_0]$ and $Cw_{x,t_0} = w_{Cx,t_0}$. In this case, we set $w_x(t) = w_{x,t_0}(t)$ for all $0 \le t \le t_0 < T_0$, then $w_x(\cdot)$ is a unique function in $C([0, T_0), \overline{D(A)})$ such that $w_x(\cdot) = S(\cdot)x + AS * C^{-1}Bw_x(\cdot)$ on $[0, T_0)$. Since $j_0 * w_x(\cdot) = j_0 * S(\cdot)x + Aj_0 * S * C^{-1}Bw_x(\cdot) = j_0 * S(\cdot)x + S * C^{-1}Bw_x(\cdot) - Bj_0 * w_x(\cdot)$ on $[0, T_0)$, we have $(I + B)j_0 * w_x(t) = j_0 * S(t)x + S * C^{-1}Bw_x(t) \in D(A)$ for all $0 \le t < T_0$. Clearly, $j_0 * w_x$ is the unique function u_x in $C^1([0, T_0), X)$ such that $u_x(\cdot) = j_0 * S(\cdot)x + Aj_0 * S * C^{-1}Bu_x(\cdot)$ on $[0, T_0)$. Since $j_0 * S(\cdot)x + S * C^{-1}Bw_x(\cdot) \in C^1([0, T_0), X)$, we obtain from Proposition 2.4 that $j_0 * S(\cdot)x + S * C^{-1}Bw_x(\cdot) = (I + B)j_0 * w_x(\cdot)$ is the unique solution of $ACP(A, Cx + Bw_x, 0)$ in $C^1([0, T_0), X)$. This implies that $A(I + B)j_0 * w_x + Cx + Bw_x = (I + B)w_x$ on $[0, T_0)$. Since

$$\begin{aligned} A(I + B)j_0 * w_x(\cdot) &= A(j_0 * S(\cdot)x + S * C^{-1}Bw_x(\cdot)) \\ &= w_x(\cdot) - Cx \end{aligned}$$

on $[0, T_0)$, we have that $j_0 * w_x$ is a solution of $ACP(A(I + B), Cx, 0)$ in $C^1([0, T_0), X)$. Next if u is a function in $C([0, T_0), X)$ such that $A(I+B)j_0*u+Cx = u$ on $[0, T_0)$, then

$$\begin{aligned} j_0 * (S * u - S * j_0Cx) &= Aj_0 * S * (I + B)j_0 * u \\ &= S * (I + B)j_0 * u - Cj_0 * (I + B)j_0 * u \end{aligned}$$

on $[0, T_0)$, and so $-S * j_1(\cdot)Cx = S * Bj_0 * u(\cdot) - Cj_0 * (I + B)j_0 * u(\cdot)$ on $[0, T_0)$. Hence

$$\begin{aligned} -S * j_0(\cdot)x &= (S * C^{-1}Bj_0 * u)'(\cdot) - (I + B)j_0 * u(\cdot) \\ &= A(S * C^{-1}Bj_0 * u)(\cdot) - j_0 * u(\cdot) \end{aligned}$$

on $[0, T_0)$, which implies that $j_0 * u(\cdot) = S * j_0(\cdot)x + A(S * C^{-1}Bj_0 * u)(\cdot)$ on $[0, T_0)$. Consequently, $j_0 * u = j_0 * w_x$ on $[0, T_0)$ or equivalently, $u = w_x$ on $[0, T_0)$. Clearly, $A(I + B)$ is closed and $A(I + B)C = CA(I + B)$ on $D(A(I + B))$. It follows from Proposition 2.3 that $C^{-1}A(I + B)C$ generates a local C-semigroup $T(\cdot)$ on X satisfying (2.7) for all $x \in X$. Since $R((I + B)C) \subseteq R(C)$ and $C^{-1}(I + B)C = I + C^{-1}BC$, we have $(I + B)C = C(I + C^{-1}BC)$, and so $A(I + B)C = AC(I + C^{-1}BC)$. Hence $C^{-1}A(I + B)C = A(I + C^{-1}BC)$. □

Just as an application of Theorem 2.5, we can obtain the next new right-multiplicative perturbation theorem concerning locally Lipschitz continuous local C-semigroups.

