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# On quadratic variation of martingales

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**Abstract.** We give a construction of an explicit mapping

$$\Psi : \mathsf{D}([0,\infty),\mathbb{R}) \to \mathsf{D}([0,\infty),\mathbb{R}),$$

where $\mathsf{D}([0,\infty),\mathbb{R})$ denotes the class of real valued r.c.l.l. functions on $[0,\infty)$ such that for a locally square integrable martingale $(M_t)$ with r.c.l.l. paths,

$$\Psi(M.(\omega)) = A.(\omega)$$

gives the quadratic variation process (written usually as $[M,M]_t$) of $(M_t)$. We also show that this process $(A_t)$ is the unique increasing process $(B_t)$ such that $M_t^2 - B_t$ is a local martingale, $B_0 = 0$ and

$$\mathbb{P}((\Delta B)_t = [(\Delta M)_t]^2,\ 0 < t < \infty) = 1.$$

Apart from elementary properties of martingales, the only result used is the Doob's maximal inequality. This result can be the starting point of the development of the stochastic integral with respect to r.c.l.l. martingales.

**Keywords.** Doob–Meyer decomposition; martingales; quadratic variation.

**Mathematics Subject Classification.** 60G44, 60G17, 60H05.

## 1. Introduction

The Doob–Meyer decomposition of square of a (square integrable) martingale was a landmark result that led to the development of stochastic integration with respect to a martingale (as outlined by Doob). Meyer [7, 8] showed that there exists a unique natural increasing process $(A_t)$ such that $A_0 = 0$ and $M_t^2 - A_t$ is a martingale. This decomposition plays a central role in the theory of stochastic integration [9, 10]. There have been several attempts, including in recent times, at giving simpler proofs of the Doob–Meyer decomposition for pedagogical reasons (see [1, 2, 11]). In every exposition of stochastic integration for semimartingales (which may have jumps), the Doob–Meyer decomposition remains the first important step. The unusual definition of the *natural increasing process* due to Meyer lead to further study and was characterized to be the same as a predictable increasing process.

Let $(M_t)$ be a square integrable martingale with respect to a filtration $(\mathcal{F}_t)$. The unique predictable increasing process $(A_t)$ such that $A_0 = 0$ and $M_t^2 - A_t$ is a martingale has been called the *compensator* of $M_t^2$ also the *predictable quadratic variation* of the martingale $M$ and has subsequently been denoted as $\langle M, M\rangle_t$.

For a simple predictable process $f$ given by

$$f_t(\omega) = \sum_{j=0}^{m-1} a_j(\omega)\mathbf{1}_{(s_j,s_{j+1}]}(t),$$

where $a_j$ is a $\mathcal{F}_{s_j}$ measurable bounded r.v. for $0 \le j < m$; $0 = s_0 < s_i < \cdots < s_m$, $m \ge 1$, the stochastic integral can be defined as

$$X_t = \int_0^t f\mathrm{d}M = \sum_{j=0}^{m-1} a_j(M_{t\wedge s_{j+1}} - M_{t\wedge s_j}).$$

It is easy to check that $(X_t)$ is a martingale and using that $M_t^2 - \langle M, M\rangle_t$ is a martingale, one checks that $X_t^2 - B_t$ is a martingale, where

$$B_t = \sum_{j=0}^{m-1} a_j^2\left(\langle M, M\rangle_{t\wedge s_{j+1}} - \langle M, M\rangle_{t\wedge s_j}\right) = \int_0^t f_s^2\mathrm{d}\langle M, M\rangle_s.$$

Using Doob's maximal inequality, one can deduce

$$\mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^s f\mathrm{d}M\right|^2\right) \le 4\mathbb{E}\left(\int_0^t f_s^2\mathrm{d}\langle M, M\rangle_s\right). \tag{1.1}$$

The process $\langle M, M\rangle_t$ can be shown to be the limit in probability

$$A_t^n = \sum_{j=0}^{\infty}\mathbb{E}((M_{t\wedge t_{j+1}^n} - M_{t\wedge t_j^n})^2 \mid \mathcal{F}_{t_j^n}),$$

where $0 = t_0^n < t_1^n < \cdots < t_m^n < \cdots$ are a sequence of partitions of $[0, \infty)$ such that $t_j^n \to \infty$ as $j \to \infty$ for each $n$ and $\delta_n = \sup_j |t_{j+1}^n - t_j^n| \to 0$ as $n \to \infty$.

Meyer [9] introduced $[M, M]_t$ via $\langle M^c, M^c\rangle$ and $\Delta M_s$ (where $M^c$ is the continuous martingale part of $M$) and showed that it is the limit in probability of

$$B_t^n = \sum_{j=0}^{\infty}(M_{t\wedge t_{j+1}^n} - M_{t\wedge t_j^n})^2.$$

He also showed that for a continuous martingale $M$, $[M, M]_t = \langle M, M\rangle_t$.

