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Stability of a simple Levi–Civitá functional equation on non-unital commutative semigroups

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Abstract. In this paper, we study the Hyers–Ulam stability of a simple Levi–Civitá functional equation $f(x+y) = f(x)h(y) + f(y)$ and its pexiderization $f(x+y) = g(x)\,h(y)+k(y)$ on non-unital commutative semigroups by investigating the functional inequalities $|f(x+y) - f(x)h(y) - f(y)| \le \epsilon$ and $|f(x+y) - g(x)h(y) - k(y)| \le \epsilon$, respectively. We also study the bounded solutions of the simple Levi–Civitá functional inequality.

Keywords. Additive function; exponential function; functional inequality; Hyers–Ulam stability; Levi–Civitá equation; non-unital semigroup; 2-divisible group.

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1. Introduction

A certain formula or equation is applicable to model a physical process if a small change in the formula or equation gives rise to a small change in the corresponding result. When this happens we say the formula or equation is stable. In an application, a functional equation like the additive Cauchy functional equation $f(x+y) - f(x) - f(y) = 0$ may not be true for all $x, y \in \mathbb{R}$ but it may be true approximately, that is

$$f(x+y) - f(x) - f(y) \approx 0$$

for all $x, y \in \mathbb{R}$. This can be stated mathematically as

$$|f(x+y) - f(x) - f(y)| \le \varepsilon \tag{1.1}$$

for some small positive ε and for all $x, y \in \mathbb{R}$. We would like to know when small changes in a particular equation like the additive Cauchy functional equation have only small effects on its solutions. This is the essence of stability theory.

In 1940, S. M. Ulam asked the following question: Given a group G, a metric group H with metric $d(\cdot,\cdot)$ and a positive number ϵ, does there exist a $\delta > 0$ such that if $f : G \to H$ satisfies

$$d(f(xy), f(x)f(y)) \leq \varepsilon$$

for all $x, y \in G$, then a homomorphism $\phi : G \to H$ exists with

$$d(f(x), \phi(x)) \leq \delta$$

for all $x \in G$? These kind of questions form the material for the stability theory of functional equations (see [11] and [14]). For Banach spaces, Ulam's problem was solved by Hyers [10] in 1941 with $\delta = \epsilon$ and the additive map

$$\phi(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}.$$

The functional equation

$$f(x+y) = f(x)h(y) + f(y) \tag{1.2}$$

is a special case of the Levi–Civitá functional equation

$$f(x+y) = g_1(x)h_1(y) + g_2(x)h_2(y) + \cdots + g_n(x)h_n(y) \tag{1.3}$$

which was solved by Levi–Civitá in [12] under differentiability conditions. A simple Levi–Civitá functional equation (1.2) was recently studied by Ebanks in [9] on non-abelian groups. This functional equation was also treated in [7], [1] and [2]. This functional equation contains the Cauchy additive functional equation $f(x+y) = f(x) + f(y)$.

The stability of the Levi-Civitá functional equation was investigated by Shulman on amenable locally compact groups in [15]. For more on functional equations and stabilities of functional equations, the interested reader should refer to the books [3], [14] and [11].

In this paper, we investigate the Hyers–Ulam stability of the functional equation (1.2) and its generalization

$$f(xy) = g(x)h(y) + k(y) \tag{1.4}$$

on non-unital commutative semigroups. We also study the bounded real-valued solutions of the functional inequality

$$|f(x+y) - f(x)h(y) - f(y)| \leq \epsilon$$

holding for all $x, y \in G$, where G is a 2-divisible commutative group and $\epsilon \geq 0$.

2. Terminology and preliminary results

A nonempty set S with an associative binary operation is called a semigroup. If in addition there is an identity element for the operation, then S is called a unital semigroup. A non-unital semigroup S is a semigroup without a neutral element. Throughout this paper we denote by S a non-unital commutative semigroup and $\mathbb{C}$ the field of complex numbers. For a commutative semigroup S with binary operation $+$, the set $S+S$ is defined as $\{s+t \mid s, t \in S\}$. A function $A : S \to \mathbb{C}$ is said to be *additive* provided $A(x+y) =$

$A(x)+A(y)$ for all $x, y \in S$, and $m : S \to \mathbb{C}$ *exponential* provided $m(x+y) = m(x)\, m(y)$ for all $x, y \in S$.

In 1941, Hyers [10] provided an answer to Ulam's problem given in the theorem below.

Theorem 2.1 [**10**]. *Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying*

$$\| f(x + y) - f(x) - f(y)\| \leq \varepsilon \tag{2.1}$$

for all x, y in X and for some $\varepsilon \geq 0$. Then the limit

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{2.2}$$

exists for all x in X and $A : X \to Y$ is the unique additive mapping satisfying

$$\| f(x) - A(x)\| \leq \varepsilon \tag{2.3}$$

for all x in X.

Hyers' theorem holds if the Banach space X is replaced by a commutative semigroup S without a neutral element.

The stability of the exponential functional equation $f(xy) = f(x)\, f(y)$ was first investigated by Baker *et al.* [6] and then subsequently improved by Baker [5] and Albert and Baker [4]. The following theorem is due to Baker (see [5]) and is well known.

Theorem 2.2. *Let $\varepsilon > 0$. Let S be a semigroup and let f be a complex-valued function defined on S such that $|f(xy) - f(x)f(y)| \leq \varepsilon$ for all $x, y \in S$. Then either $f(x) \leq (1 + \sqrt{1 + 4\varepsilon}\,)/2$ for all $x \in S$ or $f(xy) = f(x)\, f(y)$ for all $x, y \in S$.*

3. Stability of the Levi–Civitá equation (1.2)

Let $f, h : S \to \mathbb{C}$ be complex-valued functions defined on a semigroup S and $\epsilon \geq 0$. In this section, we consider the functional inequality

$$|f(x + y) - f(x)\, h(y) - f(y)| \leq \epsilon \tag{3.1}$$

for all $x, y \in S$. Throughout this section we exclude the trivial cases $f \equiv 0$ or $h \equiv 0$.