Theorem 2.6. *Let $S(\cdot)$ be a locally Lipschitz continuous local C-semigroup on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$, and for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that $(S * C^{-1}Bf)(t) \in D(A)$ and*

$$\|A(S * C^{-1}B)[f(t) - f(s)]\| \le M_{t_0} \int_s^t \|f(r)\| \mathrm{d}r \tag{2.8}$$

for all $f \in C([0, t_0], \overline{D(A)})$ and $0 \le s \le t \le t_0$. Then $A(I + C^{-1}BC)$ generates a locally Lipschitz continuous local C-semigroup $T(\cdot)$ on X satisfying (2.7) for all $x \in X$.

COROLLARY 2.7

Let $S(\cdot)$ be a locally Lipschitz continuous local C-semigroup on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$ and $C^{-1}Bx \in \overline{D(A)}$ for all $x \in \overline{D(A)}$. Then $A(I + C^{-1}BC)$ generates a locally Lipschitz continuous local C-semigroup $T(\cdot)$ on X satisfying (2.7) *for all $x \in X$. Moreover, $T(\cdot)$ is exponentially Lipschitz continuous if $S(\cdot)$ is.*

Proof. Clearly, it suffices to show that for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that (2.8) holds for all $f \in C([0, t_0], \overline{D(A)})$ and $0 \leq s < t \leq t_0$. Indeed, if $S_1(t)$ denotes the restriction of $S(t)$ to $\overline{D(A)}$ and $S_1'(t)$ is the strong derivative of $S_1(t)$ on $\overline{D(A)}$ for all $0 \leq t < T_0$, then $S_1(t)x = Cx + Aj_0 * S(t)x$ and $S_1'(t)x = AS(t)x$ for all $x \in \overline{D(A)}$ and $0 \leq t < T_0$, and so $AS(\cdot)$ is a strongly continuous family of bounded linear operators on $\overline{D(A)}$. Now if $0 < t_0 < T_0$ is given, then $S * C^{-1}Bf(\cdot)$ is continuously differentiable on $[0, t_0]$ and $D^1(S * C^{-1}Bf)(\cdot) = A(S * C^{-1}Bf)(\cdot) + Bf(\cdot) = S_1' * C^{-1}Bf(\cdot) + Bf(\cdot)$ for all $f \in C([0, t_0], \overline{D(A)})$ and $0 \leq t \leq t_0$. Here D^1 denotes the derivative of a function. Now if $0 < t_0 < T_0$ is fixed, then

$$\begin{aligned} \|A(S * C^{-1}B)[f(t) - f(s)]\| &= \|S_1' * C^{-1}B[f(t) - f(s)]\| \\ &\leq \sup_{0 \leq r \leq t_0} \|S_1'(r)\| \|C^{-1}B\| \int_s^t \|f(r)\| \mathrm{d}r, \end{aligned}$$

$f \in C([0, t_0], \overline{D(A)})$ and $0 \leq s < t \leq t_0$. It follows from Theorem 2.6 that $A(I + C^{-1}BC)$ generates a locally Lipschitz continuous local C-semigroup $T(\cdot)$ on X satisfying (2.7) for all $x \in X$. Now if $S(\cdot)$ is exponentially Lipschitz continuous, then $S_1'(\cdot)$ is an exponentially bounded family in $L(\overline{D(A)})$. Combining this with the local Lipschitz continuity of $C^{-1}BT(\cdot)$, we get that $S_1' * C^{-1}BT(\cdot)$ is exponentially Lipschitz continuous, which together with (2.7) implies that $T(\cdot)$ is exponentially Lipschitz continuous if $S(\cdot)$ is. □

COROLLARY 2.8

Let $S(\cdot)$ be a local C-semigroup on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$ and $C^{-1}Bx \in D(A)$ for all $x \in \overline{D(A)}$. Then $A(I + C^{-1}BC)$ generates a local C-semigroup $T(\cdot)$ on X satisfying

$$T(\cdot)x = S(\cdot)x + S * AC^{-1}BT(\cdot)x \quad \text{on } [0, T_0) \tag{2.9}$$

for all $x \in X$. Moreover, $T(\cdot)$ is exponentially bounded (resp., norm continuous) if $S(\cdot)$ is.

