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Diagrams for certain quotients of $PSL(2, \mathbb{Z}[i])$

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Abstract. Actions of the Picard group $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$, where $p \equiv 1 (\mathrm{mod}\ 4)$, are investigated through diagrams. Each diagram is composed of fragments of three types. A technique is developed to count the number of fragments which frequently occur in the diagrams for the action of the Picard group on $PL(F_p)$. The conditions of existence of fixed points of the transformations are evolved. It is further proved that the action of the Picard group on $PL(F_p)$ is transitive. A code in *Mathematica* is developed to perform the calculation.

Keywords. Bianchi group; Picard group; coset diagrams; quadratic number fields.

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1. Introduction

It is well known that $PSL(2, \mathbb{C}) = PGL(2, \mathbb{C})$ can be identified with the group of orientation preserving isometries of H^3 [9]. Let O_d denote the ring of integers in the imaginary quadratic number field $Q(\sqrt{-d})$, where d is a square-free positive integer [5]. An important class of discrete subgroups of $PSL(2, \mathbb{C})$ consists of the groups of the form $PSL(2, O_d)$ and $PGL(2, O_d)$, which can be considered as generalizations of the classical modular group $PSL(2, \mathbb{Z})$.

It is known that O_d has a Euclidean algorithm only if $d = 1, 2, 3, 7, 11$ while only O_1 and O_3 have units $\neq \pm 1$. The groups $\Gamma_d = PSL\ (2, O_d)$ with d and O_d as above are called Bianchi groups [3]. The group Γ_1, that is $PSL\ (2, O_1)$, where O_1 is the ring of Gaussian integers is known as the Picard group [3, 10]. The modular group, $PSL\ (2, \mathbb{Z})$, one of the most extensively studied group, is strongly related to the Picard group [4]. Group theoretically, Γ_1 is quite similar to $PSL\ (2, \mathbb{Z})$. However, Γ_1 and $PSL\ (2, \mathbb{Z})$ differ greatly in their action on the complex plane. $PSL\ (2, \mathbb{Z})$ is a Fuchsian group, Γ_1 is discontinuous in $\mathbb{C}$ and therefore has no Fuchsian subgroups [2] of finite index. As with $PSL\ (2, \mathbb{Z})$ and the other Euclidean Bianchi groups, many properties of Γ_1 depends on its decomposition as a non-trivial amalgam. Real interest in Picard and Bianchi groups, in general, began due to the famous paper of Serre [11].

It is also known (see [4] and [10]) that the Picard group has the presentation $\Gamma_1 = \langle A, B, C, D \mid A^3 = B^2 = C^3 = D^2 = (AC)^2 = (AD)^2 = (BC)^2 = (BD)^2 = 1\rangle$, where

$A(z) :\mapsto \frac{1}{z-i}$, $B(z) :\mapsto \frac{1}{z}$, $C(z) :\mapsto -\frac{1+z}{z}$, and $D(z) :\mapsto -\frac{1}{z}$. The decomposition of $\Gamma_1 = G_1 \underset{PSL(2,\mathbb{Z})}{*} G_2$ arises by letting

$$\begin{aligned} G_1 &= \langle A, C, D \mid A^3 = C^3 = D^2 = (AC)^2 = (AD)^2 = 1 \rangle, \\ G_2 &= \langle B, C, D \mid B^2 = C^3 = D^2 = (BC)^2 = (BD)^2 = 1 \rangle; \end{aligned}$$

and the group $PSL(2, \mathbb{Z})$ is precisely the modular group [1].

In this article, we have explored some group theoretic properties of the action of Picard group $\Gamma_1 = PSL(2, O_1)$ on $PL\big(F_p\big)$, where O_1 is the ring of Gaussian integers and p is a Pythagorean prime. Throughout this article, we use the symbol u for a square root of $-1 (\mathrm{mod}\ p)$.

2. The action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$

Theorem 1. *Γ_1 acts on $PL(F_p)$ only if -1 is a perfect square in F_p.*

Proof. The transformations B, C and D map all the elements of $PL(F_p)$ to the elements of $PL(F_p)$ except the transformation $A(z) = \frac{1}{z-u}$, which belongs to $PL(F_p)$ only if $-1 \equiv p - 1 (\mathrm{mod}\ p)$, that is, if -1 is a perfect square in F_p. □

Such kind of primes can also be written as a sum of squares of two integers. So, these are called Pythagorean primes [8] and, as is well-known, the odd primes with this property are precisely those which can be written in the form of $p \equiv 1 (\mathrm{mod}\ 4)$.

