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Growth of fundamental group for Finsler manifolds with integral Ricci curvature bound

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Abstract. In this paper, an upper bound on the growth of fundamental group for a class of Finsler manifolds with integral Ricci curvature bound is given. This generalizes the corresponding results with pointwise Ricci curvature in literature.

Keywords. Finsler manifold; fundamental group; integral Ricci curvature; uniformity constant; reversibility.

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1. Introduction

The growth of fundamental group for Riemannian manifolds was first discussed by Milnor [4]. By using the volume comparison theorems he was able to prove that the fundamental group of any n-dimensional compact Riemannian manifold with nonnegative Ricci curvature has polynomial growth of order $\leq n$. There have been several attempts to generalize this result in various directions, most of which extend the theorem to the case of manifolds with small 'wells of negative curvature' (see [2,10,15]). For example, Wei [10] allowed the negative lower bound on the Ricci curvature to be sufficiently small and showed that for any constant $v > 0$, there exists an $\epsilon = \epsilon(n, v) > 0$ such that if a compact n-manifold M admits a Riemannian metric satisfying the conditions $\mathbf{Ric}_M \geq -\epsilon$, $\mathrm{diam}(M) = 1$ and $\mathrm{vol}(M) \geq v$, then the fundamental group of M is of polynomial growth of order $\leq n$. It should be pointed out here that the pointwise curvature can be replaced by integral curvature bound and one can consider the growth of fundamental group under integral Ricci curvature bound [3,16].

Finsler geometry is just the Riemannian geometry without quadratic restriction. Instead of a Euclidean norm on each tangent space, one endows Minkowski norms on every tangent space of a differentiable manifold. In global Finsler geometry, it is also important to reveal the relationship between the topology and geometry invariants for Finsler manifolds. As for the fundamental group of Finsler manifolds, Milnor's result has been generalized to Finsler manifolds by Shen for reversible case [6,7], and by Shen and Zhao [8] for the general case. However, an additional condition on the te S-curvature is needed in these results. This additional condition on S-curvature was removed recently by using the maximal or minimal volume form [11,12].

Recently a relative volume comparison theorem for Finsler manifolds under integral Ricci curvature bound was established and the integral Ricci curvature and topology was studied [13]. In this paper, an upper bound on the growth of fundamental group for a class of Finsler manifolds with integral Ricci curvature bound is given. The result generalizes the corresponding results with pointwise Ricci curvature in the literature. The maximal and minimal volume forms (see §2 for the definition) are used throughout this paper. We need some notations to state our result. On a compact Finsler manifold (M, F), let $\mathrm{d}V_{\max}$ and $\mathrm{d}V_{\min}$ be the maximal volume form and minimal volume form, respectively, and $\underline{\mathbf{Ric}}$: $M \to \mathbb{R}$ the function of smallest Ricci curvature at a given point. More precisely,

$$\underline{\mathbf{Ric}}(x) = \min_{y \in T_x M \setminus \{0\}} \mathbf{Ric}(y), \quad \forall x \in M.$$

Let

$$\epsilon(p; M) := \left(\frac{\int_M (\max\{-\underline{\mathbf{Ric}}, 0\})^p \mathrm{d}V_{\max}}{\mathrm{vol}_{\min}(M)} \right)^{\frac{1}{p}}.$$

It is clear that $\epsilon(p; M) = 0$ whenever $\mathbf{Ric}_M \geq 0$. The main purpose of this paper is to prove the following.

Theorem 1.1. *Given $n \in \mathbb{N}$, $p > n/2$, $\delta \in [1, \infty)$, and $v, D \in (0, \infty)$, there exists $\alpha = \alpha(n, p, \delta, v, D) > 0$ such that if a compact n-manifold M admits a Finsler metric F satisfying the conditions*

$$\epsilon(p; M) \leq \alpha, \quad \mu_F \leq \delta^2, \quad \mathrm{vol}_{\min}(M) \geq v, \quad \mathrm{diam}(M) \leq D,$$

then the fundamental group of M is of polynomial growth of order $\leq n$. Here μ_F is the uniformity constant of F.