Proof. Clearly, $S * AC^{-1}BT(\cdot)x = SAC^{-1}B * T(\cdot)x$ on $[0, T_0)$ for all $x \in X$, and $S(\cdot)AC^{-1}B$ is also exponentially bounded (resp., norm continuous) if $S(\cdot)$ is. By (2.9), we have

$$T(\cdot)x = S(\cdot)x + SAC^{-1}B * T(\cdot)x \quad \text{on } [0, T_0) \tag{2.10}$$

for all $x \in X$, which together with Gronwall's inequality implies that $T(\cdot)$ is exponentially bounded if $S(\cdot)$ is. Similarly, we can apply (2.10) to show that $T(\cdot)$ is also norm continuous if $S(\cdot)$ is. □

Under the assumptions of Theorem 2.5, the next left-multiplicative perturbation theorem for local C-semigroups on X is also obtained in which the generator A of a perturbed local C-semigroup $S(\cdot)$ may not be densely defined, $C^{-1}(I+B)AC$ and $(I+B)A$ both may not be equal, and $\rho((I+C^{-1}BC)A)$ (the resolvent set of $(I+C^{-1}BC)A$) is nonempty. This has been established by Liang *et al.* in [4] by another method when B is a bounded linear operator from X into $R(C)$, $CA(I+B) \subset A(I+B)C$, $\rho((I+B)A)$ and the generator A of a perturbed local C-semigroup is also densely defined. We also show that $U(\cdot)$ is exponentially bounded (resp., norm continuous, locally Lipschitz continuous, or exponentially Lipschitz continuous) if $T(\cdot)$ is, where $T(\cdot)$ is given as in (2.7).

Theorem 2.9. *Under the assumptions of Theorem* 2.5, *assume that* $\rho((I+C^{-1}BC)A)$ *is nonempty. Then* $(I+C^{-1}BC)A$ *generates a local C-semigroup* $U(\cdot)$ *on* X *satisfying*

$$\begin{aligned} U(\cdot)x &= Cx + [\lambda - (I+C^{-1}BC)A] \\ &\quad \times (I+C^{-1}BC)j_0 * T(\cdot)A[\lambda - (I+C^{-1}BC)A]^{-1}x \end{aligned} \tag{2.11}$$

on $[0, T_0)$ *for all* $x \in X$. *Moreover,* $U(\cdot)$ *is exponentially bounded* (*resp., norm continuous, locally Lipschitz continuous, or exponentially Lipschitz continuous*) *if* $T(\cdot)$ *is. Here* $\lambda \in \rho((I+C^{-1}BC)A)$ *is fixed and* $T(\cdot)$ *is given as in* (2.7).

Proof. We shall first show that $(I+C^{-1}BC)ACx = C(I+C^{-1}BC)Ax$ for all $x \in D((I+C^{-1}BC)A)$. Indeed, if $x \in D((I+C^{-1}BC)A) = D(A)$ is given, then $Cx \in D(A)$ and

$$\begin{aligned} (I+C^{-1}BC)ACx &= ACx + (C^{-1}BC)ACx = CAx + C^{-1}(BC)CAx \\ &= CAx + C^{-1}(CB)CAx \\ &= CAx + B(CA)x \\ &= CAx + C(C^{-1}(BC)Ax) \\ &= C(Ax + C^{-1}BCAx) \\ &= C(I+C^{-1}BC)Ax, \end{aligned}$$

and so $(I+C^{-1}BC)AC = C(I+C^{-1}BC)A$ on $D((I+C^{-1}BC)A)$. Since $\rho((I+C^{-1}BC)A)$ is nonempty, we have $C^{-1}(I+C^{-1}BC)AC = (I+C^{-1}BC)A$. Now if $T(\cdot)$ is given as in (2.7), we set $P = I + C^{-1}BC$ and $u_x(\cdot) = Cx + (\lambda - PA)Pj_0 * T(\cdot)A(\lambda - PA)^{-1}x$ on $[0, T_0)$ for all $x \in X$, then $u_x \in C([0, T_0), X)$ and

$$\begin{aligned} A(\lambda - PA)^{-1}u_x(\cdot) &= A(\lambda - PA)^{-1}Cx + T(\cdot)A(\lambda - PA)^{-1}x \\ &\quad -CA(\lambda - PA)^{-1}x \\ &= T(\cdot)A(\lambda - PA)^{-1}x \end{aligned}$$

