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On A-nilpotent abelian groups

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Abstract. Let G be a group and $A = \mathrm{Aut}(G)$ be the group of automorphisms of G. Then, the element $[g, \alpha] = g^{-1}\alpha(g)$ is an autocommutator of $g \in G$ and $\alpha \in A$. Hence, for any natural number m the m-th autocommutator subgroup of G is defined as

$$K_m(G) = \langle [g, \alpha_1, \ldots, \alpha_m] | g \in G, \alpha_1, \ldots, \alpha_m \in A\rangle,$$

where $[g, \alpha_1, \alpha_2, \ldots, \alpha_m] = [[g, \alpha_1, \ldots, \alpha_{m-1}], \alpha_m]$. In this paper, we introduce the new notion of A-nilpotent groups and classify all abelian groups which are A-nilpotent groups.

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1. Introduction

Let G be a group and $A = \mathrm{Aut}(G)$ denote the group of automorphisms of G. As in [3], if $g \in G$ and $\alpha \in A$, then the element $[g, \alpha] = g^{-1}\alpha(g)$ is an *autocommutator* of g and α. Hence, following [5] one may define the *autocommutator of weight* $m + 1$ ($m \geq 2$) inductively as

$$[g, \alpha_1, \alpha_2, \ldots, \alpha_m] = [[g, \alpha_1, \ldots, \alpha_{m-1}], \alpha_m],$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_m \in A$.

Now for any natural number m,

$$K_m(G) = [G, \underbrace{A, \ldots, A}_{m\text{-times}}] = \langle [g, \alpha_1, \alpha_2, \ldots, \alpha_m] | g \in G, \alpha_1, \ldots, \alpha_m \in A\rangle,$$

which is called the m-th autocommutator subgroup of G. Hence, we obtain a descending chain of autocommutator subgroups of G as follows:

$$G \supseteq K_1(G) \supseteq K_2(G) \supseteq \cdots \supseteq K_m(G) \supseteq \cdots ,$$

which is called the *lower autocentral series* of G.

Throughout this paper if p is a prime, then a p-group is a group in which every element has order a power of p. Also we adopt additive notation for all abelian groups. To be brief, $([a]_n, [a']_m)$ of group $\mathbb{Z}_n \oplus \mathbb{Z}_m$ will be indicated as (a, a'), where $a \in \{0, 1, 2, \ldots, n-1\}$ and $a' \in \{0, 1, 2, \ldots, m-1\}$.

In [5, 6] some properties of autocommutator subgroups of a finite abelian group are studied. The below example shows the m-th autcommutator subgroup of a finite abelian group was incorrectly concluded in Theorem 2.5 of [5].

Example 1.1. Let $G = \mathbb{Z}_8 \oplus \mathbb{Z}_4$. Then, by Theorem 2.5 of [5], $K_2(G) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. But if we define the automorphisms α and β of G, given by $\alpha(a, b) = (a, a + b)$ and $\beta(a, b) = (a + 2b, b)$ for all $(a, b) \in G$, then we have $[(1, 0), \alpha, \beta] = (2, 0)$ and hence, $K_2(G)$ has an element of order 4. So $K_2(G) \neq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

In §3, we obtain the $K_m(\oplus_{i=1}^{k} \mathbb{Z}_{2^{n_i}})$ with $n_1 > n_2 > \cdots > n_k$ using a function which is recursively defined in terms of $n_1, \ldots, n_k$'s. In particular, we prove

$$K_{2n_1-1}(\oplus_{i=1}^{k} \mathbb{Z}_{2^{n_i}}) = \langle 0 \rangle,$$

for all natural numbers $n_1 > n_2 > \cdots > n_k$.

Now, we introduce the new notion of A-nilpotent groups.

DEFINITION 1.2

A group G is called A-nilpotent, if the lower autocentral series ends in the identity subgroup after a finite number of steps.

If a group G is A-nilpotent, then $\mathrm{Aut}(G)$ is the stability group of the lower autocentral series of G. In [2], Hall proved that the stability group is nilpotent. Hence, a group with non-nilpotent automorphism group can not be A-nilpotent.

Example 1.3. Let n be a natural number and $G = \mathbb{Z}_{2^n}$. Then, $K_m(G) = 2^m G$, for any natural number m. Hence, G is an A-nilpotent group.