Theorem 3.1. *Let $f, h : S \to \mathbb{C}$ be nonzero functions satisfying inequality (3.1). Then (f, h) satisfies one of the following*:

(i) *both f and h are bounded functions*,

(ii) *$h(x) = 1$ for all $x \in S$ and there exists an additive function $A : S \to \mathbb{C}$ such that*

$$|f(x) - A(x)| \leq \epsilon \qquad \forall\, x \in S,$$

(iii) *there exist an unbounded exponential function m and $\alpha \in \mathbb{C}$ such that*

$$h(x) = m(x) \quad \text{and} \quad f(x) = \alpha\,(m(x) - 1) \qquad \forall\, x \in S.$$

Proof. Assume that f is bounded. Since $f \not\equiv 0$, it follows from (3.1) that h is bounded, which yields (i).

If $h \equiv 1$, then by Theorem 2.1 there exists a unique additive function $A : S \to \mathbb{C}$ such that

$$|f(x) - A(x)| \leq \epsilon \tag{3.2}$$

for all $x \in S$. Hence, we have (ii).

Thus, it remains to consider the case when f is unbounded and $h \not\equiv 1$. Replacing (x, y) by (y, x) in (3.1) and using the triangle inequality we have

$$\begin{aligned}
|f(x)\,(h(y) - 1) \ - \ f(y)\,(h(x) - 1)| \\
&= |f(y + x) - f(y)\,h(x) - f(x) - f(x + y) \\
&\quad + f(x)\,h(y) + f(y)| \\
&\leq |f(y + x) - f(y)\,h(x) - f(x)| + |f(x + y) \\
&\quad - f(x)\,h(y) - f(y)| \\
&\leq 2\epsilon
\end{aligned}$$

which is

$$|f(x)\,(h(y) - 1) - f(y)\,(h(x) - 1)| \leq 2\epsilon \tag{3.3}$$

for all $x, y \in S$. Dividing both sides of (3.3) by $|h(y) - 1|$ we have

$$|f(x) - \alpha(y)(h(x) - 1)| \leq \frac{2\epsilon}{|h(y) - 1|} \tag{3.4}$$

for all $y \in J := \{y : h(y) \neq 1\}$, where $\alpha(y) = f(y)/(h(y) - 1)$. Since f is unbounded, we have $\alpha(y) \neq 0$ for all $y \in J$ and hence h is unbounded.

Replacing y by y_1 and again y by y_2 in (3.4) we have two inequalities. By the use of the triangle inequality on these two resulting inequalities we have

$$|\alpha(y_1) - \alpha(y_2)|\,|h(x) - 1| \leq 2\epsilon \left(\frac{1}{|h(y_1) - 1|} + \frac{1}{|h(y_2) - 1|} \right) \tag{3.5}$$

for all $y_1, y_2 \in J$ and $x \in S$. Thus, we must have $\alpha(y_1) = \alpha(y_2)$ and hence $\alpha(y) := \alpha$. That is, $\alpha(y)$ is independent of $y \in J$. From (3.4) we have

$$|f(x) - \alpha\,(h(x) - 1)| \leq \frac{2\epsilon}{|h(y) - 1|} \tag{3.6}$$

for all $y \in J$. Since h is unbounded we have

$$f(x) = \alpha\,(h(x) - 1) \tag{3.7}$$

for all $x \in S$.

Now, let $D(x, y) = f(x + y) - f(x)\,h(y) - f(y)$. Then, using the triangle inequality we have

$$\begin{aligned}
(\,3 + |h(z)|\,)\,\epsilon \ &\geq \ |D(x + y, z) - D(x, y + z) - D(y, z) + h(z)\,D(x, y)| \\
&= \ |f(x + y + z) - f(x + y)\,h(z) - f(z) \\
&\quad -(f(x + y + z) - f(x)h(y + z) - f(y + z)) \\
&\quad -(f(y + z) - f(y)h(z) - f(z)) \\
&\quad +h(z)(f(x + y) - f(x)h(y) - f(y))| \\
&= \ |f(x)|\,|h(y + z) - h(y)\,h(z)|
\end{aligned} \tag{3.8}$$

for all $x, y, z \in S$. Thus, from (3.8) we have

$$h(y+z) = h(y)\,h(z) \tag{3.9}$$

for all $y, z \in S$. Therefore, $h = m$, where m is an exponential function and, by (3.7), we have $f(x) = \alpha\,(m(x) - 1)$. This completes the proof of the theorem. □

In particular, if $S = G$, where G is a 2-divisible commutative group and f and h are mappings from G to $\mathbb{R}$, then we can describe the behavior of bounded solutions of inequality (3.1), that is

$$|f(x+y) - f(x)\,h(y) - f(y)| \le \epsilon, \quad x,\ y \in G, \tag{3.10}$$

in terms of the constant ϵ. If the group G is 2-divisible, then following the proof of Theorem 2 in [4] and using the inequalities

$$\frac{1-\sqrt{1-4\delta}}{2} \le 2\delta \quad \text{and} \quad \frac{\sqrt{1+4\delta}-1}{2} \le \delta$$

for $0 \le \delta \le \frac{1}{4}$, we obtain the following lemma.

Lemma 3.2. *Suppose that G is a 2-divisible commutative group and $0 \le \delta \le \frac{1}{4}$. Let $h : G \to \mathbb{R}$ be a bounded function satisfying*

$$|h(x+y) - h(x)\,h(y)| \le \delta \tag{3.11}$$

for all $x, y \in G$. Then h satisfies either

$$-\delta \le h(x) \le 2\delta \tag{3.12}$$

for all $x \in G$, or

$$-\delta \le 1 - h(x) \le 2\delta \tag{3.13}$$

for all $x \in G$.

The following theorem presents the real-valued bounded solutions $f, h : G \to \mathbb{R}$ of the functional inequality (3.10).

