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Dirichlet problem on the upper half space

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Abstract. In this paper, a solution of the Dirichlet problem on the upper half space for a fast growing continuous boundary function is constructed by the generalized Dirichlet integral with this boundary function.

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1. Introduction and results

Let $\mathbf{R}^n (n \geq 3)$ denote the n-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open set D of $\mathbf{R}^n$ are denoted by ∂D and $\bar{D}$ respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$, whose boundary is ∂H. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x,\ y \in \mathbf{R}^n$ as $x = (x', x_n),\ y = (y', y_n)$, where $y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbf{R}^{n-1}$ and putting

$$x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n, \quad |x| = \sqrt{x \cdot x}, \ \ |x'| = \sqrt{x' \cdot x'}.$$

Let $B(r)$ denote the open ball with centre at the origin and radius r and σ denote the $(n-1)$-dimensional surface area measure. Let $[d]$ denote the integer part of the positive real number d. In the sense of Lebesgue measure $\mathrm{d}y' = \mathrm{d}y_1 \ldots \mathrm{d}y_{n-1}$ and $\mathrm{d}y = \mathrm{d}y' \mathrm{d}y_n$.

Given a continuous function f on ∂H, we say that h is a solution of the (classical) Dirichlet problem on H with f, if $\Delta h = 0$ in H and $\lim_{x \in H, x \to z'} h(x) = f(z')$ for every $z' \in \partial H$.

The classical Poisson kernel for H is defined by $P(x, y') = 2x_n \omega_n^{-1} |x - y'|^{-n}$, where $\omega_n = 2\pi^{\frac{n}{2}} / \Gamma(n/2)$ is the area of the unit sphere in $\mathbf{R}^n$.

To solve the Dirichlet problem on H, as in [2, 3, 5, 7], we use the following modified Poisson kernel of order m defined by

$$P_m(x, y') = \begin{cases} P(x, y'), & \text{when } |y'| \leq 1, \\ P(x, y') - \sum_{k=0}^{m-1} \frac{2x_n |x|^k}{\omega_n |y'|^{n+k}} C_k^{\frac{n}{2}} \left(\frac{x \cdot y'}{|x||y'|} \right), & \text{when } |y'| > 1, \end{cases}$$

where m is a non-negative integer, $C_k^{n/2}(t)$ is the ultraspherical (Gegenbauer) polynomials [6]. The expression arises from the generating function for Gegenbauer polynomials

$$(1-2tr+r^2)^{-\frac{n}{2}} = \sum_{k=0}^{\infty} C_k^{\frac{n}{2}}(t) r^k,$$

where $|r| < 1$ and $|t| \leq 1$. The coefficient $C_k^{n/2}(t)$ is called the ultraspherical (Gegenbauer) polynomial of degree k associated with $n/2$, the function $C_k^{n/2}(t)$ is a polynomial of degree k in t.

Put

$$U_m(f)(x) = \int_{\partial H} P_m(x, y') f(y') \mathrm{d}y',$$

where $f(y')$ is a continuous function on ∂H.

Using the modified Poisson kernel $P_m(x, y')$, Yoshida (cf. Theorem 1 of [7]) and Siegel-Talvila (cf. Corollary 2.1 of [5]) gave classical solutions of the Dirichlet problem on H respectively. Motivated by their results, we consider the Dirichlet problem for harmonic functions of infinite order (e.g. see Definition 4.1, p. 143 of [4] for the order of harmonic functions).

To do this, we define a nondecreasing and continuously differentiable function $\rho(r) \geq 1$ on the interval $[0, +\infty)$. We assume further that

$$\varepsilon_0 = \limsup_{r\to\infty} \frac{\rho'(r) r \log r}{\rho(r)} < 1. \tag{1.1}$$

Let $F(p, \rho, \beta)$ be the set of continuous functions f on ∂H such that

$$\int_{\partial H} \frac{|f(y')|^p \mathrm{d}y'}{1+|y'|^{\rho(|y'|)+n+\beta-1}} < \infty, \tag{1.2}$$

where $1 \leq p < \infty$ and β is a positive real number.

Now we have as follows:

Theorem 1. *If $f \in F(p, \rho, \beta)$, then the integral $U_{[\rho(|y'|)+\beta]}(f)(x)$ is a solution of the Dirichlet problem on H with f.*

If we put $[\rho(|y'|) + \beta] = m$ in Theorem 1, we immediately obtain as follows (cf. Theorem 1 of [7] and Corollary 2.1 of [5]).

COROLLARY 1

If f is a continuous function on ∂H satisfying $\int_{\partial H} |f(y')|^p (1+|y'|)^{-n-m} \mathrm{d}y' < \infty$, then $U_m(f)(x)$ is a solution of the Dirichlet problem on H with f.

Theorem 2. *Let u be harmonic in H and continuous on $\bar{H}$. If $u \in F(p, \rho, \beta)$, then we have*

$$u(x) = U_{[\rho(|y'|)+\beta]}(u)(x) + h(x)$$

for all $x \in \bar{H}$, where $h(x)$ is harmonic in H and vanishes continuously on ∂H.

2. Proof of Theorem 1

We need to use the following inequality (see p. 3 of [5]):

$$|P_m(x, y')| \le M x_n |x|^m |y'|^{-n-m} \tag{2.1}$$

for any $x \in H$ and $y' \in \partial H$ satisfying $|y'| \ge \max\{1, 2|x|\}$, where M is positive constant.