on $[0, T_0)$, and so $PA(\lambda - PA)^{-1}j_0 * u_x(\cdot) = Pj_0 * T(\cdot)A(\lambda - PA)^{-1}x$ on $[0, T_0)$. Hence

$$\begin{aligned} -j_0 * u_x(\cdot) + \lambda(\lambda - PA)^{-1}j_0 * u_x(\cdot) &= PA(\lambda - PA)^{-1}j_0 * u_x(\cdot) \\ &= (\lambda - PA)^{-1}u_x(\cdot) - (\lambda - PA)^{-1}Cx \end{aligned}$$

on $[0, T_0)$, which implies that $j_0 * u_x(t) \in D(PA)$ for all $0 \le t < T_0$. Consequently, $PA(\lambda - PA)^{-1}j_0 * u_x(t) \in D(PA)$ for all $0 \le t < T_0$ and $PAj_0 * u_x = u_x - Cx$ on $[0, T_0)$. This shows that $j_0 * u_x$ is a solution of $ACP(PA, Cx, 0)$ in $C^1([0, T_0), X)$. In order to show the uniqueness, indeed, if $v \in C([0, T_0), X)$ is given and satisfies $v = PAj_0 * v$ on $[0, T_0)$. We next set $u = A(\lambda - PA)^{-1}v$ on $[0, T_0)$, then

$$\begin{aligned} Pj_0 * u &= (\lambda - PA)^{-1}PAj_0 * v \\ &= (\lambda - PA)^{-1}v \end{aligned}$$

on $[0, T_0)$, and so $APj_0 * u = A(\lambda - PA)^{-1}v = u$ on $[0, T_0)$. Hence $u = 0$ on $[0, T_0)$, which implies that $(\lambda - PA)^{-1}v = 0$ on $[0, T_0)$ or equivalently, $v = 0$ on $[0, T_0)$. We conclude from Proposition 2.3 that $(I + C^{-1}BC)A$ generates a local C-semigroup $U(\cdot)$ on X satisfying (2.11) for all $x \in X$. Clearly, for each $y \in X$, $(PA)Pj_0 * T(\cdot)y = P(AP)j_0 * T(\cdot)y = PT(\cdot)y - PCy$ on $[0, T_0)$. It follows from the right-hand side of (2.11) that $U(\cdot)$ is also exponentially bounded (resp., norm continuous, locally Lipschitz continuous, or exponentially Lipschitz continuous) if $T(\cdot)$ is. □

Under the assumptions of Theorem 2.6, the next new left-multiplicative perturbation theorem for locally Lipschitz continuous local C-semigroups on X is also obtained in which the generator A of a perturbed local C-semigroup $S(\cdot)$ may not be densely defined, $C^{-1}(I+B)AC$ and $(I+B)A$ both may not be equal, and $\rho((I+C^{-1}BC)A)$ is nonempty.

Theorem 2.10. *Under the assumptions of Theorem* 2.6, *assume that* $\rho((I + C^{-1}BC)A)$ *is nonempty. Then* $(I + C^{-1}BC)A$ *generates a locally Lipschitz continuous local* C*-semigroup* $U(\cdot)$ *on* X *satisfying* (2.11) *for all* $x \in X$.

COROLLARY 2.11

Under the assumptions of Corollary 2.7, *assume that* $\rho((I + C^{-1}BC)A)$ *is nonempty. Then* $(I + C^{-1}BC)A$ *generates a locally Lipschitz continuous local* C*-semigroup* $U(\cdot)$ *on* X *satisfying* (2.11) *for all* $x \in X$. *Moreover,* $U(\cdot)$ *is exponentially Lipschitz continuous if* $S(\cdot)$ *is.*

COROLLARY 2.12

Under the assumptions of Corollary 2.8, *assume that* $\rho((I + C^{-1}BC)A)$ *is nonempty. Then* $(I + C^{-1}BC)A$ *generates a local* C*-semigroup* $U(\cdot)$ *on* X *satisfying* (2.11) *for all* $x \in X$. *Moreover,* $U(\cdot)$ *is exponentially bounded* (*resp., norm continuous*) *if* $S(\cdot)$ *is.*

Just as an application of Theorem 2.9, the next additive perturbation theorem for local C-semigroups on X is also attained in which the generator A of a perturbed local C-semigroup may not be densely defined, and $\rho_C(A)$ (the C-resolvent set of A) and $\rho(A+B)$ are both nonempty.