Theorem 2. *Consider the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$, where p is a Pythagorean prime. Then*

(i) *D maps each fixed point of A to its other fixed point,*
(ii) *B maps each fixed point of C to its other fixed point.*

Proof. Suppose, A fixes ν, that is, $\nu = A(\nu) = \frac{1}{\nu-u}$. Therefore,

$$\nu^2 - u\nu - 1 = 0.$$

So, there are two fixed points whose product is -1, hence $D(z) = \frac{-1}{z}$ interchanges the two fixed points.

Similarly, if μ is a point fixed by C, then $\mu = C(\mu) = \frac{-1-\mu}{\mu}$, which gives

$$\mu^2 + \mu + 1 = 0.$$

There are two fixed points whose product is 1, the transformation $B(z) = \frac{1}{z}$ interchanges the two fixed points. □

3. Coset diagrams

We use special graphs propounded by Higman known as coset diagrams (see [6, 7]) to investigate the behavior of the group $PSL(2, \mathbb{Z}[i])$. A coset diagram is a graph whose vertices are the (right) cosets of a subgroup of finite index in a finitely generated group.

The vertices representing cosets α and β (say), are joined by a g_i-edge, of 'color i' directed from vertex α to vertex β,

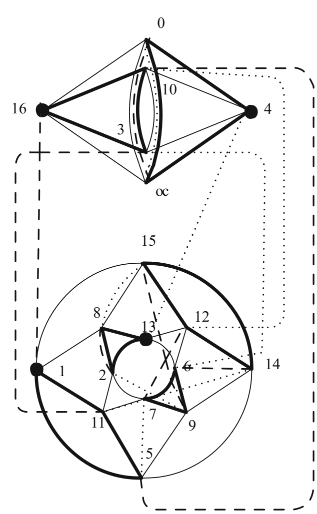
$$\alpha \longrightarrow \alpha g_i = \beta.$$

It may well happen that $\alpha g_i = \alpha$, in which case the α-vertex is joined to itself by a g_i-loop or a fixed point. The coset diagrams of the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$ are defined as follows. The coset diagram for the action of the $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$ is obtained by using linear fractional transformations A, B, C and D. These transformations give permutations and with the help of these permutations coset diagram is drawn. The three cycles of the permutations $\bar{A}$ is denoted by a triangle having bold solid lines whereas $\bar{C}$ is represented by triangles having solid lines. Any two vertices which are interchanged by the involution $\bar{B}$ and $\bar{D}$ are represented by dotted and broken line edges, respectively. The fixed points of $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are denoted by heavy dots, if they exist.

For instance, consider the action of $PSL(2, \mathbb{Z}[i])$ by $A : z \mapsto \frac{1}{z-u}$, $B : z \mapsto \frac{1}{z}$, $C : z \mapsto -\frac{1+z}{z}$ and $D : z \mapsto -\frac{1}{z}$ on $PL(F_{17})$. We obtained the following permutation representations of A, B, C and D:

$$\begin{aligned}
\bar{A} &= (0, 4, \infty)(1, 11, 5)(2, 8, 13)(3, 16, 10)(6, 9, 7)(12, 15, 14),\\
\bar{B} &= (0, \infty)(1)(2, 9)(3, 6)(4, 13)(5, 7)(8, 15)(10, 12)(11, 14)(16),\\
\bar{C} &= (0, \infty, 16)(1, 15, 8)(2, 7, 11)(3, 10, 4)(5, 9, 14)(6, 13, 12),\\
\bar{D} &= (0, \infty)(1, 16)(2, 8)(3, 11)(4)(5, 10)(6, 14)(7, 12)(9, 15)(13).
\end{aligned}$$

This action yields the following diagram:



$D(17)$

The above coset diagram represents a non-abelian and simple group $PSL(2, 17)$ of order 2448 with finite presentation $\langle A, B, C, D \mid A^3 = B^2 = C^3 = D^2 = (A^{-1}C^{-1})^2 =$

$(BC^{-1})^2 = (A^{-1}D)^2 = (BD)^2 = DABAC(BA^{-1})^2B = DA^{-1}(C^{-1}D)^3C^{-1}B = 1\rangle$.