For any ϵ $(0 < \epsilon < 1 - \epsilon_0)$, there exists a sufficiently large positive number R such that $r > R$. By (1.1) we have

$$\rho(r) < \rho(e)(\ln r)^{(\epsilon_0+\epsilon)},$$

which yields that there exists a positive constant $M(r)$ dependent only on r such that

$$k^{-\beta/2}(2r)^{\rho(k+1)+\beta+1} \le M(r) \tag{2.2}$$

for any $k > k_r = [2r] + 1$.

For any $x \in H$ and $|x| \le r$, we have by (1.2), (2.1), (2.2), $1/p + 1/q = 1$ and Hölder's inequality

$$\begin{aligned}
M \ & \textstyle\sum_{k=k_r}^{\infty} \displaystyle\int_{\{y'\in\partial H:k\le|y'|<k+1\}} \frac{(2|x|)^{[\rho(|y'|)+\beta]+1}}{|y'|^{[\rho(|y'|)+\beta]+n}}|f(y')|\mathrm{d}y' \\
&\le M\sum_{k=k_r}^{\infty}(2r)^{\rho(k+1)+\beta+1}\left(\int_{\{y'\in\partial H:k\le|y'|<k+1\}}\frac{|f(y')|^p}{|y'|^{\rho(|y'|)+n+\frac{p\beta}{2}-1}}\mathrm{d}y'\right)^{\frac{1}{p}} \\
&\quad \times\left(\int_{\{y'\in\partial H:k\le|y'|<k+1\}}|y'|^{-q\{[\rho(|y'|)+\beta]+n-\frac{\rho(|y'|)+n-1}{p}-\frac{\beta}{2}\}}\mathrm{d}y'\right)^{\frac{1}{q}} \\
&\le M\sum_{k=k_r}^{\infty}\frac{(2r)^{\rho(k+1)+\beta+1}}{k^{\beta/2}}\left(\int_{\{y'\in\partial H:k\le|y'|<k+1\}}\frac{|f(y')|^p}{|y'|^{\rho(|y'|)+n+\frac{p\beta}{2}-1}}\mathrm{d}y'\right)^{\frac{1}{p}} \\
&\le 2MM(r)\left(\int_{\{y'\in\partial H:|y'|\ge k_r\}}\frac{|f(y')|^p}{1+|y'|^{\rho(|y'|)+n+\frac{p\beta}{2}-1}}\mathrm{d}y'\right)^{\frac{1}{p}} \\
&< \infty.
\end{aligned}$$

Thus $U_{[\rho(|y'|)+\beta]}(f)(x)$ is finite for any $x \in H$. Since $P_{[\rho(|y'|)+\beta]}(x, y')$ is a harmonic function of $x \in H$ for any fixed $y' \in \partial H$, $U_{[\rho(|y'|)+\beta]}(f)(x)$ is also a harmonic function of $x \in H$.

To verify the boundary behavior of $U_{[\rho(|y'|)+\beta]}(f)(x)$, we fix a boundary point $z' \in \partial H$. Choose a large $t > |z'| + 1$, and write

$$U_{[\rho(|y'|)+\beta]}(f)(x) = X(x) - Y(x) + Z(x),$$

where

$$X(x) = \int_{\{y'\in\partial H:|y'|\le t\}} P(x, y')f(y')\mathrm{d}y',$$

$$Y(x) = \sum_{k=0}^{[\rho(|y'|+\beta)]-1}\frac{2x_n|x|^k}{\omega_n}\int_{\{y'\in\partial H:1<|y'|\le t\}}\frac{1}{|y'|^{n+k}}C_k^{\frac{n}{2}}\left(\frac{x'\cdot y'}{|x||y'|}\right)f(y')\mathrm{d}y',$$

$$Z(x) = \int_{\{y' \in \partial H : |y'| > t\}} P_{[\rho(|y'| + \beta)]}(x, y') f(y') \mathrm{d}y'.$$

Notice that $X(x)$ is the Poisson integral of $f(y')\chi_{B(t)}(y')$, where $\chi_{B(t)}$ is the characteristic function of the ball $B(t)$. So it tends to $f(z')$ as $x \to z'$. Since $Y(x)$ is a polynomial times x_n and $Z(x) = O(x_n)$, both of them tend to zero as $x \to z'$. Thus the function $U_{[\rho(|y'|)+\beta]}(f)(x)$ can be continuously extended to $\bar{H}$ such that $U_{[\rho(|y'|)+\beta]}(f)(z') = f(z')$ for any $z' \in \partial H$. Theorem 1 is proved.

3. Proof of Theorem 2

Consider the function $u(x) - U_{[\rho(|y'|)+\beta]}(u)(x)$, which is harmonic in H, can be continuously extended to $\bar{H}$ and vanishes on ∂H.

The Schwarz reflection principle (p. 68 of [1]) applied to $u(x) - U_{[\rho(|y'|)+\beta]}(u)(x)$ shows that there exists a harmonic function $h(x)$ in H such that $h(x^*) = -h(x) = -(u(x) - U_{[\rho(|y'|)+\beta]}(u)(x))$ for $x \in \bar{H}$, where $*$ denotes reflection in ∂H just as $x^* = (x', -x_n)$.

Thus $u(x) = h(x) + U_{[\rho(|y'|)+\beta]}(u)(x)$ for all $x \in \bar{H}$, where $h(x)$ is a harmonic function on H vanishing continuously on ∂H. We complete the proof of Theorem 2.

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