Theorem 3.3. *Let G be a 2-divisible commutative group and $\mathbb{R}$ be the field of real numbers. Let (f, h) be a pair of bounded functions satisfying (3.10). Then (f, h) satisfies either*

$$h(x) = 1 \quad \textit{and} \quad |f(x)| \le \epsilon \tag{3.14}$$

for all $x \in G$, or

$$|h(x+y) - h(x)\,h(y)| \le \frac{\epsilon}{M_f}\,(3 + M_h)\,, \tag{3.15}$$

$$|f(x) - \alpha\,(1 - h(x))| \le \frac{3\epsilon}{M_{h_0}} \tag{3.16}$$

for all $x, y \in G$ and for some $\alpha \in \mathbb{R}$, where

$$M_f = \sup_{y\in G} |f(y)|, \quad M_h = \sup_{y\in G} |h(y)|, \quad M_{h_0} = \sup_{y\in G} |1 - h(y)|.$$

In particular, if $\delta := \frac{\epsilon}{M_f}(3 + M_h) \le \frac{1}{4}$, *then we have*

$$-\min\left\{\frac{7\epsilon}{2M_f}, \frac{\epsilon}{|f(0)|}\right\} \le h(x) \le \min\left\{\frac{7\epsilon}{M_f}, \frac{\epsilon}{|f(0)|}\right\} \tag{3.17}$$

and

$$\left| f(x) - \frac{f(0)}{1 - h(0)} \right| \le 2\epsilon \tag{3.18}$$

for all $x \in G$.

Proof. Assume that $h \equiv 1$. Then, it follows from inequality (3.2) that the additive function A is bounded, which implies $A \equiv 0$. Thus, we have

$$|f(x)| \le \epsilon \tag{3.19}$$

for all $x \in G$. This yields (3.14). From now on, we assume that $h \not\equiv 1$. From inequality (3.8) in Theorem 3.1 we have

$$|h(y + z) - h(y)\,h(z)| \le \frac{\epsilon}{|f(x)|}(3 + M_h) \tag{3.20}$$

for all $x, y, z \in G$. Taking the infimum of both sides of (3.20) for all $x \in G$, we have

$$|h(y + z) - h(y)\,h(z)| \le \frac{\epsilon}{M_f}(3 + M_h) \tag{3.21}$$

for all $y, z \in G$. Thus, we have (3.15). Now, choosing $y_0 \in G$ such that $|h(y_0) - 1| \ge \frac{2}{3}M_{h_0}$ and putting $y = y_0$ in (3.4) we have (3.16). Now, assume that

$$\delta := \frac{\epsilon}{M_f}(3 + M_h) \le \frac{1}{4}. \tag{3.22}$$

Then by Lemma 3.2, h satisfies (3.12) or (3.13). Assume that h satisfies (3.13). Then, using the triangle inequality, (3.10), (3.13) and (3.22) we have

$$\begin{aligned} |f(x + y) - f(x) - f(y)| &\le |f(x + y) - f(x) - f(y) - f(x)\,(h(y) - 1)| \\ &\quad + |f(x)|\,|h(y) - 1| \\ &\le \epsilon + \frac{1}{2}M_f \end{aligned} \tag{3.23}$$

for all $x, y \in G$. By Hyers' theorem (see Theorem 2.1), there exists a unique additive function $A : G \to \mathbb{R}$ such that

$$|f(x) - A(x)| \le \epsilon + \frac{1}{2}M_f \tag{3.24}$$

for all $x \in G$. Since f is bounded, A is bounded, and hence, the additive function $A \equiv 0$. Thus, it follows from (3.24) that

$$M_f \le \epsilon + \frac{1}{2}M_f. \tag{3.25}$$

From (3.22) and (3.25) we have the contradiction

$$4\epsilon\,(3 + M_h) \le M_f \le 2\epsilon. \tag{3.26}$$

Therefore, h satisfies (3.12). Using (3.12) we have

$$|h(x)| \leq 2\delta \leq \frac{1}{2} \tag{3.27}$$

for all $x \in G$. Putting $y = 0$ in (3.10) and dividing the resulting inequality by $|1 - h(0)|$ we have

$$\left| f(x) - \frac{f(0)}{1 - h(0)} \right| \leq \frac{\epsilon}{|1 - h(0)|} \leq \frac{\epsilon}{1 - |h(0)|} \leq 2\epsilon \tag{3.28}$$

for all $x \in G$. Thus, we get (3.18). If $f(0) = 0$, then from (3.28) we have $M_f \leq 2\epsilon$, which contradicts (3.22). Putting $x = 0$ in (3.10) and dividing the resulting inequality by $|f(0)|$ we have

$$|h(y)| \leq \frac{\epsilon}{|f(0)|} \tag{3.29}$$

for all $y \in G$. Also, it follows from (3.12), (3.22) and (3.27) that

$$-\frac{7\epsilon}{2M_f} \leq h(x) \leq \frac{7\epsilon}{M_f} \tag{3.30}$$

for all $x \in G$. Now, (3.17) follows from (3.29) and (3.30) and the proof of the theorem is complete. □

Remark 3.4. If h and f satisfy (3.17) and (3.18) respectively, then we have

$$\begin{aligned} &|f(x+y) - f(x)\,h(y) - f(y)| \\ &\quad \leq \left| f(x+y) - \frac{f(0)}{1-h(0)} \right| + \left| \frac{f(0)}{1-h(0)} - f(y) \right| + |f(x)\,h(y)| \\ &\quad \leq 2\epsilon + 2\epsilon + 7\epsilon = 11\epsilon \end{aligned} \tag{3.31}$$

for all $x, y \in G$.