Theorem 2.13. *Let* $S(\cdot)$ *be a local* C*-semigroup on* X *with generator* A, *and let* B *be a bounded linear operator from* $[D(A)]$ *into* $R(C^2)$ *such that* $CB = BC$ *on* $D(A)$. *Assume that* $\rho_C(A)$ *and* $\rho(A + B)$ *both are nonempty, and for each* $0 < t_0 < T_0$ *there exists an* $M_{t_0} > 0$ *such that*

$$|S * C^{-2}Bf(t)| \leq M_{t_0} \int_0^t |f(s)|\mathrm{d}s \tag{2.12}$$

for all $f \in C([0, t_0], [D(A)])$ *and* $0 \le t \le t_0$. *Then* $A+B$ *generates a local C-semigroup* $V(\cdot)$ *on* X.

Proof. Indeed, if $\lambda \in \rho_C(A)$ is fixed, then $A-\lambda$ is the generator of the local C-semigroup $\tilde{S}(\cdot)$ on X defined by $\tilde{S}(t) = e^{-\lambda t}S(t)$ for all $0 \le t < T_0$ and $(A-\lambda)^{-1}C^2 = C(A-\lambda)^{-1}C$ on X. Clearly, $\tilde{S}(\cdot)$ is also exponentially bounded (resp., norm continuous, locally Lipschitz continuous, or exponentially Lipschitz continuous) if $S(\cdot)$ is. Since the norm $|\cdot|_{A-\lambda}$ on $D(A)$ defined by $|x|_{A-\lambda} = \|x\| + \|(A-\lambda)x\|$ for all $x \in D(A)$ is equivalent to $|\cdot|$, we may assume that (2.12) holds under $|\cdot|_{A-\lambda}$. Now let $\tilde{B}$ denote the restriction of $C^{-1}B(A-\lambda)^{-1}C$ to $\overline{D(A)}$, then $\tilde{B}$ is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $\tilde{B}C = C^{-1}B(A-\lambda)^{-1}CC = B(A-\lambda)^{-1}C = CC^{-1}B(A-\lambda)^{-1}C = C\tilde{B}$ on $\overline{D(A)}$. Since

$$\begin{aligned}
(I + C^{-1}\tilde{B}C)(A-\lambda) &= (A-\lambda) + (C^{-1}\tilde{B}C)(A-\lambda) \\
&= (A-\lambda) + C^{-1}(C^{-1}B(A-\lambda)^{-1}C)C(A-\lambda) \\
&= (A-\lambda) + C^{-1}(C^{-1}BC(A-\lambda)^{-1}C)(A-\lambda) \\
&= (A-\lambda) + C^{-1}(B(A-\lambda)^{-1}C)(A-\lambda) \\
&= (A-\lambda) + C^{-1}B((A-\lambda)^{-1}C)(A-\lambda) \\
&= (A-\lambda) + C^{-1}BC = A - \lambda + B
\end{aligned}$$

and $\rho(A+B)$ is nonempty, we have that $\rho((I + C^{-1}\tilde{B}C)(A-\lambda))$ is nonempty. Next if $0 < t_0 < T_0$ and $f \in C([0, t_0], \overline{D(A)})$ are given. Applying integration by parts, we get that

$$\begin{aligned}
\tilde{S} * C^{-1}\tilde{B}f(t) &= S * C^{-1}\tilde{B}f(t) + \lambda\int_0^t e^{-\lambda(t-s)} S * C^{-1}\tilde{B}f(s)ds \\
&= S * C^{-2}B(A-\lambda)^{-1}Cf(t) \\
&\quad + \lambda\int_0^t e^{-\lambda(t-s)} S * C^{-2}B(A-\lambda)^{-1}Cf(s)ds \qquad (2.13)
\end{aligned}$$