Theorem 3. *There does not exist any T in $PSL(2, \mathbb{Z}[i])$, such that $T^2 = (AT)^2 = (BT)^2 = (CT)^2 = 1$.*

Proof. Suppose, on the contrary, there exists T satisfying $T^2 = (AT)^2 = (BT)^2 = (CT)^2 = 1$. Suppose, $T(z) = \frac{az+b}{cz+b}$, i.e., $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The matrix T in $GL(2, \mathbb{Z}[i])$ satisfies $T^2 = 1$ if and only if the trace of the matrix T is zero or $T = \pm 1$. So, the matrix T becomes $T = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ because we can take $a + d = 0$. Now $AT = \begin{pmatrix} uc & ud \\ ua + c & ub + d \end{pmatrix}$, since $AT \neq \pm 1 \neq T$ because of $A^3 = 1$. So the trace of AT must be zero, hence $u(b + c) + d = 0$. Again $BT = \begin{pmatrix} uc & ud \\ ua & ub \end{pmatrix}$, since $BT \neq \pm 1$ because $B = B^{-1}$ and $T \neq B \neq \pm 1$. Thus trace of BT is zero and we obtain $u(b + c) = 0$. Now $CT = \begin{pmatrix} -a - c & -b - d \\ a & b \end{pmatrix}$; since $C^3 = 1$, hence $CT \neq \pm 1 \neq T$. This means trace of CT should be zero, which is $-a - c + b = 0$. Indeed, $T^2 = (AT)^2 = (BT)^2 = (CT)^2 = 1$ gives $T = \begin{pmatrix} 2b & b \\ -b & -2b \end{pmatrix}$. But the determinant of T is $-3b^2$ which can never be ± 1. □

PROPOSITION 1

The coset diagram of the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$ is not a bipartite graph.

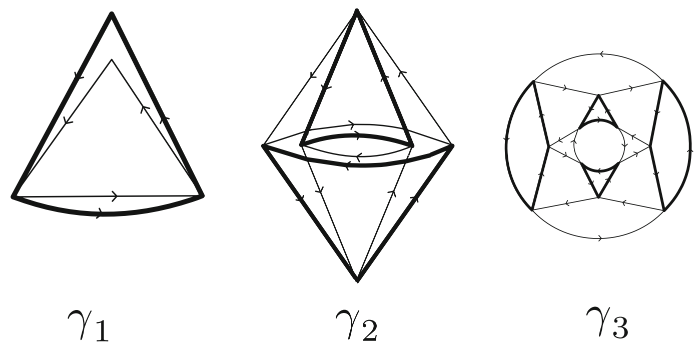
Proof. By definition, a graph is bipartite if and only if it does not contain an odd cycle. Since the coset diagram contains 3-cycles due to the generators A and C of $PSL(2, \mathbb{Z}[i])$ therefore, it is not a bipartite graph. □

Remark 1. Fixed points of A and C exist when 3 is a perfect square in F_p and fixed points of B and D exist for Pythagorean primes only.

It can be proved as: let z be a point of F_p fixed by A if and only if $z^2 - uz - 1 = 0$; this has a solution if and only if $u^2 + 4(= 3)$ is a perfect square. Similarly, a point z is fixed by C if and only if $z^2 + z + 1 = 0$; this has a solution only if -3 or 3 is a perfect square (By Theorem 1). Suppose, B fixes z, i.e., $z = B(z) = \frac{1}{z}$. Therefore, $z^2 = 1$, this has a solution for all p, where p is a Pythagorean prime. Similarly for $z = D(z) = \frac{-1}{z}$ implies that $z^2 = -1$, by Theorem 1 this has a solution for only Pythagorean primes.

4. Occurrence of fragments

The coset diagrams, which depict the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$ where p is a Pythagorean prime, are composed of various types of fragments. There are certain fragments which frequently occur in these coset diagrams and it is worthwhile to know under what conditions they exist in them. In the coset diagrams for the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$, where p is a Pythagorean prime, the following special fragments [7], namely γ_1, γ_2 and γ_3 occur frequently. We describe them here respectively.



Each fragment represents A_4.

Theorem 4. *In the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$,*

(i) *fragment γ_1 occurs if and only if -3 is a perfect square in F_p, and*
(ii) *fragments γ_2 and γ_3 occur for all Pythagorean primes p.*

Proof.