Remark 3.5. We can find the behavior of f when h is near 1. Assume that h satisfies

$$|h(x) - 1| \leq r < 1$$

for all $x \in G$. Then, using inequalities similar to (3.23), (3.24) and (3.25) we have

$$M_f \leq \frac{\epsilon}{1 - r}.$$

Example 3.6. Consider the functional inequality

$$|f(x+y) - f(x)h(y) - f(y)| \leq 10^{-10} \tag{3.32}$$

for all $x, y \in G$, where G is 2-divisible. We can determine the behavior of (f, h) satisfying (3.32) when $|f|$ is not extremely small and $|h|$ is not extremely large. For example, we assume that

$$M_f \geq 10^{-4} \quad \text{and} \quad M_h \leq 10^5. \tag{3.33}$$

Then, $\frac{\epsilon}{M_f}(3 + M_h) \leq \frac{10^{-10}(3+10^5)}{10^{-4}} < \frac{1}{4}$. Thus, by Theorem 3.3, $|h(0)| \leq \frac{1}{2}$ and f satisfies

$$|f(x) - \frac{f(0)}{1 - h(0)}| \leq 2 \times 10^{-10} \tag{3.34}$$

for all $x \in G$. Using the triangle inequality with (3.34), and using (3.27), we have

$$2|f(0)| \geq \frac{|f(0)|}{1-h(0)} \geq |f(x)| - 2 \times 10^{-10} \tag{3.35}$$

for all $x \in G$. Thus, using (3.35) and (3.33) we have

$$|f(0)| \geq \frac{1}{2}(M_f - 2 \times 10^{-10}) \geq \frac{1}{2}(10^{-4} - 2 \times 10^{-10}). \tag{3.36}$$

Now, using (3.17) and (3.36) we have

$$-\frac{2}{10^6 - 2} \leq -\frac{10^{-10}}{|f(0)|} \leq h(x) \leq \frac{10^{-10}}{|f(0)|} \leq \frac{2}{10^6 - 2} \tag{3.37}$$

for all $x \in G$. Also, using (3.17) and (3.33) we have

$$-\frac{3.5}{10^6} \leq \frac{7 \times 10^{-10}}{2M_f} \leq h(x) \leq \frac{7 \times 10^{-10}}{M_f} \leq \frac{7}{10^6} \tag{3.38}$$

for all $x \in G$. Therefore, h satisfies (3.37). We can also find the behavior of f when h is near 1. Assume that

$$|1 - h(x)| \leq \frac{1}{2} \tag{3.39}$$

for all $x \in G$. Then, by Remark 3.5 we have

$$|f(x)| \leq 2 \times 10^{-10} \tag{3.40}$$

for all $x \in G$. If h and f satisfy (3.39) and (3.40) respectively, then we have

$$\begin{aligned} |f(x+y) - f(x)h(y) - f(y)| &\leq |f(x+y)| + |f(y)| + |f(x)h(y)| \\ &\leq 2 \times 10^{-10} + 2 \times 10^{-10} + 3 \times 10^{-10} \\ &= 7 \times 10^{-10} \end{aligned}$$

for all $x, y \in G$.

4. Pexider generalization

In this section, we consider the pexiderized version of the inequality (3.1). Let $f, g, h, k : S \to \mathbb{C}$ and $\epsilon \geq 0$. We consider the stability of the functional inequality

$$|f(x+y) - g(x)h(y) - k(y)| \leq \epsilon \tag{4.1}$$

for all $x, y \in S$. Note that if h is a constant function (without loss of generality we may assume $h \equiv 1$), the inequality (4.1) is reduced to the Hyers–Ulam stability of the Pexider functional equation $f(x+y) = g(x) + k(y)$ on non-unital commutative semigroups (see [8]). The following theorem was proved in [8].

Theorem 4.1. *Let S be a non-unital commutative semigroup and $\mathbb{C}$ be the field of complex numbers. Assume that $\epsilon \geq 0$ and $f, g, k : S \to \mathbb{C}$ satisfy the functional inequality*

$$|f(x+y) - g(x) - k(y)| \leq \epsilon \tag{4.2}$$

for all $x, y \in S$. *Then, there exists a unique additive function* $A : S \to \mathbb{C}$ *and* $\alpha, \beta \in \mathbb{C}$ *such that*

$$|g(x)-A(x)-\alpha| \leq 8\epsilon, \quad |k(x)-A(x)-\beta| \leq 8\epsilon, \quad |f(t)-A(t)-\alpha-\beta| \leq 18\epsilon$$

for all $x \in S,\ t \in S\backslash(S+S)$.

From now on, we assume that h is a nonconstant function.

Theorem 4.2. *Let S be a non-unital commutative semigroup and* $\mathbb{C}$ *be the field of complex numbers. Let* $f, g, h, k : S \to \mathbb{C}$ *satisfy the functional inequality* (4.1). *Then* (f, g, h, k) *satisfies one of the following*:

(i) g, h, k *are bounded on S and* f *is bounded on* $S + S$;
(ii) *there exist an unbounded exponential function* $m : S \to \mathbb{C}$ *and* $\alpha, \beta, \gamma, \mu \in \mathbb{C}$ *with* $\alpha\beta \neq 0$ *such that*

$$\begin{aligned}
&h(x) = \beta m(x), \qquad g(x) = \alpha\beta m(x) + \gamma,\\
&|k(x) + \beta\gamma m(x) - \mu| \leq \frac{2}{\sqrt{3}}\epsilon, \qquad |f(t) - \alpha\beta^2 m(t) - \mu| \leq \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon,\\
&f(w) : \textit{arbitrary}
\end{aligned}$$

for all $x \in S,\ t \in S+S,\ w \in S\backslash(S+S)$*;*
(iii) *there exist* $\alpha\ (\neq 0), \gamma, \mu \in \mathbb{C}$ *such that*

$$\begin{aligned}
&h(t) = 0,\\
&|h(p)h(q) - h(r)h(s)| \leq \frac{2(2+\sqrt{3})\epsilon}{\sqrt{3}|\alpha|}, \ \ p + q = r + s,\\
&g(x) = \alpha h(x) + \gamma,\\
&|k(x) + \gamma h(x) - \mu| \leq \frac{2}{\sqrt{3}}\epsilon,\\
&|f(x+y) - \alpha h(x)h(y) - \mu| \leq \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon,\\
&f(w) : \textit{arbitrary}
\end{aligned}$$

for all $x, y, p, q, r, s \in S,\ t \in S+S,\ w \in S\backslash(S+S)$.
(iv) *there exist* $\gamma, \mu \in \mathbb{C}$ *such that*

$$\begin{aligned}
&g(x) \equiv \gamma,\\
&|k(x) + \gamma h(x) - \mu| \leq \frac{2}{\sqrt{3}}\,\epsilon,\\
&|f(t) - \mu| \leq \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon,\\
&f(w) : \textit{arbitrary}
\end{aligned}$$

for all $x \in S,\ t \in S+S,\ w \in S\backslash(S+S)$.