Example 1.4. Let $G = D_{2n}$ be a dihedral group of order $2n$. Then, one can check that $K_m(G) \cong 2^{m-1}\mathbb{Z}_n$, for any natural number m. Hence, $G = D_{2n}$ is A-nilpotent if and only if n has no odd factors.

The A-nilpotent groups are nilpotent, but the converse is not true in general. For example, the generalized quaternion group of order 8, Q_8, is a nilpotent group, but is not A-nilpotent.

Remark 1.5. The set of elements $L(G) = \{g \in G \mid [g, \alpha] = 1, \forall \alpha \in A\}$ is called the autocentre of G. Clearly, it is a characteristic subgroup of G. Following [6], we define the upper autocentral series of G as follows:

$$\langle 1 \rangle = L_0(G) \subseteq L_1(G) = L(G) \subseteq L_2(G) \subseteq \cdots \subseteq L_m(G) \subseteq \cdots,$$

where $\frac{L_m(G)}{L_{m-1}(G)} = L(\frac{G}{L_{m-1}(G)})$. In [6], a group G is said to be autonilpotent group, if the upper autocentral series ends in the group G after a finite number of steps. It is easy to check that any autonilpotent group is A-nilpotent, but the converse is not true in general. The dihedral group of order 8, D_8, is A-nilpotent, but is not autonilpotent.

In §4, we classify all abelian groups which are A-nilpotent groups.

2. Preliminary results

We begin with some useful results that will be used in the proof of our main results.

Lemma 2.1

(i) *Let H and T be two arbitrary groups. Then, for any natural number m, $K_m(H) \times K_m(T) \subseteq K_m(H \times T)$.*
(ii) *Let H and T be finite groups such that $(|H|, |T|) = 1$. Then, for any natural number m, $K_m(H) \times K_m(T) = K_m(H \times T)$.*

Proof.

(i) For $\alpha \in \mathrm{Aut}(H)$ and $\beta \in \mathrm{Aut}(T)$, we define the automorphism $\alpha \times \beta$ of group $H \times T$, given by $(\alpha \times \beta)((h, t)) = (\alpha(h), \beta(t))$ for all $h \in H$ and $t \in T$. Now by induction on m, it is easy to check that $([h, \alpha_1, \ldots, \alpha_m], [t, \beta_1, \ldots, \beta_m]) = [(h, t), \alpha_1 \times \beta_1, \ldots, \alpha_m \times \beta_m]$ for all $h \in H$, $t \in T$, $\alpha_1, \ldots, \alpha_m \in \mathrm{Aut}(H)$ and $\beta_1, \ldots, \beta_m \in \mathrm{Aut}(T)$. This implies the result.
(ii) It is sufficient to prove that $K_m(H \times T) \subseteq K_m(H) \times K_m(T)$. It is easy to check that $\gamma|_H \in \mathrm{Aut}(H)$ and $\gamma|_T \in \mathrm{Aut}(T)$, for all $\gamma \in \mathrm{Aut}(H \times T)$. Now by induction on m, we have $[(h, t), \gamma_1, \ldots, \gamma_m] = ([h, \gamma_1|_H, \ldots, \gamma_m|_H], [t, \gamma_1|_T, \ldots, \gamma_m|_T])$, for all $h \in H$, $t \in T$ and $\gamma_1, \ldots, \gamma_m \in \mathrm{Aut}(H \times T)$. This implies the result. □

The following corollary is an immediate result of the above lemma.

COROLLARY 2.2

If H or T is not A-nilpotent group, then $H \times T$ is not A-nilpotent.

Lemma 2.3 (*Lemma* 2.2 *of* [5]). *If G is a finite cyclic group, then $K_m(G) = 2^m G$, for any natural number m.*

COROLLARY 2.4

If G is a finite abelian group of odd order, then $K_m(G) = G$, for any natural number m.