(i) Suppose, fragment γ_1 occurs in the coset diagram. Let υ be a vertex of γ_1, it means that AAC fixes the vertex υ, that is,

$$\upsilon = \upsilon(AAC) = -1 - \frac{\upsilon}{u\upsilon + 1}$$

implies

$$\upsilon^2 + (1 - 2u)\upsilon - u = 0$$

which has a solution only when $(1 - 2u)^2 + 4u = -3$ is a perfect square in F_p.

Conversely, suppose that -3 is a perfect square in F_p, so AAC fixes an element υ in F_p but the element υ is represented by a vertex in the coset diagram which implies that the fragment occurs in the coset diagram.

(ii) Let γ_2 and γ_3 be two fragments. Since every vertex fixed by $(AC)^2$ is in both the fragments, $(AC)^2$ is also a relator of $PSL(2, \mathbb{Z}[i])$ and the coset diagram depicts the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$. So, γ_2 and γ_3 occur together in every coset diagram for the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$. □

We denote the total number of fragments of type γ_1 by $N(\gamma_1)$, type γ_2 by $N(\gamma_2)$ and type γ_3 by $N(\gamma_3)$.

Theorem 5. *A Pythagorean prime p can be expressed as $12l + 4m + 5$, where $l \in \mathbb{Z}^+$, $m = 0$ or 2, $N(\gamma_1) = m$, $N(\gamma_2) = 1$ and $N(\gamma_3) = l$.*

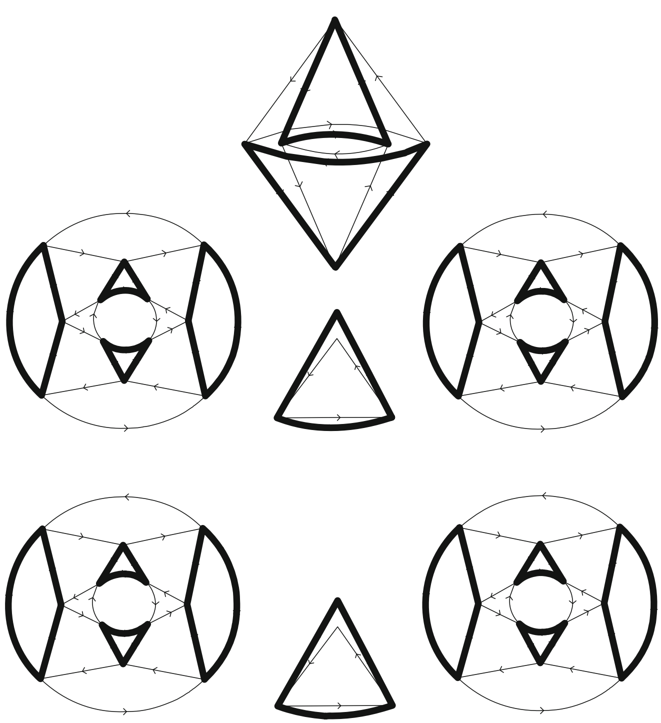
Proof. The coset diagrams for the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$ contain three types of fragments namely γ_1, γ_2 and γ_3. The fragment γ_1 has four vertices and two copies of γ_1

exist in the diagrams only when -3 is a perfect square mod p. Let, m denote the number of fragments γ_1. Since each vertex of a coset diagram is an element of $PL(F_p)$, therefore the elements of $PL(F_p)$, occupied by γ_1 are $4m$, where $m = 0$ or 2. The fragment γ_2 has six vertices, and this fragment exists once in every coset diagram of the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$. The fragment γ_3 has twelve vertices and this fragment exists also in every coset diagram. If l is the number of fragments of type γ_3, then the elements of $PL(F_p)$, occupied by γ_3 are $12l$. Thus the total number of vertices in the coset diagram are $12l+4m+6$, where $l \in \mathbb{Z}^+$and $m = 0$ or 2. Hence, $|PL(F_p)| = 12l+4m+6$, where $l \in \mathbb{Z}^+$ and $m = 0$ or 2.