Proof. First, we assume that g is a nonconstant bounded function. Fix $y_0 \in S$ and let $D(x, y) = f(x+y) - g(x)\,h(y) - k(y)$. Then, using the triangle inequality we have

$$\begin{aligned}
4\epsilon &\geq |D(y,x) - D(x,y) - D(y_0,x) + D(x,y_0)|\\
&= |(h(y) - h(y_0))\,g(x) - (g(y) - g(y_0))\,h(x) + k(y) - k(y_0)|. \quad (4.3)
\end{aligned}$$

Thus, it follows from (4.3) that h is bounded. Replacing x by y and y by x in (4.1) and using the triangle inequality we have

$$|k(x)| \leq 2\epsilon + |g(x)h(y_0) - h(y_0)g(x) + k(y_0)|$$

for all $x \in G$. Thus, k is bounded and consequently, f is bounded on $S+S$, which yields (i).

Secondly, we assume that g is unbounded. Since h is nonconstant, it follows from (4.3) that h is unbounded. Now, dividing both sides of (4.3) by $|h(y) - h(y_0)|$ we have

$$|g(x) - \alpha(y)\, h(x) + \eta(y)| \leq \frac{4\epsilon}{|h(y) - h(y_0)|} \tag{4.4}$$

for all $x, y \in S$ such that $h(y) - h(y_0) \neq 0$, where

$$\alpha(y) = \frac{g(y) - g(y_0)}{h(y) - h(y_0)} \quad \text{and} \quad \eta(y) = \frac{k(y) - k(y_0)}{h(y) - h(y_0)}. \tag{4.5}$$

Replacing y by y_1 and y by y_2 in (4.4) and using the triangle inequality with the results we have

$$\begin{aligned} |\alpha(y_1) - \alpha(y_2)|\, |h(x)| \;\leq\; & |\eta(y_1) - \eta(y_2)| \\ & + 4\epsilon \left(\frac{1}{|h(y_1) - h(y_0)|} + \frac{1}{|h(y_2) - h(y_0)|} \right) \end{aligned} \tag{4.6}$$

for all $x \in S$. Since h is unbounded, it follows from (4.6) that $\alpha(y_1) = \alpha(y_2)$. Thus $\alpha(y) := \alpha$ is independent of $y \in J := \{y \in S : h(y) - h(y_0) \neq 0\}$ and from (4.5),

$$g(y) - g(y_0) = \alpha(h(y) - h(y_0)) \tag{4.7}$$

for all $y \in J$. From (4.3) it is easy to see that $g(y) - g(y_0) = 0$ if and only if $h(y) - h(y_0) = 0$ since both g and h are unbounded. Thus, we have

$$g(x) = \alpha\, h(x) + \gamma \tag{4.8}$$

for all $x \in S$, where $\gamma = g(y_0) - \alpha h(y_0)$. Putting (4.8) in (4.1) we have

$$|f(x+y) - \alpha h(x)\, h(y) - \gamma h(y) - k(y)| \leq \epsilon. \tag{4.9}$$

Replacing (x, y) by (y, x) in (4.9) and using the triangle inequality we have

$$|k(x) + \gamma\, h(x) - k(y) - \gamma h(y)| \leq 2\epsilon. \tag{4.10}$$

Let $q(x) := k(x) + \gamma h(x)$ and let $d := \sup_{x,y \in G} |q(x) - q(y)|$, the diameter of $q(S)$. Then, by Jung's theorem [13], there exists a circle with radius $r \leq \frac{1}{\sqrt{3}} d$ containing $q(S)$. Let μ be the center of the circle. Then we have

$$|k(x) + \gamma h(x) - \mu| \leq \frac{2}{\sqrt{3}}\epsilon \tag{4.11}$$

for all $x \in S$. Using the triangle inequality, (4.9), and (4.11), we have

$$|f(x+y) - \alpha h(x) h(y) - \mu| \leq \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon. \tag{4.12}$$

Now, using the triangle inequality and (4.12) we have

$$\begin{aligned} |\alpha h(x)h(y) - \alpha h(u)h(v)| &\le |-f(x+y) + \alpha h(x)h(y) + \mu| \\ &\quad + |f(u+v) - \alpha h(u)h(v) - \mu| \\ &\le \frac{2(2+\sqrt{3})\epsilon}{\sqrt{3}} \end{aligned} \tag{4.13}$$

for all $x, y, u, v \in S$ satisfying $x + y = u + v$. Dividing both sides of (4.13) by $|\alpha|$ we have

$$|h(x)\,h(y) - h(u)\,h(v)| \le \frac{2(2+\sqrt{3})\epsilon}{\sqrt{3}|\alpha|} := M. \tag{4.14}$$

Replacing (x, y) by $(x + y, z)$ and (u, v) by $(x, y + z)$ in (4.14) we have

$$|h(x+y)\,h(z) - h(x)\,h(y+z)| \le M. \tag{4.15}$$

Now, using the triangle inequality and (4.15) we can write

$$\begin{aligned} |h(x+y)h(z) - h(x)h(y+z)| &\le \left| \frac{h(x+y)h(z)h(v)}{h(v)} - \frac{h(x)h(y+v)h(z)}{h(v)} \right| \\ &\quad + \left| \frac{h(x)h(y+v)h(z)}{h(v)} - \frac{h(x)h(y+z)h(v)}{h(v)} \right| \\ &\le M \left(\frac{|h(z)| + |h(x)|}{|h(v)|} \right) \end{aligned} \tag{4.16}$$

for all $x, y, z, v \in S$. Since h is unbounded we have

$$h(x+y)\,h(z) = h(x)\,h(y+z) \tag{4.17}$$

for all $x, y, z \in S$. Multiplying both sides of (4.17) by $h(v)$ we have

$$\begin{aligned} h(x+y)\,h(z)\,h(v) &= h(x)\,h(y+z)\,h(v) \\ &= h(x)\,h(y)\,h(z+v). \end{aligned} \tag{4.18}$$