for all $0 \le t \le t_0$, and so

$$\begin{aligned}
&\|(A-\lambda)\tilde{S} * C^{-1}\tilde{B}f(t)\| \\
&\quad \le \|(A-\lambda)S * C^{-2}B(A-\lambda)^{-1}Cf(t)\| \\
&\qquad + \|\lambda\int_0^t e^{-\lambda(t-s)} S * C^{-2}B(A-\lambda)^{-1}Cf(s)ds\| \\
&\quad \le M_{t_0}\int_0^t |(A-\lambda)^{-1}Cf(s)|_{A-\lambda}ds \\
&\qquad + |\lambda|\int_0^t e^{-\lambda(t-s)} M_{t_0}\int_0^s |(A-\lambda)^{-1}Cf(r)|_{A-\lambda}drds \\
&\quad \le M_{t_0}(\|(A-\lambda)^{-1}C\| + \|C\|)\int_0^t \|f(s)\|ds \\
&\qquad + |\lambda|M_{t_0}(\|(A-\lambda)^{-1}C\| + \|C\|)\int_0^t e^{-\lambda(t-s)}\int_0^s \|f(r)\|drds \\
&\quad \le \widetilde{M_{t_0}}\int_0^t \|f(s)\|ds \qquad (2.14)
\end{aligned}$$

for all $0 \le t \le t_0$. Here $\widetilde{M_{t_0}} = M_{t_0}(1 + 1 + e^{|\lambda|t_0})(\|(A-\lambda)^{-1}C\| + \|C\|)$. It follows from Theorem 2.9 that $A + B - \lambda$ generates a local C-semigroup $\tilde{U}(\cdot)$ on X, which implies that $A + B$ generates a local C-semigroup $V(\cdot)$ on X defined by $V(t) = e^{\lambda t}\tilde{U}(t)$ for all $0 \leq t < T_0$. □

Combining Theorem 2.6 with Theorem 2.13, we can apply (2.13) and a similar estimation of (2.14) to obtain the next new additive perturbation theorem concerning locally Lipschitz continuous local C-semigroups on X in which the generator A of a perturbed local C-semigroup may not be densely defined, and $\rho_C(A)$ and $\rho(A + B)$ are both nonempty.

Theorem 2.14. *Let $S(\cdot)$ be a locally Lipschitz continuous local C-semigroup on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C^2)$ such that $CB = BC$ on $D(A)$. Assume that $\rho_C(A)$ and $\rho(A + B)$ are both nonempty, and for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that*

$$|S * C^{-2}B[f(t) - f(s)]| \leq M_{t_0} \int_s^t |f(r)|\mathrm{d}r \tag{2.15}$$

for all $f \in C([0, t_0], [D(A)])$ and $0 \leq s < t \leq t_0$. Then $A + B$ generates a locally Lipschitz continuous local C-semigroup $V(\cdot)$ on X.

Just as in the proof of Corollary 2.7, we can apply Theorem 2.14 to obtain the next additive perturbation result for locally Lipschitz continuous and exponentially Lipschitz continuous local C-semigroups on X.

COROLLARY 2.15

Let $S(\cdot)$ be a locally Lipschitz continuous local C-semigroup on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C^2)$ such that $CB = BC$ on $D(A)$ and $C^{-2}Bx \in \overline{D(A)}$ for all $x \in D(A)$. Assume that $\rho_C(A)$ and $\rho(A + B)$ are both nonempty. Then $A + B$ generates a locally Lipschitz continuous local C-semigroup $V(\cdot)$ on X given as in the proof of Theorem 2.13. *Moreover, $V(\cdot)$ is exponentially Lipschitz continuous if $S(\cdot)$ is.*

Just as in the proof of Corollary 2.8, we can apply Theorem 2.13 to obtain the next corollary.

COROLLARY 2.16

Let $S(\cdot)$ be a local C-semigroup on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C^2)$ such that $CB = BC$ on $D(A)$ and $C^{-2}Bx \in D(A)$ for all $x \in D(A)$. Assume that $\rho_C(A)$ and $\rho(A + B)$ are both nonempty. Then $A + B$ generates a local C-semigroup $V(\cdot)$ on X given as in the proof of Theorem 2.13. *Moreover, $V(\cdot)$ is exponentially bounded* (*resp.*, *norm continuous*) *if $S(\cdot)$ is.*

By slightly modifying the proof of Theorem 2.13, the next additive perturbation result is attained with the assumption that B is a bounded linear operator from $[D(A)]$ into $R(C^2)$, and $\rho_C(A)$ is nonempty are replaced by assuming that B is a bounded linear operator from $[D(A)]$ into $R(C)$ and $\rho(A)$ is nonempty.