For a continuous local martingale $M$, it was shown in Karandikar [3] that for suitably chosen sequence of random partitions $\{\tau_i^n : i \ge 0\}$ the quadratic variation

$$Q_t^n = \sum_{j=0}^{\infty}(M_{t\wedge\tau_{j+1}^n} - M_{t\wedge\tau_j^n})^2 \tag{1.2}$$

converges *almost surely* to $\langle M, M\rangle_t$. The random partitions were defined as follows: $\tau_0^n = 0$ and for each $n$, $\{\tau_i^n : i \ge 1\}$ is defined inductively by

$$\tau_{i+1}^n = \inf\{t \ge \tau_i^n : |M_t - M_{\tau_i^n}| \ge 2^{-n}\}.$$

The proof relied on the theory of stochastic integration. Subsequently, in Karandikar [4], the formula was derived using only Doob's maximal inequality. Thus this could be the starting point for the development of stochastic calculus for continuous semimartingales without bringing in any results from general theory of processes (see [5]).

The almost sure convergence of $Q_t^n$ to $\langle M, M\rangle_t$ also gives a pathwise formula for the quadratic variation of a continuous local martingale. It also directly shows that for a continuous local martingale $M$, the process $\langle M, M\rangle_t$ does not depend upon the underlying filtration and nor does it depend upon the underlying probability measure (see [6]). Indeed, in [6], a pathwise formula for $[M, M]_t$ when $M$ is an r.c.l.l. martingale was obtained, but the proof depended upon the theory of stochastic integration.

In this article, we show (once again using only Doob's maximal inequality) that for any square integrable r.c.l.l. martingale $(M_t)$, the processes $Q_t^n$ defined by

$$Q_t^n = \sum_{j=0}^{\infty}(M_{t\wedge\tau_{j+1}^n} - M_{t\wedge\tau_j^n})^2$$

converge almost surely uniformly on $t \in [0, T]$ for all $T < \infty$ to the quadratic variation $[M, M]_t$ and that $X_t = M_t^2 - [M, M]_t$ is a martingale.

Once we have shown the existence of $[M, M]_t$, one can get an estimate analogous to (1.1) for simple predictable processes $f$:

$$\mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^s f\,\mathrm{d}M\right|^2\right) \le 4\mathbb{E}\left(\int_0^t f_s^2\mathrm{d}[M, M]_s\right) \tag{1.3}$$

and this instead of (1.1) could be used as a starting point for developing stochastic calculus for locally square integrable martingales, bypassing completely the Doob Meyer decomposition. This approach is being taken in a book under preparation.

We give an explicit construction of a mapping $\Psi$ on the set of r.c.l.l. functions on $[0, \infty)$ such that for a r.c.l.l. martingale $M$,

$$\Psi(M.(\omega)) = [M, M].(\omega)$$

yields the quadratic variation of $M$.

## 2. The quadratic variation map

Let $\mathsf{D}([0, \infty), \mathbb{R})$ denote the space of r.c.l.l. functions on $[0, \infty)$. For $\rho \in \mathsf{D}([0, \infty), \mathbb{R})$, $\rho(t-)$ denotes the left limit at $t$ (for $t > 0$) and $\rho(0-) = 0$ and $\Delta\rho(t) = \rho(t) - \rho(t-)$. We will now define *quadratic variation* $\Psi(\rho)$ of a function $\rho \in \mathsf{D}([0, \infty), \mathbb{R})$.

For each $n \ge 1$, let $\{t_i^n(\rho) : i \ge 1\}$ be defined inductively as follows: $t_0^n(\rho) = 0$ and having defined $t_i^n(\rho)$, let

$$\begin{aligned} t_{i+1}^n(\rho) = \inf\big\{&t > t_i^n(\rho) : |\rho(t) - \rho(t_i^n(\rho))| \ge 2^{-n} \\ \text{or}\quad &|\rho(t-) - \rho(t_i^n(\rho))| \ge 2^{-n}\big\}. \end{aligned}$$

Note that for each $\rho \in \mathsf{D}([0, \infty), \mathbb{R})$ and for $n \ge 1$, $t_i^n(\rho) \uparrow \infty$ as $i \uparrow \infty$ (if $\lim_i t_i^n(\rho) = t^* < \infty$, then the function $\rho$ cannot have a left limit at $t^*$). Let

$$\Psi_n(\rho)(t) = \sum_{i=0}^{\infty}\big(\rho(t_{i+1}^n(\rho) \wedge t) - \rho(t_i^n(\rho) \wedge t)\big)^2.$$

Since $t_i^n(\rho)$ increases to infinity, for each $\rho$ and $t$ fixed, the infinite sum appearing above is essentially a finite sum and hence $\Psi_n(\rho)$ is itself an r.c.l.l. function. The space $\mathsf{D} = \mathsf{D}([0,\infty),\mathbb{R})$ is equipped with the topology of uniform convergence on compact subsets (abbreviated as ucc). Let $\tilde{\mathsf{D}}$ denote the set of $\rho \in \mathsf{D}$ such that $\psi_n(\rho)$ converges in the ucc topology and

$$\Psi(\rho) = \begin{cases} \lim_n \psi_n(\rho), & \text{if } \rho \in \tilde{\mathsf{D}}, \\ 0, & \text{if } \rho \notin \tilde{\mathsf{D}}. \end{cases}$$

Here are some basic properties of the quadratic variation map $\Psi$.