If $h \not\equiv 0$ on $S + S$, i.e., there exist $s_1, s_2 \in S$ such that $h(s_1 + s_2) \neq 0$, and putting $z = s_1, v = s_2,\ x = y = s_1 + s_2$ in (4.18) we have

$$h(2s_1 + 2s_2)\,h(s_1)\,h(s_2) = [h(s_1 + s_2)]^3 \neq 0.$$

Thus, we have

$$h(s_1)\,h(s_2) \neq 0.$$

Putting $z = s_1,\ v = s_2$ in (4.18) and dividing the result by $h(s_1)^2 h(s_2)^2 / h(s_1 + s_2)$ we have

$$\frac{h(x+y)}{\beta} = \frac{h(x)}{\beta} \cdot \frac{h(y)}{\beta} \tag{4.19}$$

for all $x, y \in S$, where $\beta = h(s_1)h(s_2)/h(s_1 + s_2)$. This implies

$$h(x) = \beta\, m(x) \tag{4.20}$$

for all $x \in S$, where m is an unbounded exponential function. Putting (4.20) in (4.8), (4.11) and (4.12) respectively, we get (ii). Similarly, (iii) follows from (4.8), (4.11), (4.12) and (4.14).

Finally, we assume that g is a constant function. Let $g(x) \equiv \gamma$. Replacing (x, y) by (y, x) in (4.1), and using the triangle inequality with (4.1) and the result we have

$$|k(x) + \gamma h(x) - k(y) - \gamma h(y)| \leq 2\epsilon \tag{4.21}$$

for all $x, y \in S$. By Jung's theorem, there exists $\mu \in \mathbb{C}$ such that

$$|k(x) + \gamma h(x) - \mu| \leq \frac{2}{\sqrt{3}}\epsilon \tag{4.22}$$

for all $x \in S$. Using the triangle inequality with (4.1) and (4.22) we have

$$|f(x+y) - \mu| \leq \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon \tag{4.23}$$

for all $x, y \in S$, which gives (iv). This completes the proof. □

Remark 4.3. If we take $f = g = k$ in Theorem 4.2, then (i), (iii) and (iv) are reduced to case (i) of Theorem 3.1. Also, item (ii) of Theorem 4.2 is reduced to

$$h(x) = \beta m(x), \quad x \in S, \tag{4.24}$$

$$f(x) = \alpha\beta m(x) + \gamma, \tag{4.25}$$

$$|f(x) + \beta\gamma m(x) - \mu| \leq \frac{2}{\sqrt{3}}\epsilon, \tag{4.26}$$

$$|f(t) - \alpha\beta^2 m(t) - \mu| \leq \left(1 + \frac{2}{\sqrt{3}}\right). \tag{4.27}$$

Putting (4.25) in (4.26) and using the triangle inequality we have

$$|\beta(\alpha + \gamma)|\,|m(x)| \leq |\gamma - \mu| + \frac{2}{\sqrt{3}}\epsilon$$

for all $x \in S$. Since m is unbounded, we get $\gamma = -\alpha$. Similarly, putting (4.25) in (4.27) we have $\beta = 1$. Thus, Theorem 4.2 includes Theorem 3.1.

In particular, if $S + S = S$, then as a direct consequence of the above result we have the following.

COROLLARY 4.4

Assume that S is a commutative semigroup such that $S = S + S$ and $f, g, h, k : S \to \mathbb{C}$ satisfy the functional inequality (4.1). *Then (f, g, h, k) satisfies one of the following*:

(i) *f, g, h, k are all bounded on S*;
(ii) *there exist an unbounded exponential function m and $\alpha, \beta, \gamma, \mu \in \mathbb{C}$ with $\alpha\beta \neq 0$ such that*

$$h(x) = \beta m(x), \qquad g(x) = \alpha\beta m(x) + \gamma,$$
$$|k(x) + \beta\gamma m(x) - \mu| \leq \frac{2}{\sqrt{3}}\epsilon, \qquad |f(x) - \alpha\beta^2 m(x) - \mu| \leq \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon$$

for all $x \in S$;

(iii) *there exist* $\gamma, \mu \in \mathbb{C}$ *such that*

$$g(x) \equiv \gamma, \quad |k(x) + \gamma h(x) - \mu| \le \frac{2}{\sqrt{3}}\epsilon, \quad |f(x) - \mu| \le \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon$$

for all $x \in S$.

Now, we give two examples of regular, as well as irregular solutions of functional inequality (4.1). In the following example, we denote the polynomials P of finite degree by the infinite sum $P(x) = \sum_{j=0}^{\infty} a_j x^j$ to avoid confusion, where the coefficients $a_j = 0$ for all but a finite number of $j \in \mathbb{N} \cup \{0\}$.

Example 4.5. Let S be the set of all nonzero polynomials $P(x) = \sum_{j=0}^{\infty} a_j x^j$ of finite degree with $a_j \in \mathbb{N} \cup \{0\}$ for all $j = 0, 1, 2, 3, \ldots$, and $f, g, h, k : S \to \mathbb{C}$. It is easy to see that every nonzero exponential function $m : S \to \mathbb{C}$ has the form

$$m\left(\sum_{j=0}^{\infty} a_j x^j\right) = \prod_{j=0}^{\infty} c_j^{a_j}, \tag{4.28}$$

where $c_0, c_1, c_2, \ldots, c_n, \ldots$ is an arbitrary sequence of nonzero complex numbers. Thus, the regular solution (f, g, h, k) of the inequality (4.1) has the form

$$\begin{aligned}
&h\left(\sum_{j=0}^{\infty} a_j x^j\right) = \beta \prod_{j=0}^{\infty} c_j^{a_j}, \\
&g\left(\sum_{j=0}^{\infty} a_j x^j\right) = \alpha\,\beta \prod_{j=0}^{\infty} c_j^{a_j} + \gamma, \\
&\left|k\left(\sum_{j=0}^{\infty} a_j x^j\right) + \beta\,\gamma \prod_{j=0}^{\infty} c_j^{a_j} - \mu\right| \le \frac{2}{\sqrt{3}}\,\epsilon, \\
&\left|f\left(\sum_{j=0}^{\infty} a_j x^j\right) - \alpha\,\beta^2 \prod_{j=0}^{\infty} c_j^{a_j} - \mu\right| \le \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon, \quad \sum_{j=1}^{\infty} a_j \ge 2, \\
&f(x^j) : \text{arbitrary}, \quad j = 0, 1, 2, \ldots.
\end{aligned}$$