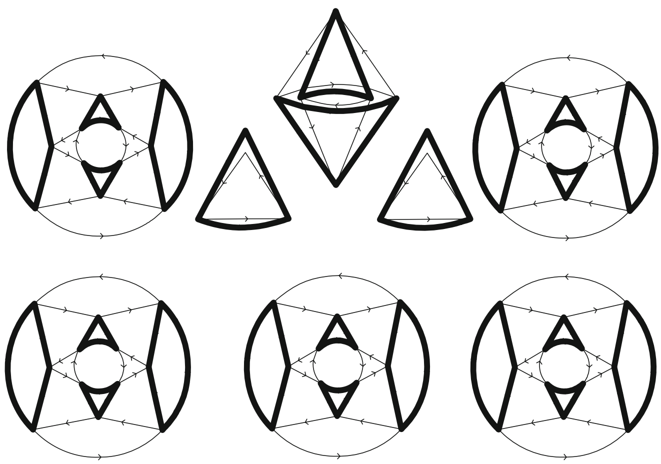
Now $|PL(F_p)| = p + 1$ implies that $p = 12l + 4m + 5$, where $l \in \mathbb{Z}^+$ and $m = 0$ or 2. It further implies that $N(\gamma_1) = m$, $N(\gamma_2) = 1$ and $N(\gamma_3) = l$, where $l \in \mathbb{Z}^+$and $m = 0$ or 2. □

The fragment γ_1 occurs twice when 3 is a perfect square mod$p$, γ_2 fragment exists once and γ_3 fragment occurs $\frac{p-13}{12}$ times if 3 is a perfect square mod$p$ and occurs $\frac{p-5}{12}$ times otherwise. Thus the total number of copies of A_4 are $l + m + 1$.

Example 1. Since $p = 61$ is a Pythagorean prime, therefore by Theorem 5, it can be expressed as $12l + 4m + 5$, where $l \in \mathbb{Z}^+$ and $m = 0$ or 2. That is, $61 = 12(4) + 4(2) + 5$, where $l = 4$, $m = 2$ implies that $N(\gamma_1) = 2$, $N(\gamma_2) = 1$ and $N(\gamma_3) = 4$. Hence the total number of copies of A_4 is 7 and these are:



Example 2. The prime $p = 73$ can be expressed as $73 = 12(5) + 4(2) + 5$, where $l = 5$ and $m = 2$. Therefore by Theorem 5, $N(\gamma_1) = 2$, $N(\gamma_2) = 1$ and $N(\gamma_3) = 5$. Hence the total number of copies of A_4 is 8, as shown:



DEFINITION 1

If $PSL(2, \mathbb{Z}[i])$ acts on $PL(F_p)$, then two elements of $PL(F_p)$ are equivalent if there exists $g \in PSL(2, \mathbb{Z}[i])$, of the form

$$g : z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$$

with $\alpha\delta - \beta\gamma = \pm 1$ and $\alpha, \beta, \gamma, \delta \in \mathbb{Z}[i]$, such that $(\mu)\, g = \nu$.

Theorem 6. *Action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$ is transitive.*

Proof. The group $PSL(2, \mathbb{Z}[i])$ is generated by the linear fractional transformations A, B, C and D. Therefore, $BA(z) = B\left(\frac{1}{z-u}\right) = \frac{z}{1-zu}$. Also $(BA)^2(z) = \frac{z}{1-2zu}$ and $(BA)^3(z) = \frac{z}{1-3zu}$. With the help of mathematical induction, we get $(BA)^n(z) = \frac{z}{1-nzu}$, where $n = 1, 2, \ldots, p-1$. Similarly, $(DC)^n(z) = \frac{z}{1+nz}$, where $n = 1, 2, \ldots, p-1$.

Since, $PSL(2, \mathbb{Z}[i])$ is generated by the linear fractional transformations $A : z \mapsto \frac{1}{z-u}$, $B : z \mapsto \frac{1}{z}$, $C : z \mapsto \frac{-1-z}{z}$ and $D : z \mapsto -\frac{1}{z}$. And we have inductively

$$(BA)^n(z) = \frac{z}{1-nzu}, \quad \text{for } n = 1, 2, \ldots, p-1$$

Thus, we get any $\upsilon \in F_p^*$ as $\upsilon = ((BA)^n)1$, where $n = (\upsilon^{-1} - 1)u$ in F_p. Also, $((BA)^m))1 = \infty$, where $m = -u$. Thus, we have obtained in the orbit of 1 every point on $PL(F_p)$ other than 0. Finally, 0 is also obtained by using $B : z \mapsto \frac{1}{z}$. $\square$

5. Outline of the *Mathematica* code

The symbolic code is written in *mathematica*. The code is constructed in the following way:

(1) One should give a Pythagorean prime as an input.
(2) The main features of the code:

  (i) Solve the expression for a given Pythagorean prime, which is proved in Theorem 5.
  (ii) Count the number of fragments of type γ_1, γ_2 and γ_3.
  (iii) Count the total number of copies of A_4.
  (iv) Take the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$, which gives the permutation group as a homomorphic image of $PSL(2, \mathbb{Z}[i])$.