*Lemma* 2.1. *For $\rho \in \tilde{\mathsf{D}}$,*
(i) $\Psi(\rho)$ *is an increasing function*,
(ii) $\Delta\Psi(\rho)(t) = (\Delta\rho(t))^2$ *for all* $t \in (0,\infty)$,
(iii) $\sum_{s\le t}(\Delta\rho(s))^2 < \infty$ *for all* $t \in (0,\infty)$,
(iv) *let* $\Phi(\rho)(t) = \Psi(\rho)(t) - \sum_{0<s\le t}(\Delta\rho(s))^2$. *Then* $\Phi(\rho)$ *is a continuous function.*

*Proof.* For (i), note that for $s \le t$, if $t_j^n \le s < t_{j+1}^n$, then $|(\rho(s) - \rho(t_j^n))| \le 2^{-n}$, and

$$\begin{aligned} \Psi_n(\rho)(s) &= \sum_{i=0}^{j-1}(\rho(t_{i+1}^n(\rho)) - \rho(t_i^n(\rho)))^2 + (\rho(s) - \rho(t_j^n))^2, \\ \Psi_n(\rho)(t) &= \sum_{i=0}^{j-1}(\rho(t_{i+1}^n(\rho)) - \rho(t_i^n(\rho)))^2 \\ &\quad + \sum_{i=j}^{\infty}(\rho(t_{i+1}^n(\rho)\wedge t) - \rho(t_i^n(\rho)\wedge t))^2 \end{aligned}$$

and hence

$$\Psi_n(\rho)(s) \le \Psi_n(\rho)(t) + 2^{-2n}. \tag{2.4}$$

Thus (2.4) is valid for all $n \ge 1$ and $s \le t$ and hence it follows that the limiting function $\Psi(\rho)$ is an increasing function.

For (ii), it is easy to see that the points of discontinuity of $\Psi_n(\rho)$ are contained in the points of discontinuity of $\rho$ and uniform convergence of $\Psi_n(\rho)(t)$ to $\Psi(\rho)(t)$ for $t \in [0,T]$ for every $T < \infty$ implies that the same is true for $\Psi(\rho)$ i.e. for $t > 0$, $\Delta\Psi(\rho)(t) \neq 0$ implies that $\Delta\rho(t) \neq 0$.

On the other hand, let $t > 0$ be a discontinuity point for $\rho$. Let us note that by the definition of $t_j^n(\rho)$,

$$|\rho(u) - \rho(v)| \le 2.2^{-n} \quad \forall u, v \in [t_j^n(\rho), t_{j+1}^n(\rho)). \tag{2.5}$$

Thus for $n$ such that $2.2^{-n} < \Delta(\rho)(t)$, $t$ must be equal to $t_k^n(\rho)$ for some $k \ge 1$ since (2.5) implies $\Delta\rho(v) \le 2.2^{-n}$ for any $v \in \cup_j(t_j^n(\rho), t_{j+1}^n(\rho))$. Let $s_n = t_{k-1}^n(\rho)$, where $t = t_k^n(\rho)$ and $s^* = \liminf_n s_n$. We will prove that

$$\lim_n \rho(s_n) = \rho(t-), \quad \lim_n \Psi_n(\rho)(s_n) = \Psi(\rho)(t-). \tag{2.6}$$

If $s^* = t$, then $s_n \le t$ for all $n \ge 1$ implies $s_n \to t$ and (2.6) follows from the uniform convergence of $\Psi_n(\rho)$ to $\Psi(\rho)$ on $[0, t]$.

If $s^* < t$, using (2.5) it follows that $|\rho(u) - \rho(v)| = 0$ for $u, v \in (s^*, t)$ and hence the function $\rho(u)$ is constant on the interval $(s^*, t)$ and implies that $s_n \to s^*$. Also, $\rho(s^*) = \rho(t-)$ and $\Psi(\rho)(s^*) = \Psi(\rho)(t-)$. So if $\rho$ is continuous at $s^*$, once again uniform convergence of $\Psi_n(\rho)$ to $\Psi(\rho)$ on $[0, t]$ shows that (2.6) is valid in this case too.

It remains to consider the case $s^* < t$ and $\Delta\rho(s^*) = \delta > 0$. In this case, for $n$ such that $2.2^{-n} < \delta$, $s_n = s^*$ and uniform convergence of $\Psi_n(\rho)$ to $\Psi(\rho)$ on $[0, t]$ shows that (2.6) is true in this case as well.

We have, for large $n$,

$$\Psi_n(\rho)(t) = \Psi_n(\rho)(s_n) + (\rho(s_n) - \rho(t))^2 \tag{2.7}$$

and hence (2.6) yields

$$\Psi(\rho)(t) = \Psi(\rho)(t-) + [\Delta\rho(t)]^2$$

completing the proof of (ii).