Also it is easy to see that the irregular solutions are given by

$$\begin{aligned}
&h(x^j) = c_j, \quad j = 0, 1, 2, \ldots, \\
&h\left(\sum_{j=0}^{\infty} a_j x^j\right) = 0, \qquad \sum_{j=0}^{\infty} a_j \ge 2, \\
&g(x^j) = \alpha\, c_j + \gamma, \quad j = 0, 1, 2, \ldots, \\
&g\left(\sum_{j=0}^{\infty} a_j x^j\right) = \gamma, \qquad \sum_{j=0}^{\infty} a_j \ge 2, \\
&|k(x^j) + \gamma\, c_j - \mu| \le \frac{2}{\sqrt{3}}\epsilon, \quad j = 0, 1, 2, \ldots,
\end{aligned}$$

$$\left|k\left(\sum_{j=0}^{\infty} a_j x^j\right) - \mu\right| \le \frac{2}{\sqrt{3}}\epsilon, \qquad \sum_{j=0}^{\infty} a_j \ge 2,$$
$$|f(x^p + x^q) - \alpha\, c_p\, c_q - \mu| \le \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon, \quad p, q = 0, 1, 2, \dots,$$
$$\left|f\left(\sum_{j=0}^{\infty} a_j x^j\right) - \mu\right| \le \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon, \qquad \sum_{j=0}^{\infty} a_j \ge 3,$$
$$f(x^j) : \text{arbitrary}, \ \ j = 0, 1, 2, \dots,$$

where $\alpha, \beta, \gamma, \mu \in \mathbb{C}$.

Example 4.6. Let $a_j > 0,\ j = 0, 1, 2, \dots, n$, and $S = [a_1, \infty) \times \cdots \times [a_n, \infty)$. Then every exponential function $m : S \to \mathbb{C}$ is given by

$$m(x_1, \dots, x_n) = e^{p(x_1,\dots,x_n)+iq(x_1,\dots,x_n)}, \ \ x_j \ge a_j, \ \ j = 0, 1, 2, \dots, n, \tag{4.29}$$

where p is an additive function on the semigroup S and the function q satisfies

$$q(x_1 + y_1, \dots, x_n + y_n) \equiv q(x_1, \dots, x_n) + q(y_1, \dots, y_n) \ (mod\ 2\pi)$$

for all $x_j, y_j \ge a_j, \ \ j = 0, 1, 2, ..., n$.

Thus, the regular solution of the functional inequality (4.1) is given by

$$h(x_1, \dots, x_n) = \beta\, e^{p(x_1,\dots,x_n)+iq(x_1,\dots,x_n)},$$
$$g(x_1, \dots, x_n) = \alpha\, \beta\, e^{p(x_1,\dots,x_n)+iq(x_1,\dots,x_n)} + \gamma,$$
$$\left|k(x_1, \dots, x_n) + \beta\, \gamma\, e^{p(x_1,\dots,x_n)+iq(x_1,\dots,x_n)} - \mu\right| \le \frac{2}{\sqrt{3}}\epsilon,$$
$$\left|f(t_1, \dots, t_n) - \alpha\, \beta^2\, e^{p(x_1,\dots,x_n)+iq(x_1,\dots,x_n)} - \mu\right| \le \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon,$$
$$f(w_1, \dots, w_n) : \text{arbitrary}$$

for all $x_j \ge a_j,\ t_j \ge 2a_j,\ a_j \le s_j < 2a_j,\ j = 0, 1, 2, \dots n$, and some $\alpha, \beta, \gamma, \mu \in \mathbb{C}$.

Next we find the irregular solutions. Define $h_0 : [0, \infty) \times \cdots \times [0, \infty) \to \mathbb{C}$ by

$$h_0(x_1, \dots, x_n) = h(x_1 + a_1, \dots, x_n + a_n), \ \ x_j \ge 0, \ \ j = 0, 1, 2, \dots, n. \tag{4.30}$$

From (4.14), we have

$$\left|h_0\left(\frac{x_1 + y_1}{2}, \dots, \frac{x_n + y_n}{2}\right)^2 - h_0(x_1, \dots, x_n)\, h_0(y_1, \dots, y_n)\right| \le M$$

for all $x_j, y_j \ge 0,\ j = 0, 1, 2, \dots, n$. Let $k \equiv 0$ and replace g, h by h_0 and $f(x_1, \dots, x_n)$ by $h_0(\frac{x_1}{2}, \dots, \frac{x_n}{2})$ in (4.1). Then by Corollary 4.4 we have

$$h_0(x_1, \dots, x_n) = \beta\, m(x_1, \dots, x_n) \tag{4.31}$$

for some $\beta \in \mathbb{C}$ and an exponential function $m : [0, \infty) \times \cdots \times [0, \infty) \to \mathbb{C}$. Since $h \equiv 0$ on $S + S$, we have $h(y_1, \dots, y_n) = 0$ for all $y_j \ge 2a_j,\ j = 0, 1, 2, \dots, n$.