COROLLARY 1.2 [14]

Given $n \in \mathbb{N}$, $\delta \in [1, \infty)$, and $v, D \in (0, \infty)$, there exist $\varepsilon = \varepsilon(n, \delta, v, D) > 0$ such that if a compact n-manifold M admits a Finsler metric F satisfying the conditions

$$\mathbf{Ric}_M \geq -\varepsilon, \quad \mu_F \leq \delta^2, \quad \mathrm{vol}_{\min}(M) \geq v, \quad \mathrm{diam}(M) \leq D,$$

then the fundamental group of M is of polynomial growth of order $\leq n$. Particularly, the fundamental group of any n-dimensional compact Finsler manifold with nonnegative Ricci curvature has polynomial growth of order $\leq n$.

Remark 1.1. Corollary 1.2 generalizes the corresponding results in [9,11,12]. By comparing the result in [9], we remove the additional condition on S-curvature.

2. Preliminaries

In this section, we give a brief description of some basic materials that are needed to prove Theorem 1.1, for more details one is referred to [11,13]. Let (M, F) be a Finsler n-manifold with Finsler metric $F : TM \to [0, \infty)$. The *fundamental tensor* g_{ij} is defined by

$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}.$$

A *volume form* $\mathrm{d}\mu$ on Finsler manifold (M, F) is nothing but a global non-degenerate n-form on M. In local co-ordinates we can express $\mathrm{d}\mu$ as $\mathrm{d}\mu = \sigma(x)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. The frequently used volume forms in Finsler geometry are the so-called Busemann-Hausdorff volume form and Holmes–Thompson volume form. Other useful volume forms are the maximal and minimal volume forms which can be defined as follows. Let

$$\mathrm{d}V_{\max} = \sigma_{\max}(x)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

and

$$\mathrm{d}V_{\min} = \sigma_{\min}(x)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

with

$$\sigma_{\max}(x) := \max_{y \in T_xM\setminus\{0\}} \sqrt{\det(g_{ij}(x, y))}, \quad \sigma_{\min}(x) := \min_{y \in T_xM\setminus\{0\}} \sqrt{\det(g_{ij}(x, y))}.$$

Then it is easy to check that the n-forms $\mathrm{d}V_{\max}$ and $\mathrm{d}V_{\min}$ are well-defined on M. We call $\mathrm{d}V_{\max}$ and $\mathrm{d}V_{\min}$ the *maximal volume form* and the *minimal volume form* of (M, F), respectively. We note that both maximal and minimal volume forms play a crucial role in comparison techniques in Finsler geometry [11–13]. The *uniformity function* $\mu : M \to \mathbb{R}$ is defined by

$$\mu(x) = \max_{y,z,u \in T_xM\setminus 0} \frac{\mathbf{g}_y(u, u)}{\mathbf{g}_z(u, u)}.$$

$\mu_F = \max_{x\in M} \mu(x)$ is called the *uniformity constant* [1]. Similarly, the *reversible function* $\lambda : M \to \mathbb{R}$ is defined by

$$\lambda(x) = \max_{y \in T_xM\setminus 0} \frac{F(y)}{F(-y)}.$$

$\lambda_F = \max_{x\in M} \lambda(x)$ is called the *reversibility* of (M, F) [5], and (M, F) is called *reversible* if $\lambda_F = 1$. It is clear that $\lambda(x)^2 \leq \mu(x)$ and thus $\lambda_F^2 \leq \mu_F$.

In order to prove Theorem 1.1, we need to use the relative volume comparison theorem for star-shaped subset with integral Ricci curvature bound. Let $T \subset M$, we say that T is star-shaped at $x \in T$ if for all $y \in T$ there exists a minimal geodesic from x to y contained in T. For $r > 0$, let $T(r) = T \cap B_x(r)$, where $B_x(r)$ is the forward geodesic ball of radius r centered at x. For any star-shaped subset T of finite minimal volume consider

$$\epsilon(p; T) := \left(\frac{\int_T (\max\{-\underline{\mathbf{Ric}}, 0\})^p \mathrm{d}V_{\max}}{\mathrm{vol}_{\min}(T)} \right)^{\frac{1}{p}}.$$

The following theorem is a special case of Theorem 1.1 in [13].

