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An Engel condition with an additive mapping in semiprime rings

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Abstract. The main purpose of this paper is to prove the following result: Let $n > 1$ be a fixed integer, let R be a $n!$-torsion free semiprime ring, and let $f : R \to R$ be an additive mapping satisfying the relation $[f(x), x]_n = [[\ldots[[f(x), x], x], \ldots], x] = 0$ for all $x \in R$. In this case $[f(x), x] = 0$ is fulfilled for all $x \in R$. Since any semisimple Banach algebra (for example, C^* algebra) is semiprime, this purely algebraic result might be of some interest from functional analysis point of view.

Keywords. Prime ring; semiprime ring; additive mapping; derivation; commuting mapping; centralizing mapping; functional identity.

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Throughout, R will represent an associative ring with a centre $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, in case $nx = 0$, $x \in R$, implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $\left[x, y\right]$. Set $[y, x]_1 = [y, x]$ for any $x, y \in R$, and for $n > 1$, let $[y, x]_n = [[y, x]_{n-1}, x]$. Recall that a ring R is prime if for $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies $a = 0$. We denote by C, Q and $RC \subseteq Q$ the extended centroid, the maximal right ring of quotients and a central closure of a semiprime ring R, respectively. For the explanation of C, Q and RC we refer the reader to [3]. We denote by char(R) the characteristic of a prime ring R. An additive mapping D of a ring R into itself is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. A mapping f of a ring R into itself is called centralizing on R if $\left[f(x), x\right] \in Z(R)$ holds for all $x \in R$. In the special case, when $\left[f(x), x\right] = 0$ holds for all $x \in R$ the mapping f is said to be commuting on R. A classical result of Posner (Posner's second theorem) [12] states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Posner's second theorem in general cannot be proved for semiprime rings as the following example shows. Take prime rings R_1, R_2, where R_1 is commutative, and set $R = R_1 \oplus R_2$. Let $D_1 : R_1 \to R_1$ be a nonzero derivation. A mapping $D : R \to R$, defined by $D((r_1, r_2)) = (D_1(r_1), 0)$ is then a nonzero commuting derivation. It is not difficult to show that if $D : R \to R$ is a commuting derivation on a semiprime ring R, then D maps R into $Z(R)$.

It is our aim in this paper to prove the following result.

Theorem 1. *Let $n > 1$ be a fixed integer, let R be a $n!$-torsion free semiprime ring, and let $f : R \to R$ be an additive mapping satisfying the relation*

$$[f(x), x]_n = 0 \tag{1}$$

for all $x \in R$. In this case, f is commuting on R.

In case $n = 2$, the above result reduces to Theorem 4 of [16]. Since any semisimple Banach algebra is semiprime (for example, C^* algebra), Theorem 1 might be of some interest from functional analysis point of view.

Let us see in some more details the background and the motivation of Theorem 1. Vukman [13] proved extensions of Posner's second theorem. More precisely, he proved the following result: Let R be a prime ring with $\mathrm{char}(R) \neq 2$. Suppose there exists a nonzero derivation $D : R \to R$, such that the mapping $x \mapsto [D(x), x]$ is commuting on R. In this case R is commutative. In [14], Vukman proved the following result: Let R be a prime ring with $\mathrm{char}(R) \neq 2, 3, 5$. Suppose there exists a nonzero derivation $D : R \to R$, such that the mapping $x \mapsto [[D(x), x], x]$ is centralizing on R. In this case, R is commutative (see also [15]).