Example 3. The action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_{17})$ by $A : z \mapsto \frac{1}{z-u}$, $B : z \mapsto \frac{1}{z}$, $C : z \mapsto -\frac{1+z}{z}$ and $D : z \mapsto -\frac{1}{z}$ on $PL(F_{17})$ yields the following:

INPUT
$p = 17$
OUTPUT
By Theorem 5,
$17 = 12(1) + 4(0) + 5$
$N(\gamma_1) = 0$, $N(\gamma_2) = 1$ and $N(\gamma_3) = 1$
Total number of copies of subscript A_4 are 2.
Action of $PSL(2, \mathbb{Z}[i])$ by $A :\mapsto \frac{1}{z-u}$, $B :\mapsto \frac{1}{z}$, $C :\mapsto -\frac{1+z}{z}$ and $D :\mapsto -\frac{1}{z}$ on $PL(F_{17})$ to give the following permutation representations:

$$
\begin{aligned}
\bar{A} &= (0, 13, \infty)(1, 7, 14)(2, 3, 5)(4, 15, 9)(6, 12, 16)(8, 10, 11),\\
\bar{B} &= (0, \infty)(1)(2, 9)(3, 6)(4, 13)(5, 7)(8, 15)(10, 12)(11, 14)(16),\\
\bar{C} &= (0, \infty, 16)(1, 15, 8)(2, 7, 11)(3, 10, 4)(5, 9, 14)(6, 13, 12),\\
\bar{D} &= (0, \infty)(1, 16)(2, 8)(3, 11)(4)(5, 10)(6, 14)(7, 12)(9, 15)(13).
\end{aligned}
$$

Appendix

The code is developed in *Mathematica* software to generate the permutation group of the action of $PSL(2, \mathbb{Z}[i])$ on $PL(F_p)$ and for finding the number fragments which frequently occur in the coset diagrams.

```
ClearAll;
P =?;
Print["By theorem5"]
Ex=Solve[{P == 12 * l + 4 * m + 5, m == 0||m == 2}, {l, m}, Integers];
l = Ex[[1, 1, 2]];
m = Ex[[1, 2, 2]];
Print[P, " = 12(", l, ") + 4(", m, ") + 5"]
Print["N(γ1) = ", m, ", "N(γ2) = 1", "N(γ3) = ", l]
```

Print["Total Number of Copies of A_4 are", $l + m + 1$]
temp = FindInstance[n^2 + 1 == 0, n, Modulus−> P];
iota = temp[[1, 1, 2]];
Print["A := {"] For [$i = 0, i < P/2$,
i + +, {tt = Solve[$x * i$ − iota $* x - 1$ == 0, Modulus−> P];
$t = tt$[[1, 1, 2]];
ttt = Solve[$x * t$ − iota $* x - 1$ == 0, Modulus−> P];
$t2 = ttt$[[1, 1, 2]]; If[Integer$Q$[$t2$], $t2$, $t2 = \infty$]}
Print["(", i, ", ", t, ", ", $t2$, ")"]]Print["}"]
Print["B := {"]For[$i = 0, i < P/2, i$ + +,
tt = Solve[$x * i - 1$ == 0, Modulus−> P]; $t = tt$[[1, 1, 2]];
If[Integer$Q$[t], t, $t = \infty$]
Print["(", i, ", ", t, ")"]]Print["}"]
Print["C := {"]For[$i = 0, i < P/2$,
i + +, {tt = Solve[$(-x - 1) * i - 1$ == 0, Modulus−> P];
$t = tt$[[1, 1, 2]]; ttt = Solve[$(x + 1)t + 1$ == 0, Modulus−> P];
$t2 = ttt$[[1, 1, 2]]; If[Integer$Q$[t], t, $t = \infty$]}
If[i == 0, Print["(0, ", t, ", ", $P - 1$, ")"],
Print["(", i, ", ", t, ", ", $t2$, ")"]]]Print["}"]
Print["D := {"]For[$i = 0, i < P/2, i$ + +,
tt = Solve[$x * i + 1$ == 0, Modulus−> P]; $t = tt$[[1, 1, 2]];
If[Integer$Q$[t], t, $t = \infty$]
Print["(", i, ", ", t, ")"]]Print["}"]

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