Theorem 2.17. *Let $S(\cdot)$ be a local C-semigroup on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C)$ such that $CB = BC$ on $D(A)$. Assume that $\rho(A)$ and $\rho(A+B)$ are both nonempty, and for each $0 < t_0 < T_0$, there exists an $M_{t_0} > 0$ such that*

$$|S * C^{-1}Bf(t)| \leq M_{t_0} \int_0^t |f(s)|\mathrm{d}s \tag{2.16}$$

for all $f \in C([0, t_0], [D(A)])$ and $0 \leq t \leq t_0$. Then $A+B$ generates a local C-semigroup on X.

Similarly, the next theorem and corollaries are also attained.

Theorem 2.18. *Let $S(\cdot)$ be a locally Lipschitz continuous local C-semigroup on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C)$ such that $CB = BC$ on $D(A)$. Assume that $\rho(A)$ and $\rho(A+B)$ are both nonempty, and for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that*

$$|S * C^{-1}B[f(t) - f(s)]| \leq M_{t_0} \int_s^t |f(r)|\mathrm{d}r \tag{2.17}$$

for all $f \in C([0, t_0], [D(A)])$ and $0 \leq s < t \leq t_0$. Then $A + B$ generates a locally Lipschitz continuous local C-semigroup on X.

COROLLARY 2.19

Let $S(\cdot)$ be a locally Lipschitz continuous local C-semigroup on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C)$ such that $CB = BC$ on $D(A)$ and $C^{-1}Bx \in \overline{D(A)}$ for all $x \in D(A)$. Assume that $\rho(A)$ and $\rho(A+B)$ are both nonempty. Then $A+B$ generates a locally Lipschitz continuous local C-semigroup on X, which is also exponentially Lipschitz continuous if $S(\cdot)$ is.

Similarly, we can combine Corollary 2.8 with Theorem 2.17 to obtain the next corollary.

COROLLARY 2.20

Let $S(\cdot)$ be a local C-semigroup on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C)$ such that $CB = BC$ on $D(A)$ and $C^{-1}Bx \in D(A)$ for all $x \in D(A)$. Assume that $\rho(A)$ and $\rho(A+B)$ are both nonempty. Then $A+B$ generates a local C-semigroup on X, which is also exponentially bounded (*resp.*, *norm continuous*) *if $S(\cdot)$ is.*

Remark 2.21. The conclusions of Corollaries 2.7, 2.11 and 2.19 are still true with the assumption that $R(C^{-1}B) \subset \overline{D(A)}$ is replaced by assuming that $R(C^{-1}B) \subset \{x \in X \mid S(\cdot)x \in C^1([0, T_0), X)\}$, and the conclusion of Corollary 2.15 is still true when the assumption that $R(C^{-2}B) \subset \overline{D(A)}$ is replaced by assuming that $R(C^{-2}B) \subset \{x \in X \mid S(\cdot)x \in C^1([0, T_0), X)\}$.

We end this paper with a simple illustrative example: Let $X = L^{\infty}(\mathbb{R})$ and $A : D(A) \subset X \to X$ be defined by $D(A) = W^{1,\infty}(\mathbb{R})$ and $Af = -f'$ for all $f \in D(A)$. Then A

generates a locally Lipschitz continuous local C-semigroup $S(\cdot)(= \{S(t)|0 \leq t < T_0\})$ on X and $\overline{D(A)} = C_0(\mathbb{R})$ (see Example 3.3.10 of [1] and Theorem 18.3 of [2]). Here $C = (\lambda - A)^{-1}$ with $\lambda \in \rho(A)$ and $0 < T_0 \leq \infty$ are fixed. Applying Corollary 2.7, we get that $A(I + C^{-1}BC)$ generates a locally Lipschitz continuous local C-semigroup $T(\cdot)$ on $L^\infty(\mathbb{R})$ satisfying (2.7) when B is a bounded linear operator from $C_0(\mathbb{R})$ into $W^{1,\infty}(\mathbb{R})$ such that $(\lambda - A)^{-1}B = B(\lambda - A)^{-1}$ on $C_0(\mathbb{R})$ and $R((\lambda - A)B) \subset C_0(\mathbb{R})$.

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