(iii) follows from (i) and (ii) since for an increasing function that is non-negative at zero, the sum of jumps up to $t$ is at most equal to its value at $t$:

$$\sum_{0<s\le t} (\Delta\rho(s))^2 \le \Psi(\rho)(t).$$

The last part, (iv) follows from (ii), (iii). □

*Remark*. $\Psi$ is the *quadratic variation* map. It may depend upon the choice of the partitions. If instead of $2^{-n}$, we had used any other sequence $\{\varepsilon_n\}$, it would yield another mapping $\tilde{\Psi}$ which will have similar properties. Our proof will show that if $\sum_n \varepsilon_n < \infty$, then for a square integrable local martingale $(M_t)$,

$$\Psi(M.(\omega)) = \tilde{\Psi}(M.(\omega)) \ \text{ a.s. } \mathbb{P}.$$

We note two more properties of the quadratic variation map $\Psi$. Recall that the total variation $\mathrm{Var}_T(\rho)$ of $\rho$ on the interval $[0, T]$ is defined by

$$\mathrm{Var}_T(\rho) = \sup\left\{\sum_{j=0}^{m-1} |\rho(s_{j+1}) - \rho(s_j)| : 0 \le s_1 \le s_2 \le \cdots \le s_m = T, \quad m \ge 1\right\}.$$

If $\mathrm{Var}_T(\rho) < \infty$, $\rho$ is said to have bounded variation on $[0, T]$ and then on $[0, T]$ it can be written as difference of two increasing functions.

*Lemma* 2.2. *The quadratic variation map $\Psi$ satisfies the following*:
(i) *For $\rho \in \mathsf{D}$ and $s_k \uparrow \infty$, let $\rho^k$ be defined via $\rho^k(t) = \rho(t \wedge s_k)$. If $\rho^k \in \tilde{\mathsf{D}}$ for all $k$, then $\rho \in \tilde{\mathsf{D}}$ and*

$$\Psi(\rho)(t \wedge s_k) = \Psi(\rho^k)(t), \ \forall t < \infty, \ \forall k \ge 1. \tag{2.8}$$

(ii) *Suppose $\rho$ is continuous, and* $\mathrm{Var}_T(\rho) < \infty$, *then* $\Psi(\rho)(t) = 0, \ \forall t \in [0, T]$.

*Proof.* For (i), it can be checked from the definition that

$$\Psi_n(\rho)(t \wedge s_k) = \Psi_n(\rho^k)(t), \ \forall t. \tag{2.9}$$

Since $\rho^k \in \tilde{\mathsf{D}}$, it follows that $\Psi_n(\rho)(t)$ converges uniformly on $[0, s_k]$ for every $k$ and hence using (2.9) we conclude that $\rho \in \tilde{\mathsf{D}}$ and that (2.8) holds.

For (ii), note that $\rho$ being a continuous function,

$$\left|\rho(t^n_{i+1}(\rho) \wedge t) - \rho(t^n_i(\rho) \wedge t)\right| \le 2^{-n}$$

for all $i, n$ and hence we have

$$\begin{aligned}\Psi_n(\rho)(t) &= \sum_{i=0}^{\infty}\left(\rho(t^n_{i+1}(\rho) \wedge t) - \rho(t^n_i(\rho) \wedge t)\right)^2 \\ &\le 2^{-n} \times \sum_{i=0}^{\infty}\left|\rho(t^n_{i+1}(\rho) \wedge t) - \rho(t^n_i(\rho) \wedge t)\right| \\ &\le 2^{-n} \times \mathrm{Var}_{[0,T]}(\rho).\end{aligned}$$

This shows that $\Psi(\rho)(t) = 0$ for $t \in [0, T]$. □

## 3. Quadratic variation of a martingale

The next lemma connects the quadratic variation map $\Psi$ and r.c.l.l. martingales.

*Lemma* 3.3. *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $(\mathcal{F}_t)$ be a filtration with $\mathcal{F}_0$ containing all null sets in $\mathcal{F}$.*

*Let $(N_t, \mathcal{F}_t)$ be an r.c.l.l. martingale such that $\mathbb{E}(N_t^2) < \infty$ for all $t > 0$. Suppose there is a constant $C < \infty$ such that with*

$$\tau = \inf\{t \ge 0 : \ |N_t| \ge C \quad or \quad |N_{t-}| \ge C\}$$

*one has*

$$N_t = N_{t\wedge\tau}.$$

*Let*

$$[N, N]_t(\omega) = [\Psi(N.(\omega))](t).$$

*Then $([N, N]_t)$ is an $(\mathcal{F}_t)$ adapted r.c.l.l. increasing process such that $X_t := N_t^2 - [N, N]_t$ is also a martingale.*

*Proof.* Let $\Psi_n(\rho)$ and $t^n_i(\rho)$ be as in the previous section.

$$\begin{aligned}A^n_t(\omega) &= \Psi_n(N.(\omega))(t), \\ \sigma^n_i(\omega) &= t^n_i(N.(\omega)), \\ Y^n_t(\omega) &= N^2_t(\omega) - N^2_0(\omega) - A^n_t(\omega).\end{aligned} \tag{3.10}$$

It is easy to see that $\{\sigma_i^n : i \geq 1\}$ are stopping times (for each $n$ by induction on $i$) and that

$$A_t^n = \sum_{i=0}^{\infty} (N_{\sigma_{i+1}^n \wedge t} - N_{\sigma_i^n \wedge t})^2.$$

Further, for each $n$, $\sigma_i^n(\omega)$ increases to $\infty$ as $i \uparrow \infty$. $\square$