Proof. It is obvious by Lemma 2.1(i) and 2.3. □

Lemma 2.5 *Suppose that G is an abelian group, $G = \langle X \rangle$ and $A = \langle \Gamma \rangle$. Then, $[G, A] = \langle [x, \alpha] | x \in X, \alpha \in \Gamma \rangle$. In particular, if G is an abelian group and $K_m(G) = \langle Y \rangle$, then $K_{m+1}(G) = \langle [y, \alpha] | y \in Y, \alpha \in \Gamma \rangle$.*

Proof. Let $x \in G$, $\alpha, \beta \in \Gamma$ and suppose that $x = \sum_{i=1}^{s} n_i x_i$ and $\beta(x) = \sum_{j=1}^{t} m_j x'_j$ with for $i = 1, \ldots, s$, $j = 1, \ldots, t$ and $x_i, x'_j \in X$. Then,

$$[x, \alpha\beta] = [x, \beta] + [\beta(x), \alpha] = \sum_{i=1}^{s} n_i [x_i, \beta] + \sum_{j=1}^{t} m_j [x'_j, \alpha].$$

The result follows by induction. □

Remark 2.6. Recall for any natural number n, if $G = \mathbb{Z}_{2^n}$, then $\mathrm{Aut}(G)$ consists of all automorphisms $\alpha_i : g \mapsto ig$, where $1 \leq i < 2^n$ and i is an odd number. We know that the automorphism groups of finitely generated abelian groups are well-understood (see, for example [4]).

Now we need to describe the automorphisms of $G = \sum_{i=1}^{k} \mathbb{Z}_{2^{n_i}}$ with $n_1 > n_2 > \cdots > n_k$. Let $\epsilon_1 = (1, 0, \ldots, 0)$, $\epsilon_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\epsilon_k = (0, 0, \ldots, 1)$. Now $\{\epsilon_1, \ldots, \epsilon_k\}$ is a generating set for G. Thus, an automorphism of G is completely determined by its action on this generating set. Using the fact that for $i = 1, \ldots, k$, we have $|\epsilon_i| = 2^{n_i}$ and that the height of ϵ_i in G is 0 for all $\alpha \in \mathrm{Aut}(G)$, we must have

$$
\begin{aligned}
\alpha(\epsilon_1) &= (a_{11}, a_{12}, \ldots, a_{1k}) \\
\alpha(\epsilon_2) &= (2^{n_1 - n_2} a_{21}, a_{22}, \ldots, a_{2k}) \\
\alpha(\epsilon_3) &= (2^{n_1 - n_3} a_{31}, 2^{n_2 - n_3} a_{32}, a_{33}, \ldots, a_{3k}) \\
&\ \ \vdots \\
\alpha(\epsilon_k) &= (2^{n_1 - n_k} a_{k1}, 2^{n_2 - n_k} a_{k2}, \ldots, 2^{n_{k-1} - n_k} a_{k(k-1)}, a_{kk}),
\end{aligned}
$$

where $a_{ij} \in \mathbb{Z}$ for all i, j and for all i, we must have a_{ii} is odd. It is also easy to see that the automorphisms of G must arise in this manner for appropriate choices of a_{ij}'s.

3. Autocommutator subgroups of finite abelian groups

Let G be a finite abelian group and m be a natural number. Then $G = H \oplus T$, where T is a finite abelian group of odd order and H is trivial group or a finite abelian 2-group. Hence, $K_m(G) = K_m(H) \oplus T$, by Lemma 2.1(ii) and Corollary 2.4. Therefore in this section, without loss of generality, we restrict attention to the case where G is a finite abelian 2-group.

Lemma 3.1 [1]. *For all natural numbers $k, n_1, n_2, \ldots, n_k$ such that $n_1 > n_2 \geq \cdots \geq n_k$,*

$$K_1(\mathbb{Z}_{2^{n_1}} \oplus \mathbb{Z}_{2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}}) = 2\mathbb{Z}_{2^{n_1}} \oplus \mathbb{Z}_{2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}}.$$

Lemma 3.2 [7]. *Suppose that $G = \oplus_{i=1}^{k} \mathbb{Z}_{2^{n_i}}$ with $k > 1$ and $n_1 = n_2 \geq n_3 \geq \cdots \geq n_k$. Then,*

$$K_m(G) = G, \quad \textit{for any natural number } m.$$

COROLLARY 3.3

Suppose that $G = \oplus_{i=1}^{k} \mathbb{Z}_{2^{n_i}}$ with $k > 1$ and $n_1 \geq n_2 \geq \cdots \geq n_t = n_{t+1} \geq \cdots \geq n_k$, for some natural number $1 \leq t < k$. Then,

$$K_m(\oplus_{i=1}^{k} \mathbb{Z}_{2^{n_i}}) \neq \langle 0 \rangle, \quad \textit{for any natural number } m.$$

That is, G is not A-nilpotent.