Using the theory of differential identities, Lanski [11] fairly generalizes the results we have just mentioned by proving the following result: Let I be a nonzero ideal of a prime ring R, and let $D : R \to R$ be a nonzero derivation, such that $[D(x), x]_n = 0$ holds for all $x \in I$ and some fixed integer $n \geq 1$. In this case, R is commutative. In the same paper, Lanski proved the result which tells: Let R be a prime ring, L a noncommutative Lie ideal of R and $D : R \to R$ a nonzero derivation. If $[D(x), x]_n = 0$ holds for all $x \in L$ and some fixed integer $n \geq 1$, then $\mathrm{char}(R) = 2$ and $R \subseteq M_2(\mathbb{F})$ for $\mathbb{F}$ a field, so $[D(x), x]_2 = 0$. Brešar [5] has proved that in case an additive mapping $f : R \to R$, where R is a noncommutative prime ring, is commuting on R, then f is of the form $f(x) = \lambda x + \zeta(x)$, where $\lambda \in C$ is some fixed element, and $\zeta : R \to C$ is an additive mapping. With this result, the development of the theory of functional identities (Brešar–Beidar–Chebotar theory) started. We refer the reader to [7] for an introductory account of the Brešar–Beidar–Chebotar theory. For full treatment of this sophisticated and powerful theory, we refer the reader to [8]. In [4], Brešar proved that in case we have an additive mapping $f : R \to R$, where R is a prime ring with $\mathrm{char}(R) \neq 2$, such that $\left[f(x), x\right]_2 = 0$ for all $x \in R$, then f is commuting on R. This result has been generalized to 2-torsion free semiprime rings in [16]. Using the theory of functional identities Brešar [6] has proved the following theorem: Let R be a prime ring and let $f : R \to R$ be an additive mapping. Suppose there is a fixed integer $n > 1$ such that $[f(x), x]_n = 0$ holds for all $x \in R$. If $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) > n$, then f is commuting on R.

The work of Beidar *et al.* [2] should be mentioned. They studied an additive mapping $f : I \to A$ satisfying the relation

$$[[\ldots[[f(x), x^{n_1}], x^{n_2}], \ldots], x^{n_k}] = 0$$

for some fixed positive integers $n_1, n_2, \ldots, n_k$, and all $x \in I$, where I is a right ideal of a prime ring R and $A = RC$ is a central closure of R. They showed that in case either $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) > n_1 + n_2 + \cdots + n_k$ a mapping f is commuting on R.

Proof of Theorem 1. As observed above, we may assume that $n > 2$. In the light of comments referring to [16] we shall begin the proof. According to the semiprimeness of R there exists a family of prime ideals $\{P_\alpha \mid \alpha \in I\}$ such that $\cap_\alpha P_\alpha = \{0\}$. Without loss of generality, we may assume that for prime rings $R_\alpha = R/P_\alpha$, $\mathrm{char}(R_\alpha) > n$ (see page 459 of [1]). By C we shall denote the extended centroid of the prime ring R/P, for $P = P_\alpha$ for some α, and by A the central closure of R/P. One can consider A as a vector space over the field C, which can be regarded as a subspace of A. Thus there exists a subspace B of A such that $A = B + C$. We shall denote by π the canonical projection of A onto B. Let us fix some $P = P_\alpha$, $\alpha \in I$. We will show that $[f(x), x] \in P$ for all $x \in R$. For $x \in R$, we shall write $\bar{x}$ for the coset $x + P \in R/P$. Write $x + p$, $x \in R$, $p \in P$, instead of x in (1). It follows that $[f(p), x]_n \in P$ for all $x \in R$ and $p \in P$. Therefore $[\overline{f(p)}, \bar{x}]_n = 0$ for all $x \in R$. In particular,

$$[[\overline{f(p)}, \bar{x}], \bar{x}]_{n-1} = 0. \tag{2}$$

Define a mapping $D : R/P \to R/P$ by $D(\bar{x}) = [\overline{f(p)}, \bar{x}]$, which is called an inner derivation on R/P. Hence (2) can be written as

$$[D(\bar{x}), \bar{x}]_{n-1} = 0$$

for all $\bar{x} \in R/P$. Using [6, Theorem], it follows that $[D(\bar{x}), \bar{x}] = 0$. Posner's second theorem implies $[\overline{f(p)}, \bar{x}] = 0$ for all $x \in R$, $p \in P$, which means that $\overline{f(p)}$ lies in the centre of R/P. In particular, we have $\pi\overline{f(p)} = 0$. It follows that the mapping $\bar{f} : R/P \to A$, $\bar{f}(\bar{x}) = \pi\overline{f(x)}$ is well defined. It is easy to verify that $\bar{f}$ is additive and satisfies $[\bar{f}(\bar{x}), \bar{x}]_n = 0$ for all $x \in R$. Using Theorem 1.1 of [2], it follows that $[\bar{f}(\bar{x}), \bar{x}] = 0$, which in turn implies $[f(x), x] \in P$, as desired. The proof is completed.