We will first prove that for each $n$, $(Y_t^n)$ is an $(\mathcal{F}_t)$ martingale. Using the identity $b^2 - a^2 - (b-a)^2 = 2a(b-a)$, we can write

$$\begin{aligned} Y_t^n &= N_t^2 - N_0^2 - \sum_{i=0}^{\infty} (N_{\sigma_{i+1}^n \wedge t} - N_{\sigma_i^n \wedge t})^2 \\ &= \sum_{i=0}^{\infty} (N^2_{\sigma_{i+1}^n \wedge t} - N^2_{\sigma_i^n \wedge t}) - \sum_{i=0}^{\infty} (N_{\sigma_{i+1}^n \wedge t} - N_{\sigma_i^n \wedge t})^2 \\ &= 2 \sum_{i=0}^{\infty} N_{\sigma_i^n \wedge t} (N_{\sigma_{i+1}^n \wedge t} - N_{\sigma_i^n \wedge t}). \end{aligned}$$

Let us define

$$X_t^{n,i} = N_{\sigma_i^n \wedge t} (N_{\sigma_{i+1}^n \wedge t} - N_{\sigma_i^n \wedge t}).$$

Then

$$Y_t^n = 2 \sum_{i=0}^{\infty} X_t^{n,i}. \tag{3.11}$$

Noting that for $s < \tau$, $|N_s| \leq C$ and for $s \geq \tau$, $N_s = N_\sigma$, it follows that

$$(N_{\sigma_{i+1}^n \wedge t} - N_{\sigma_i^n \wedge t}) > 0 \text{ implies that } |N_{\sigma_i^n \wedge t}| \leq C.$$

Thus, writing $\Gamma_C(x) = \max\{\min\{x, C\}, -C\}$ ($x$ truncated at $C$), we have

$$X_t^{n,i} = \Gamma_C(N_{\sigma_i^n \wedge t})(N_{\sigma_{i+1}^n \wedge t} - N_{\sigma_i^n \wedge t}) \tag{3.12}$$

and hence, $X_t^{n,i}$ is a martingale. Using the fact that $X_t^{n,i}$ is $\mathcal{F}_{t \wedge \sigma_{i+1}^n}$ measurable and that $\mathbb{E}(X_t^{n,i} | \mathcal{F}_{t \wedge \sigma_i^n}) = 0$, it follows that for $i \neq j$,

$$\mathbb{E} X_t^{n,i} X_t^{n,j} = 0 \tag{3.13}$$

Also, using (3.12) and the fact that $N$ is a martingale, we have

$$\begin{aligned} \mathbb{E}(X_t^{n,i})^2 &\leq C^2 \mathbb{E}\{N_{\sigma_{i+1}^n \wedge t} - N_{\sigma_i^n \wedge t}\}^2 \\ &= C^2 \mathbb{E}\{N^2_{\sigma_{i+1}^n \wedge t} - N^2_{\sigma_i^n \wedge t}\}. \end{aligned} \tag{3.14}$$

Using (3.13) and (3.14), it follows that for $s \leq r$,

$$\mathbb{E}\left(\sum_{i=s}^{r} X_t^{n,i}\right)^2 \leq C^2\mathbb{E}\{N^2_{\sigma^n_{r+1}\wedge t} - N^2_{\sigma^n_s \wedge t}\}. \tag{3.15}$$

Since $\sigma_i^n$ increases to $\infty$ as $i$ tends to infinity, $\mathbb{E}(N^2_{\sigma^n_s\wedge t})$ and $\mathbb{E}(N^2_{\sigma^n_{r+1}\wedge t})$ both tend to $\mathbb{E}[N_t^2]$ as $r, s$ tend to $\infty$ and hence $\sum_{i=1}^{r} X_t^{n,i}$ converges in $L^2(\mathbb{P})$. In view of (3.11), one has

$$2\sum_{i=1}^{r} X_t^{n,i} \to Y_t^n \text{ in } L^2(\mathbb{P}) \text{ as } r \to \infty$$

and hence $(Y_t^n)$ is an $(\mathcal{F}_t)$-martingale for each $n \geq 1$.

For $n \geq 1$, define a process $(N^n)$ by

$$N_t^n = N(\sigma_i^n) \text{ if } \sigma_i^n \leq t < \sigma_{i+1}^n.$$

Observe that by the choice of $\{\sigma_i^n : i \geq 1\}$, one has

$$|N_t - N_t^n| \leq 2^{-n} \quad \text{for all } t. \tag{3.16}$$

For now, let us fix $n$. For each $\omega \in \Omega$, let us define

$$E(\omega) = \{\sigma_i^n(\omega) \;:\; i \geq 1\} \cup \{\sigma_i^{n+1}(\omega) \;:\; i \geq 1\}. \tag{3.17}$$

It may be noted that for $\omega$ such that $t \mapsto N_t(\omega)$ is continuous, each $\sigma_j^n(\omega)$ is necessarily equal to $\sigma_{i+1}^n(\omega)$ for some $i$, but this need not be the case when $t \mapsto N_t(\omega)$ has jumps. Let $\zeta_0(\omega) = 0$ and for $j \geq 0$, let

$$\zeta_{j+1}(\omega) = \inf\{s > \zeta_j(\omega) \;:\; s \in E(\omega)\}.$$

It can be verified that

$$\{\zeta_i \;:\; i \geq 1\} = \{\sigma_i^n \;:\; i \geq 1\} \cup \{\sigma_i^{n+1} \;:\; i \geq 1\}. \tag{3.18}$$