Proof. It is obvious by Lemmas 3.2 and 2.1(i). □

DEFINITION 3.4

We define for $i = 1, 2, \ldots, k$, $T_{0,i} = 0$ and for $m = 1, 2, \ldots$, that $T_{m,0} = \infty$. Now for $m \geq 0$ and $i = 1, 2, \ldots, k$, we have

$$T_{m+1,i} = \min\{T_{m,i-1}, T_{m,i} + 1, n_i - n_{i+1} + T_{m,i+1}, \ldots, n_i - n_k + T_{m,k}\}.$$

With this notation, we can determine the structure of $K_m(\oplus_{i=1}^k \mathbb{Z}_{2^{n_i}})$ with $n_1 > n_2 > \cdots > n_k$.

Theorem 3.5. *Suppose that $G = \oplus_{i=1}^k \mathbb{Z}_{2^{n_i}}$ with $n_1 > n_2 > \cdots > n_k$. Then,*

$$K_m(G) = \oplus_{i=1}^k 2^{T_{m,i}} \mathbb{Z}_{2^{n_i}},$$

for any natural number m.

Proof. We prove the result by induction on m. First we note that $T_{1,1} = 1$ and $T_{1,j} = 0$ for $j = 2, 3, \ldots, k$, as required.

Now suppose we have determined that

$$K_m(G) = \oplus_{j=1}^k 2^{T_{m,j}} \mathbb{Z}_{2^{n_j}},$$

the induction hypothesis. Considering the automorphisms of G, as described earlier, it is easy to see that

$$T_{m+1,1} = \min\{T_{m,1} + 1, n_1 - n_2 + T_{m,2}, \ldots, n_1 - n_k + T_{m,k}\},$$

and if $j = 2, \ldots, k$, then

$$T_{m+1,j} = \min\{T_{m,j-1}, T_{m,j} + 1, n_j - n_{j+1} + T_{m,j+1}, \ldots, n_j - n_k + T_{m,k}\}.$$

Note that there is an automorphism α of G so that $\alpha(\epsilon_j) = \epsilon_j + \epsilon_{j+1}$. It follows that since $K_m(G)$ is a characteristic subgroup of G, $T_{m,j+1} \leq T_{m,j}$ for $j = 1, \ldots, k-1$. This completes the proof. □

In order to simplify the statements of the following corollaries, we introduce the following notations:

The ceiling function of x, also called the smallest integer function, gives the smallest integer not less than to x. The ceiling function is written in a number of different notations. In this paper, we use of the symbol $\lceil x \rceil$. Also for each real number x, we define

$$x^+ = \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

COROLLARY 3.6

Suppose that $G = \oplus_{i=1}^k \mathbb{Z}_{2^{n_i}}$ with $n_1 - 1 = n_2 > \cdots > n_k$. Then,

$$K_m(G) = \oplus_{i=1}^k 2^{\lceil \frac{m-i+1}{2} \rceil^+} \mathbb{Z}_{2^{n_i}},$$

for any natural number m.

Proof. We use mathematical induction to prove that for $m = 1, 2, \ldots,$

$$T_{m,j} = \left\lceil \frac{m-j+1}{2} \right\rceil^+, \quad \text{for} \quad j = 1, 2, \ldots, k.$$

First we note that $T_{1,1} = 1$ and $T_{1,j} = 0$, for $j = 2, 3, \ldots, k,$ as required.

Case 1. Here we assume that $m \le k$. Now by Theorem 3.5 and the induction hypothesis, we have

$$\begin{aligned} T_{m+1,1} &= \min\{T_{m,1}+1, n_1-n_2+T_{m,2}, n_1-n_3+T_{m,3}, \ldots, \\ &\quad n_1-n_{m+1}+T_{m,m+1}, \ldots, n_1-n_k+T_{m,k}\} \\ &= \min\left\{\left\lceil\frac{m}{2}\right\rceil+1, 1+\left\lceil\frac{m-1}{2}\right\rceil, n_1-n_3+\left\lceil\frac{m-2}{2}\right\rceil, \ldots, \right. \\ &\quad \left. n_1-n_{m+1}+0, \ldots, n_1-n_k+0\right\} \\ &= \left\lceil\frac{m+1}{2}\right\rceil = \left\lceil\frac{(m+1)-1+1}{2}\right\rceil, \end{aligned}$$