Theorem 2.1. *Let (M, F) be a forward complete Finsler n-manifold, T be a star-shaped subset of M at $x \in M$, and $T \subset B_x(R_T)$ for some $R_T > 0$. For $p > n/2$, there exists a constant $C(n, p, R_T) > 0$ such that when*

$$\epsilon = \epsilon(p; T) < \left(\frac{1}{2C(n, p, R_T)} \right)^{\frac{2p-1}{p}},$$

then one has for all $0 < r \le R \le R_T$,

$$\frac{\mathrm{vol}_{\min}(T(R))}{\mathrm{vol}(\mathbb{B}^n(R))} \le \left(\frac{1-C(n,\, p,\, R_T)\epsilon^{\frac{p}{2p-1}}}{1-2C(n,\, p,\, R_T)\epsilon^{\frac{p}{2p-1}}}\right)^{2p-1} \cdot \max_{z\in B_x(r)} (\mu(z))^{\frac{n}{2}} \cdot \frac{\mathrm{vol}_{\min}(T(r))}{\mathrm{vol}(\mathbb{B}^n(r))}.$$

Here μ *is the uniformity function, and* $\mathbb{B}^n(R)$ *the ball of radius* R *in Euclidean n-space* $\mathbb{R}^n$. *In particular, for all* $0 < R \le R_T$,

$$\mathrm{vol}_{\min}(T(R)) \le \mu(x)^{\frac{n}{2}} \left(\frac{1 - C(n,\, p,\, R_T)\epsilon^{\frac{p}{2p-1}}}{1 - 2C(n,\, p,\, R_T)\epsilon^{\frac{p}{2p-1}}}\right)^{2p-1} \mathrm{vol}(\mathbb{B}^n(R)).$$

Remark 2.1 [13]. The quantity $\left(\frac{1-C(n,p,R_T)\epsilon^{\frac{p}{2p-1}}}{1-2C(n,p,R_T)\epsilon^{\frac{p}{2p-1}}}\right)^{2p-1}$ in Theorem 2.1 is nondecreasing both in R_T and in ϵ. This property will be used in §3.

For a given compact Finsler manifold (M, F), let $f : (\tilde{M}, \tilde{F}) \to (M, F)$ be the universal covering with pulled-back metric. Then it is known that the fundamental group is isomorphic to the deck transformation group Γ and each deck transformation is an isometry of $(\tilde{M}, \tilde{F})$ (see [8,12] for details). We also need the following two lemmas.

Lemma 2.2 [13]. *Let* $f : (\tilde{M}, \tilde{F}) \to (M, F)$ *be the universal covering space of* (M, F), *then for any forward geodesic ball* $\tilde{B}_{\tilde{x}}(r) \subset \tilde{M}$ *with* $r > \mathrm{diam}(M)$ *there exists a star-shaped subset* T *satisfying* $\tilde{B}_{\tilde{x}}(r) \subset T \subset \tilde{B}_{\tilde{x}}((2+\lambda_F)r)$ *and*

$$\frac{\int_T (\max\{-\underline{\mathbf{Ric}}, 0\})^p \mathrm{d}V_{\max}}{\mathrm{vol}_{\min}(T)} = \frac{\int_M (\max\{-\underline{\mathbf{Ric}}, 0\})^p \mathrm{d}V_{\max}}{\mathrm{vol}_{\min}(M)}. \tag{1}$$

Lemma 2.3 [9]. *Let* (M, F) *be a compact Finsler n-manifold of reversibility* λ_F *and* $\tilde{M}$ *be its universal covering space. For each* $x \in M$, *there always exists a generating set* $\{\gamma_1, \ldots, \gamma_m\}$ *for the fundamental group* $\Gamma \cong \pi_1(M, x)$ *such that* $d(\tilde{x}, \gamma_i(\tilde{x})) \le (1 + \lambda_F)\mathrm{diam}(M)$ (*where* $\tilde{x} \in f^{-1}(x)$ *is in the fiber over* $x \in M$) *and such that all relations for* Γ *in these generators are of the form* $\gamma_i\gamma_j\gamma_k^{-1} = 1$.