To see that each $\zeta_i$ is a stop time, fix $i \geq 1$, $t < \infty$. Let $A_{kj} = \{(\sigma_k^n \wedge t) \neq (\sigma_j^{n+1} \wedge t)\}$. Since $\sigma_k^n, \sigma_j^{n+1}$ are stopping times, $A_{kj} \in \mathcal{F}_t$ for all $k, j$. It is not difficult to see that

$$\{\zeta_i \leq t\} = \bigcup_{k=0}^{i} (\{\sigma_{i-k}^n \leq t\} \bigcap B_k),$$

where $B_0 = \Omega$ and for $1 \leq k \leq i$,

$$B_k = \bigcup_{0<j_1<j_2<\cdots<j_k} \left(\left(\bigcap_{l=0}^{i-k} \bigcap_{m=1}^{k} A_{lj_m}\right) \cap \{\sigma_{j_k}^{n+1} \leq t\}\right)$$

and hence $\zeta_i$ is a stopping time.

Using (3.18) and using the fact that $N^n_t = N_{t\wedge\sigma^n_j}$ for $\sigma^n_j \le t < \sigma^n_{j+1}$, one can write $Y^n$ and $Y^{n+1}$ as

$$\begin{aligned} Y^n_t &= \sum_{j=0}^{\infty} 2N^n_{t\wedge\zeta_j}\{N_{t\wedge\zeta_{j+1}} - N_{t\wedge\zeta_j}\}, \\ Y^{n+1}_t &= \sum_{j=0}^{\infty} 2N^{n+1}_{t\wedge\zeta_j}\{N_{t\wedge\zeta_{j+1}} - N_{t\wedge\zeta_j}\}. \end{aligned}$$

Hence

$$Y^{n+1}_t - Y^n_t = 2\sum_{j=0}^{\infty} Z^{n,j}_t, \tag{3.19}$$

where

$$Z^{n,j}_t = (N^{n+1}_{t\wedge\zeta_j} - N^n_{t\wedge\zeta_j})(N_{t\wedge\zeta_{j+1}} - N_{t\wedge\zeta_j}).$$

Also, using (3.16) one has

$$|N^{n+1}_t - N^n_t| \le |N^{n+1}_t - N_t| + |N_t - N^n_t| \le 2^{-(n+1)} + 2^{-n} \le 2.2^{-n} \tag{3.20}$$

and hence (using that $(N_s)$ is a martingale), one has

$$\mathbb{E}[(Z^{n,j}_t)^2] \le \frac{4}{2^{2n}}\mathbb{E}[(N_{t\wedge\zeta_{j+1}} - N_{t\wedge\zeta_j})^2] = \frac{4}{2^{2n}}\mathbb{E}[(N_{t\wedge\zeta_{j+1}})^2 - (N_{t\wedge\zeta_j})^2]. \tag{3.21}$$

It is easy to see that $\mathbb{E}(Z^{n,j}_t|\mathcal{F}_{t\wedge\sigma^n_j}) = 0$ and $Z^{n,j}_t$ is $\mathcal{F}_{t\wedge\sigma^n_{j+1}}$ measurable. It then follows that for $i \ne j$,

$$\mathbb{E}[Z^{n,j}_t . Z^{n,i}_t] = 0$$

and hence (using (3.21))

$$\begin{aligned} \mathbb{E}(Y^{n+1}_t - Y^n_t)^2 &= 4\mathbb{E}\left[\left(\sum_{j=0}^{\infty} Z^{n,j}_t\right)^2\right] \\ &= 4\mathbb{E}\left[\sum_{j=0}^{\infty}(Z^{n,j}_t)^2\right] \\ &\le \frac{16}{2^{2n}}\sum_{j=0}^{\infty}\mathbb{E}[(N_{t\wedge\zeta_{j+1}})^2 - (N_{t\wedge\zeta_j})^2] \\ &\le \frac{16}{2^{2n}}\mathbb{E}[(N_t)^2]. \end{aligned} \tag{3.22}$$

Thus, recalling that $Y_t^{n+1}$, $Y_t^n$ are martingales, it follows that $Y_t^{n+1} - Y_t^n$ is also a martingale and thus invoking Doob's maximal inequality, one has (using (3.22))

$$\begin{aligned}\mathbb{E}[\sup_{s\le T}|Y_s^{n+1} - Y_s^n|^2] &\le 4\mathbb{E}(Y_T^{n+1} - Y_T^n)^2 \\ &\le \frac{64}{2^{2n}}\mathbb{E}N_T^2. \end{aligned} \tag{3.23}$$

Thus, for each $n \ge 1$ (writing $\|X\|_2$ for the $L^2(\mathbb{P})$ norm : $\|X\|_2 = \sqrt{(\mathbb{E}[X^2])}$),

$$\|\,[\sup_{s\le T}|Y_s^{n+1} - Y_s^n|]\,\|_2 \le \frac{8}{2^n}\|N_T\|_2. \tag{3.24}$$