Now, for each $x_j > 0$, $j = 0, 1, 2, \ldots, n$, if we choose a positive integer r such that $rx_j \geq a_j$, $j = 0, 1, 2, \ldots, n$, then

$$\begin{aligned} m(x_1, \ldots, x_n)^r &= m(rx_1, \ldots, rx_n) \\ &= \beta^{-1} h_0(rx_1, \ldots, rx_n) \\ &= \beta^{-1} h(rx_1 + a_1, \ldots, rx_n + a_n) \\ &= 0. \end{aligned} \tag{4.32}$$

Thus, it follows that

$$m(x_1, \ldots, x_n) = 0 \tag{4.33}$$

for all $x_j > 0$, $j = 0, 1, 2, \ldots, n$. Since m is written in the form

$$m(x_1, \ldots, x_n) = m_1(x_1) \cdots m_n(x_n) \tag{4.34}$$

for some nonzero exponential functions $m_j : [0, \infty) \to \mathbb{C}$, $j = 0, 1, 2, \ldots, n$, it follows from (4.33) and (4.34) that there exists $j_0 \in \{1, 2, \ldots, n\}$ such that

$$m_{j_0}(x) = 0 \tag{4.35}$$

for all $x > 0$. Note that for all $j \in \{1, 2, \ldots, n\}$, m_j satisfies $m_j(0) = 1$ and one of the following:

$$m_j(x) = 0 \tag{4.36}$$

for all $x > 0$, or

$$m_j(x) \neq 0 \tag{4.37}$$

for all $x > 0$. Without loss of generality, we assume that $J = \{1, 2, \ldots, r\}$ is the set of all $j \in \{1, 2, \ldots, n\}$ such that m_j satisfies (4.36). Since m is unbounded, we have $r < n$. Thus, we can write

$$\begin{aligned} h(x_1, \ldots, x_n) &= h_0(x_1 - a_1, \ldots, x_n - a_n) \\ &= \beta\, m(x_1 - a_1, \ldots, x_n - a_n) \\ &= \beta \prod_{i=1}^{r} m_i(x_i - a_i) \prod_{i=r+1}^{n} m_i(x_i - a_i) \\ &= \beta \prod_{i=1}^{r} m_i(x_i - a_i) \prod_{i=r+1}^{n} m_i(x_i) \prod_{i=r+1}^{n} m_i(a_i)^{-1} \\ &= \beta \prod_{i=r+1}^{n} m_i(a_i)^{-1} \prod_{i=1}^{r} m_i(x_i - a_i) \prod_{i=r+1}^{n} m_i(x_i) \\ &= \mathcal{M}_1(x_1, \ldots, x_r)\, \mathcal{M}_2(x_{r+1}, \ldots, x_n) \\ &= \mathcal{M}_1 \otimes \mathcal{M}_2(x_1, \ldots, x_r; x_{r+1}, \ldots, x_n) \end{aligned}$$

for all $x_j \geq a_j$, $j = 1, 2, \ldots, n$, where

$$\mathcal{M}_1(x_1, \ldots, x_r) := \beta \prod_{i=r+1}^{n} m_i(a_i)^{-1} \prod_{i=1}^{r} m_i(x_i - a_i),$$

$$\mathcal{M}_2(x_{r+1}, \ldots, x_n) := \prod_{i=r+1}^{n} m_i(x_i)$$

and

$$\mathcal{M}_1 \otimes \mathcal{M}_2(x_1, \ldots, x_r; x_{r+1}, \ldots, x_n) := \mathcal{M}_1(x_1, \ldots, x_r)\ \mathcal{M}_2(x_{r+1}, \ldots, x_n).$$

It is easy to see that

$$\mathcal{M}_1(x_1, \ldots, x_r) \quad \begin{cases} = 0, & \text{if } \forall\, x_j > a_j,\ \ j = 1, 2, ..., r \\ \neq 0, & \text{if } \forall\, x_j = a_j,\ \ j = 1, 2, ..., r, \end{cases}$$

and $\mathcal{M}_2$ is an unbounded exponential function on $[a_{r+1}, \infty) \times \cdots \times [a_n, \infty)$.

Now, the irregular solution of inequality (4.1) has the form

$$\begin{aligned}
&h(x_1, \ldots, x_n) = \mathcal{M}_1 \otimes \mathcal{M}_2(x_1, \ldots, x_r; x_{r+1}, \ldots, x_n), \\
&g(x_1, \ldots, x_n) = \alpha\, \mathcal{M}_1 \otimes \mathcal{M}_2(x_1, \ldots, x_r; x_{r+1}, \ldots, x_n) + \gamma, \\
&|k(x_1, \ldots, x_n) + \gamma\, \mathcal{M}_1 \otimes \mathcal{M}_2(x_1, \ldots, x_r; x_{r+1}, \ldots, x_n) - \mu| \leq \frac{2}{\sqrt{3}}\epsilon, \\
&|f(2x_1, \ldots, 2x_n) - \alpha\, \mathcal{M}_1 \otimes \mathcal{M}_2(2x_1, \ldots, 2x_r; 2x_{r+1}, \ldots, 2x_n) \\
&-\mu| \leq \left(1 + \frac{2}{\sqrt{3}}\epsilon\right), \\
&f(w_1, \ldots, w_n) : \text{arbitrary}
\end{aligned}$$

for all $x_j \geq a_j,\ a_j \leq w_j < 2a_j,\ \ j = 1, 2, \ldots, n$, and some $\alpha (\neq 0), \gamma, \mu \in \mathbb{C}$.

We end this paper with the following question.

Question 4.7. We do not know if every unbounded function $h : S \to \mathbb{C}$ satisfying the following functional inequality

$$|h(p)\, h(q) - h(r)\, h(s)| \leq M, \quad \forall\, p, q, r, s \in S \ \text{ with } \ p + q = r + s \tag{4.38}$$

for some $M > 0$, also satisfies the functional equation

$$h(p)\, h(q) = h(r)\, h(s), \quad \forall\, p, q, r, s \in S \ \text{ with } \ p + q = r + s. \tag{4.39}$$

If there exists a function satisfying (4.38) but not (4.39), then under what conditions on h and/or the semigroup S does the above functional inequality imply the functional equation?

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