3. Proof of Theorem 1.1

In this section we shall complete the proof of Theorem 1.1. Let us first recall the notion of algebraic norm. Let G be a finitely generated group and $S = \{g_i\}$ be a generating set for G. For each $g \in G$, define $\|g\|_{\mathrm{alg}}$ to be the smallest length of the word in terms of g_i and their inverse that represents g. We call $\|\cdot\|_{\mathrm{alg}}$ the *algebraic norm* associated with the generating set S. One is referred to [4,8] for more details of the algebraic norm.

Proof of Theorem 1.1. Given $n \in \mathbb{N}$, $p > n/2$, $\delta \in [1, \infty)$, and $v, D \in (0, \infty)$. Suppose on the contrary that for any

$$\left(\frac{1}{3C(n,\, p,\, (2+\delta)^2 D)}\right)^{\frac{2p-1}{p}} \ge \alpha > 0,$$

there exists a Finsler metric F on compact n-manifold M satisfying

$$\epsilon(p; M) \leq \alpha, \quad \mu_F \leq \delta^2, \quad \mathrm{vol}_{\min}(M) \geq v, \quad \mathrm{diam}(M) \leq D$$

such that $\pi_1(M)$ is not of polynomial growth of order $\leq n$, here the constant C is given by Theorem 2.1. Choose the generating set $\{\gamma_1, \ldots, \gamma_m\}$ of $\pi_1(M, x)$ as in Lemma 2.3. Let $\Omega_x \subset \tilde{M}$ be a fundamental domain constructed as in [8]. The sets $\gamma_i(\Omega_x)$, $1 \leq i \leq m$ are mutually disjoint and have same minimal volume as M, since γ_i acts isometrically. Notice that $\lambda_F^2 \leq \mu_F \leq \delta^2$, one has $d(\tilde{x}, \gamma_i(\tilde{x})) \leq (1+\delta)D$, $\gamma_i(\Omega_x) \subset \tilde{B}_{\tilde{x}}((2+\delta)D)$, $\forall 1 \leq i \leq m$. By Lemma 2.2, there is a star-shaped subset $T \subset \tilde{M}$ satisfying (1) and $\tilde{B}_{\tilde{x}}((2+\delta)D) \subset T \subset \tilde{B}_{\tilde{x}}((2+\delta)^2 D)$. Since

$$\epsilon = \epsilon(p; M) \leq \alpha \leq \left(\frac{1}{3C(n, p, (2+\delta)^2 D)} \right)^{\frac{2p-1}{p}},$$

by Theorem 2.1 and Remark 2.1 we have

$$\begin{aligned} &\mathrm{vol}_{\min}(\tilde{B}_{\tilde{x}}((2+\delta)D)) \leq \mathrm{vol}_{\min}(T) \\ &\leq \delta^n \cdot \left(\frac{1 - C(n, p, (2+\delta)^2 D)\epsilon^{\frac{p}{2p-1}}}{1 - 2C(n, p, (2+\delta)^2 D)\epsilon^{\frac{p}{2p-1}}} \right)^{2p-1} \cdot \mathrm{vol}(\mathbb{B}^n((2+\delta)^2 D)) \\ &\leq \delta^n \cdot 2^{2p-1} \cdot \mathrm{vol}(\mathbb{B}^n((2+\delta)^2 D)). \end{aligned}$$

Recall that $\mathrm{vol}_{\min}(\Omega_x) = \mathrm{vol}_{\min}(M)$, we get

$$m \leq \frac{\mathrm{vol}_{\min}(\tilde{B}_{\tilde{x}}((2+\delta)D))}{\mathrm{vol}_{\min}(\Omega_x)} \leq \frac{\delta^n \cdot 2^{2p-1} \cdot \mathrm{vol}(\mathbb{B}^n((2+\delta)^2 D))}{v} < \infty.$$