It follows that

$$\xi = \sum_{n=1}^{\infty}\sup_{s\le T}|Y_s^{n+1} - Y_s^n| < \infty \text{ a.s.}$$

(as $\|\xi\|_2 < \infty$ by (3.24)). Hence $(Y_s^n)$ converges uniformly in $s \in [0, T]$ for every $T$ a.s. to an r.c.l.l. process, say $(Y_s)$. As a result, $(A_s^n)$ also converges uniformly in $s \in [0, T]$ for every $T < \infty$ a.s. to say $(A_s)$ with $Y_t = N_t^2 - N_0^2 - A_t$. Further, (3.4) also implies that for each $s$, convergence of $Y_s^n$ to $Y_s$ is also in $L^2$ and thus $(Y_t)$ is a martingale.

Since $A_s^n$ converges uniformly in $s \in [0, T]$ for all $T < \infty$ a.s., it follows that

$$\mathbb{P}(\omega:\ N.(\omega) \in \tilde{\mathsf{D}}) = 1$$

and $A_t = [N, N]_t$. We have already proven that $Y_t = N_t^2 - N_0^2 - [N, N]_t$ is a martingale. This completes the proof.

We are now in a position to prove an analogue of the Doob–Meyer decomposition theorem for the square of an r.c.l.l. locally square integrable martingale.

**Theorem 3.4.** *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $(\mathcal{F}_t)$ be a filtration with $\mathcal{F}_0$ containing all null sets in $\mathcal{F}$. Let $(M_t, \mathcal{F}_t)$ be an r.c.l.l. locally square integrable martingale i.e. there exist stopping times $\zeta_n$ increasing to $\infty$ such that for each $n$, $M_t^n = M_{t\wedge\zeta_n}$ is a martingale with $\mathbb{E}[(M_{t\wedge\zeta_n})^2] < \infty$ for all $t, n$.*

*Let*

$$[M, M]_t(\omega) = [\Psi(M.(\omega))](t).$$

*Then*

(i) *$([M, M]_t)$ is an $(\mathcal{F}_t)$ adapted r.c.l.l. increasing process such that $X_t = M_t^2 - [M, M]_t$ is also a local martingale.*

(ii)

$$\mathbb{P}(\Delta[M, M]_t = (\Delta M_t)^2,\ \forall t > 0) = 1.$$

(iii) *If $(B_t)$ is an r.c.l.l. adapted increasing process such that $B_0 = 0$ and*

$$\mathbb{P}(\Delta B_t = (\Delta M_t)^2,\ \forall t > 0) = 1$$

*and $V_t = M_t^2 - B_t$ is a local martingale, then $\mathbb{P}(B_t = [M, M]_t,\ \forall t) = 1$.*

(iv) *If $M$ is a martingale and $\mathbb{E}\left(M_t^2\right) < \infty$ for all $t$, then $\mathbb{E}\left([M, M]_t\right) < \infty$ for all $t$ and $X_t = M_t^2 - [M, M]_t$ is a martingale.*
(v) *If $\mathbb{E}\left([M, M]_t\right) < \infty$ for all $t$ then $\mathbb{E}\left(M_t^2\right) < \infty$ for all $t$, $(M_t)$ is a martingale and $X_t = M_t^2 - [M, M]_t$ is a martingale.*

*Proof.* For $k \geq 1$, let $\tau_k$ be the stopping time defined by

$$\tau_k = \inf\{t > 0 : |M_t| \geq k\} \wedge \zeta_k \wedge k.$$

Then $M_t^k = M_{t\wedge\tau_k}$ is a martingale satisfying conditions of Lemma 3.3 with $C = k$ and $\tau = \tau_k$ and hence $X_t^k = (M_t^k)^2 - [M^k, M^k]_t$ is a martingale, where $[M^k, M^k]_t = \Phi(M_\cdot^k(\omega))_t$. Also,

$$\mathbb{P}(\{\omega : M_\cdot^k(\omega) \in \tilde{\mathsf{D}}\}) = 1, \quad \forall k \geq 1. \tag{3.25}$$

Since $M_t^k = M_{t\wedge\tau_k}$, it follows from Lemma 2.2 that

$$\mathbb{P}(\{\omega : M_\cdot(\omega) \in \tilde{\mathsf{D}}\}) = 1 \tag{3.26}$$

and

$$\mathbb{P}(\{\omega : [M^k, M^k]_t(\omega) = [M, M]_{t\wedge\tau_k(\omega)}(\omega)\}) = 1.$$