Posner's first theorem [12], which states that compositum of two nonzero derivations on a prime ring with characteristic different from two cannot be a derivation, in general cannot be proved for semiprime rings. However, in case we have a semiprime ring, one can prove the following result (Lemma 1.1.9 of [10]): Let R be a 2-torsion free semiprime ring, and let $D, G : R \to R$ be derivations such that $D^2(x) = G(x)$ holds for all $x \in R$. In this case, $D = 0$. This result was the motivation for the following result proved by Vukman [15]: Let R be a 2-torsion free semiprime ring, and let $D, G : R \to R$ be derivations, such that the mapping $x \longmapsto D^2(x) + G(x)$ is centralizing on R. In this case, D and G are commuting on R. The corollary below generalizes Vukman's result we have just mentioned.

COROLLARY 2

Let $n > 1$ be a fixed integer, let R be a $n!$-torsion free semiprime ring, and let $D, G : R \to R$ be derivations satisfying the relation $[D^2(x) + G(x), x]_n = 0$ for all $x \in R$. In this case, D and G map R into $Z(R)$.

Proof. Since all the assumptions of Theorem 1 are fulfilled, one can conclude that the mapping $x \longmapsto D^2(x) + G(x)$ is commuting on R, whence it follows by Theorem 4 in [15] that D and G are both commuting on R. Since any commuting derivation on a semiprime ring maps the ring into its centre (see, for example, the end of the proof of Theorem 2.1 in [17]), the proof of the corollary is complete. □

In case $n = 2$, Corollary 2 has been recently proved by Fošner and Vukman in [9].

References

[1] Baxter W E and Martindale III W S, Jordan homomorphisms of semiprime rings, *J. Algebra* **56** (1979) 457–471

[2] Beidar K I, Fong Y, Lee P-H and Wong T-L, On additive maps of prime rings satisfying the Engel condition, *Commun. Algebra* **25** (1997) 3889–3902

[3] Beidar K I, Martindale III W S and Mikhalev A V, Rings with generalized identities (1996) (New York: Marcel Dekker, Inc.)

[4] Brešar M, On a generalization of the notion of centralizing mappings, *Proc. Amer. Math. Soc.* **114** (1992) 641–649

[5] Brešar M, Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings, *Trans. Amer. Math. Soc.* **335** (1993) 525–546

[6] Brešar M, Applying the theorem on functional identities, *Nova J. Math. Game Th. Algebra* **4** (1995) 43–54

[7] Brešar M, Functional identities: a survey, *Contemp. Math.* **259** (1999) 93–103

[8] Brešar M, Chebotar M and Martindale III W S, Functional identities (2007) (Schwitzerland: Birkäuser Verlag)

[9] Fošner A and Vukman J, Some results concerning additive mappings and derivations on semiprime rings, *Publ. Math. Debrecen* **78** (2011) 575–581

[10] Herstein I N, Rings with involution, Chicago Lectures in Math. (1976) (Chicago London: Univ. of Chicago Press)

[11] Lanski C, An Engel condition with derivation, *Proc. Amer. Math. Soc.* **118** (1993) 731–734

[12] Posner E C, Derivations in prime rings, *Proc. Amer. Math. Soc.* **8** (1957) 1093–1100

[13] Vukman J, Commuting and centralizing mappings in prime rings, *Proc. Amer. Math. Soc.* **109** (1990) 47–52

[14] Vukman J, On derivations in prime rings and Banach algebras, *Proc. Amer. Math. Soc.* **116** (1992) 877–884

[15] Vukman J, Derivations on semiprime rings, *Bull. Austral. Math. Soc.* **53** (1996) 353–35

[16] Vukman J and Kosi-Ulbl, On some equations related to derivations in rings, *Internat. J. Math. and Math. Soc.* **17** (2005) 2703–2710

[17] Vukman J, Identities with derivations on rings and Banach algebras, *Glasnik Mat. Vol.* **40** (2005) 189–199