as required. Note that the last equality uses the fact that for $j \ge 3$, since $n_1 - n_j \ge j-1$, we have that $n_1 - n_j + \lceil\frac{m-j+1}{2}\rceil \ge \lceil\frac{m+j-1}{2}\rceil \ge \lceil\frac{m+1}{2}\rceil$. Next we assume that $j \ge 2$. We have

$$\begin{aligned} T_{m+1,j} &= \min\{T_{m,j-1}, T_{m,j}+1, n_j-n_{j+1}+T_{m,j+1}, \ldots, n_j-n_k+T_{m,k}\} \\ &= \min\left\{\left\lceil\frac{m-j+2}{2}\right\rceil^+, \left\lceil\frac{m-j+1}{2}\right\rceil^+ +1, n_j-n_{j+1}+\left\lceil\frac{m-j}{2}\right\rceil, \ldots, \right. \\ &\quad \left. n_j-n_{m+1}+0, \ldots, n_j-n_k+0\right\} \\ &= \left\lceil\frac{m-j+2}{2}\right\rceil^+ = \left\lceil\frac{(m+1)-j+1}{2}\right\rceil^+, \end{aligned}$$

as required. Here again we are using the fact that $n_j - n_l \ge l - j$ for $l > j$.

Case 2. Here we assume that $m > k$. Now we have

$$\begin{aligned} T_{m+1,1} &= \min\{T_{m,1}+1, n_1-n_2+T_{m,2}, n_1-n_3+T_{m,3}, \ldots, n_1-n_k+T_{m,k}\} \\ &= \min\left\{\left\lceil\frac{m}{2}\right\rceil+1, 1+\left\lceil\frac{m-1}{2}\right\rceil, n_1-n_3+\left\lceil\frac{m-2}{2}\right\rceil, \ldots, \right. \\ &\quad \left. n_1-n_k+\left\lceil\frac{m-k+1}{2}\right\rceil\right\} \\ &= \left\lceil\frac{m+1}{2}\right\rceil, \end{aligned}$$

as required. Here again we need to use the fact that for $j \geq 3$, we have $n_1 - n_j \geq j - 1$. Now for $j \geq 2$, we have

$$\begin{aligned} T_{m+1,j} &= \min\{T_{m,j-1}, T_{m,j} + 1, n_j - n_{j+1} + T_{m,j+1}, \ldots, n_j - n_k + T_{m,k}\} \\ &= \min\left\{\left\lceil \frac{m-j+2}{2} \right\rceil, \left\lceil \frac{m-j+1}{2} \right\rceil + 1, n_j - n_{j+1} + \left\lceil \frac{m-j}{2} \right\rceil, \ldots, \right. \\ &\qquad \left. n_j - n_k + \left\lceil \frac{m-k+1}{2} \right\rceil \right\} \\ &= \left\lceil \frac{m-j+2}{2} \right\rceil = \left\lceil \frac{(m+1)-j+1}{2} \right\rceil, \end{aligned}$$

as required. Again we use the fact that $n_j - n_l \geq l - j$ for $l > j$. This completes the proof. □

With a similar proof one can prove as follows:

COROLLARY 3.7

Suppose that $G = \oplus_{i=1}^{k} \mathbb{Z}_{2^{n_i}}$ *with* $n_1 > n_2 > \cdots > n_k$. *Further, suppose that for all* $s = 1, 2, \ldots, k-1$, *we have* $n_s - n_{s+1} \geq 2$. *Then,*

$$K_m(G) = \oplus_{i=1}^{k} 2^{(m-i+1)^+} \mathbb{Z}_{2^{n_i}},$$

for any natural number m.

Finally we prove the following corollary which is very useful in the proof of our main result.