It follows that $X_{t\wedge\tau_k} = X_t^k$ a.s. and since $X^k$ is a martingale for all $k$, it follows that $X_t$ is a local martingale. This completes the proof of part (i). Part (ii) follows from Lemma 2.1. For (iii), note that

$$U_t = [M, M]_t - B_t$$

is a continuous process and recalling $X_t = M_t^2 - [M, M]_t$ and $V_t = M_t^2 - B_t$ are local martingales, it follows that $U_t = V_t - X_t$ is also a local martingale with $U_0 = 0$. By part (i) above, $W_t = U_t^2 - [U, U]_t$ is a local martingale. On the other hand, $U_t$ being a difference of two increasing functions has bounded variation, $\mathrm{Var}_T(U) < \infty$. Since $U$ is continuous, by Lemma 2.2,

$$[U, U]_t = 0 \ \ \forall t.$$

Hence $W_t = U_t^2$ is a local martingale. Now if $\sigma_k$ are stop times increasing to $\infty$ such that $W_{t\wedge\sigma_k}$ is a martingale for $k \geq 1$, then we have

$$\mathbb{E}[W_{t\wedge\sigma_k}] = \mathbb{E}[U_{t\wedge\sigma_k}^2] = \mathbb{E}[U_0^2] = 0$$

and hence $U_{t\wedge\sigma_k}^2 = 0$ for each $k$. This yields $U_t = 0$ a.s. for every $t$. This completes the proof of (iii).

For (iv), we have proven in (i) that $X_t = M_t^2 - [M, M]_t$ is a local martingale. Let $\sigma_k$ be stop times increasing to $\infty$ such that $X_t^k = X_{t\wedge\sigma_k}$ are martingales. Hence, $\mathbb{E}[X_t^k] = 0$, or

$$\mathbb{E}\left([M, M]_{t\wedge\sigma_k}\right) = \mathbb{E}(M_{t\wedge\sigma_k}^2) - \mathbb{E}(M_0^2). \tag{3.27}$$

Invoking Doob's maximal inequality, we have

$$\mathbb{E}\Big[\sup_{s\le t}|M_s^2|\Big] \le 4\mathbb{E}[M_t^2] < \infty. \tag{3.28}$$

Now, $M^2_{t\wedge\sigma_k}$ are dominated by the integrable function $\sup_{s\le t}|M_s^2|$ and hence $M^2_{t\wedge\sigma_k}$ converges to $M_t^2$ in $L^1$. By monotone convergence theorem,

$$\mathbb{E}([M,M]_{t\wedge\sigma_k}) \to \mathbb{E}\left([M,M]_t\right). \tag{3.29}$$

Thus from (3.27),

$$\mathbb{E}\left([M,M]_t\right) = \mathbb{E}[M_t^2] - \mathbb{E}[M_0^2]$$

and convergence in (3.29) is in $L^1$. It follows that $X_t^k$ converges to $X_t$ in $L^1(\mathbb{P})$ and hence $(X_t)$ is a martingale.

For (v) let $\sigma_k$ be as in part (iv). One has using (3.27) and that $[M,M]_t$ is increasing,

$$\begin{aligned}\mathbb{E}[M^2_{t\wedge\sigma_k}] &= \mathbb{E}[M_0^2] + \mathbb{E}([M,M]_{t\wedge\sigma_k})\\ &\le \mathbb{E}[M_0^2] + \mathbb{E}([M,M]_t).\end{aligned}$$

Now using Fatou's lemma, one gets

$$\mathbb{E}[M_t^2] \le \mathbb{E}[M_0^2] + \mathbb{E}([M,M]_t) < \infty.$$

Now we can invoke part (iv) to complete the proof.

It may be noted that we have not assumed that the underlying filtration is right-continuous.

*Remark*. We conclude with an observation that we cannot get a mapping

$$\Phi : D([0,\infty),\mathbb{R}) \to D([0,\infty),\mathbb{R})$$

such that for any martingale $M$,

$$\Phi(M_{\cdot}(\omega)) = \langle M,M\rangle_{\cdot}(\omega). \tag{3.30}$$

Let $\Omega = \mathbb{R}$ and $\mathcal{F} = \mathcal{B}(\mathbb{R})$, the Borel $\sigma$-field on $\mathbb{R}$. Let $X(\omega) = \omega$ and let $\mathbb{P}_\theta$ denote the probability measure on $(\Omega,\mathcal{F})$ such that $X$ has Gaussian distribution with mean 0 and variance $\theta^2$. Let $M$ be defined by

$$M_t(\omega) = X(\omega)\mathbf{1}_{[1,\infty)}(t).$$

It is easy to see that $M$ is a martingale with respect to its canonical filtration and

$$[M,M]_t(\omega) = X^2(\omega)\mathbf{1}_{[1,\infty)}(t)$$

while

$$\langle M,M\rangle_t(\omega) = \theta^2\mathbf{1}_{[1,\infty)}(t).$$

Thus if a mapping $\Phi$ satisfying (3.30) exists, then

$$\Phi(M.(\omega))_t = \theta^2 \mathbf{1}_{[1,\infty)}(t) \quad \text{a.s.}\, \mathbb{P}_\theta. \tag{3.31}$$

On the other hand, it is well known and easy to see that for any $0 < \theta_1, \theta_2 < \infty$, the probability measures $\mathbb{P}_{\theta_1}$ and $\mathbb{P}_{\theta_2}$ are mutually absolutely continuous contradicting (3.31).

Thus for a martingale $M$, while the $\omega$-*path* of the quadratic variation process $[M, M] \cdot (\omega)$ can be *computed* using only the path $M.(\omega)$ of the martingale, the path of predictable quadratic variation $\langle M, M\rangle.(\omega)$ cannot be *computed* or *expressed* in a similar fashion.

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