In summary, one can choose a finite generating set $\{\gamma_1, \ldots, \gamma_m\}$ of $\pi_1(M, x)$ such that

(i) $m \leq \frac{\delta^n \cdot 2^{2p-1} \cdot \mathrm{vol}(\mathbb{B}^n((2+\delta)^2 D))}{v} := N(n, p, \delta, v, D)$,
(ii) $d(\tilde{x}, \gamma_i(\tilde{x})) \leq (1+\delta)D$, for each $1 \leq i \leq m$,
(iii) every relation is of the form $\gamma_i \gamma_j \gamma_k^{-1} = 1$.

For any $s \geq 1$ define $\Gamma(s) = \{\gamma \in \pi_1(M, x) : \|\gamma\|_{alg} \leq s\}$, and choose the fundamental domain Ω_x as before. By (ii) and triangle inequality, one easily has $\gamma(\Omega_x) \subset \tilde{B}_{\tilde{x}}(s(1+\delta)D + D) \subset \tilde{B}_{\tilde{x}}(s(2+\delta)D)$ for each $\gamma \in \Gamma(s)$, and consequently,

$$\sharp\Gamma(s) \leq \frac{\mathrm{vol}_{\min}(\tilde{B}_{\tilde{x}}(s(2+\delta)D)}{\mathrm{vol}_{\min}(M)}. \tag{2}$$

Since $\pi_1(M)$ is not of polynomial growth of order $\leq n$, for each $j \in \mathbb{N}$, there exists $s_j \in \mathbb{N}$ such that

$$\sharp\Gamma(s_j) > j \cdot (s_j)^n. \tag{3}$$

It is crucial that this relation is independent of α, as follows from (i) and (iii).

Now by Lemma 2.2, there is a star-shaped subset $T \subset \tilde{M}$ satisfying (1) and $\tilde{B}_{\tilde{x}}(s(2+\delta)D) \subset T \subset \tilde{B}_{\tilde{x}}(s(2+\delta)^2 D)$, and from Theorem 2.1 and (2) one has

$$\sharp\Gamma(s) \leq \frac{\delta^n}{v} \cdot \left(\frac{1 - C(n, p, s(2+\delta)^2 D)\epsilon^{\frac{p}{2p-1}}}{1 - 2C(n, p, s(2+\delta)^2 D)\epsilon^{\frac{p}{2p-1}}} \right)^{2p-1} \cdot \mathrm{vol}(\mathbb{B}^n(s(2+\delta)^2 D))$$

$$\leq \frac{\delta^n (2+\delta)^{2n} D^n}{v} \cdot 2^{2p-1} \cdot \mathrm{vol}(\mathbb{B}^n(1)) \cdot s^n$$

when

$$\epsilon(p; M) < \left(\frac{1}{3C(n, p, s(2+\delta)^2 D)} \right)^{\frac{2p-1}{p}}.$$

In summary, for any fixed, sufficiently large s_0, there is

$$\alpha_0 = \alpha_0(s_0, n, p, \delta, D) := \left(\frac{1}{3C(n, p, s_0(2+\delta)^2 D)} \right)^{\frac{2p-1}{p}}$$

such that for each $s \leq s_0$ and $\alpha \leq \alpha_0(s_0, n, p, \delta, D)$,

$$\begin{aligned} &\sharp\Gamma(s) \leq A(n, p, \delta, v, D) s^n, \\ &A(n, p, \delta, v, D) := \frac{\delta^n (2+\delta)^{2n} D^n}{v} \cdot 2^{2p-1} \cdot \mathrm{vol}(\mathbb{B}^n(1)). \end{aligned} \tag{4}$$

Now let $j_0 > A(n, p, \delta, v, D)$, by (3), there exists s_{j_0} such that

$$\sharp\Gamma(s_{j_0}) > A(n, p, \delta, v, D)(s_{j_0})^n.$$

But we get a contradiction by taking $\alpha \leq \alpha_0(s_{j_0}, n, p, \delta, D)$ and (4). □

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