COROLLARY 3.8

Suppose that $G = \oplus_{i=1}^{k} \mathbb{Z}_{2^{n_i}}$ *with* $n_1 > n_2 > \cdots > n_k$. *Then,*

$$K_{2n_1-1}(G) = \langle 0 \rangle.$$

That is, G *is A-nilpotent.*

Proof. Let

$$H = \begin{cases} G, & \text{if } n_1 - n_2 = 1 \\ G \oplus \mathbb{Z}_{2^{n_1-1}}, & \text{otherwise} \end{cases}.$$

In other cases, G is a direct summand of H, so it is clear that $K_m(G) \subseteq K_m(H)$. Now from the result of Corollary 3.6 being applied to H, we get

$$\begin{aligned} T_{2n_1-1,2l-1} &= \left\lceil \frac{(2n_1-1)-(2l-1)+1}{2} \right\rceil^+ = \left\lceil \frac{2n_1-2l+1}{2} \right\rceil^+ \\ &= n_1 - l + 1 \geq n_l \geq n_{2l-1} \end{aligned}$$

and

$$T_{2n_1-1,2l} = \left\lceil \frac{(2n_1-1)-2l+1}{2} \right\rceil^+ = \lceil n_1 - l \rceil^+ = n_1 - l \geq n_{l+1} \geq n_{2l}.$$

Note that we use the fact that $n_1 \geq k$. This completes the proof. □

4. *A*-nilpotent abelian groups

In this section, we classify all abelian groups which are A-nilpotent groups.

Lemma 4.1 *Let G be an abelian group. Then for any natural number m,*

$$2^m G \subseteq K_m(G).$$

Proof. It is proved in a similar way to Lemma 2.3. □

Remark 4.2. An additively written group is called *bounded* if its elements have boundedly finite orders. Of course multiplicative groups with this property are said to have finite exponent but this term is inappropriate in the context of additive groups.

Theorem 4.3 (Prüfer-Baer, Corollary 10.37 of [8]). *If G is a bounded, abelian p-group, then G is a direct sum of cyclic groups.*

COROLLARY 4.4

Suppose that G is a bounded, infinite abelian 2-group. Then, $G \cong \mathbb{Z}_{2^t} \oplus \mathbb{Z}_{2^t} \oplus H$, for some natural number t and an abelian group H. That is, G is not A-nilpotent.

Proof. Let G be a bounded, infinite abelian 2-group. Then, by Theorem 4.3 we write $G = \oplus_{i\in I} X_i$ as the direct sum of cyclic subgroups. Note that there is a positive integer n so that each $X_i \cong \mathbb{Z}_{2^{t_i}}$ for some $t_i \leq n$. If there was only (at most) one copy of each X_i, then G would be finite. So, the result is true. Also G is not A-nilpotent by Corollary 3.3. □

Now, we are able to classify all abelian groups, which are A-nilpotent.

Theorem 4.5. *A non trivial abelian group G is A-nilpotent if and only if G is $\mathbb{Z}_{2^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}}$ with $n_1 > \cdots > n_k$.*

Proof. The necessary condition follows from Corollary 3.8. Now for the reverse conclusion, we assume that G is a non-trivial, A-nilpotent, abelian group. By Lemma 4.1, it follows that $2^m G = \langle 0 \rangle$ for some positive integer m. Thus, G has no elements of odd order and G is a bounded, abelian 2-group. As above, it follows that G is the direct sum of cyclic 2-groups. However, since G is A-nilpotent, no two direct summands can be isomorphic. Thus, G is isomorphic to a finite, direct sum of cyclic 2-groups, all with distinct orders. The result follows. □

The following corollary is an immediate result of the above theorem.

COROLLARY 4.6

Let G be an infinite abelian group. Then G is not A-nilpotent.

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References

[1] Chiš C, Chiš M and Silberberg G, Abelian groups as autocommutator groups, *Arch. Math. (Basel)* **90(6)** (2008) 490–492
[2] Hall P, Some sufficient conditions for a group to be nilpotent, *Illinois J. Math.* **2** (1958) 787–801
[3] Hegarty P, The absolute centre of a group, *J. Algebra* **169(3)** (1994) 929–935
[4] Hillar C and Rhea D, Automorphisms of finite abelian groups, *Amer. Math. Monthly.* **114(10)** (2007) 917–922
[5] Moghaddam M R R, Parvaneh F and Naghshineh M, The lower autocentral series of abelian groups, *Bull. Korean Math. Soc.* **48(1)** (2011) 79–83
[6] Moghaddam M R R, Some properties of autocommutator groups, The first two-days group theory seminar in Iran, University of Isfahan, 12–13 March (2009)
[7] Nasrabadi M M and Gholamian A, On finite A-perfect abelian groups, *Int. J. Group Theory.* **01(3)** (2012) 11–14
[8] Rotman J J, An introduction to the theory of groups, 4th ed. (1995) (Springer-Verlag)