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On a generalization of $B_1(\Omega)$ on C^*-algebras

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Abstract. We discuss the unitary classification problem of a class of holomorphic curves on C^*-algebras. It can been regarded as a generalization of Cowen–Doulgas operators with index one.

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1. Introduction

Let $\mathscr{H}$ be a complex and separable Hilbert space and let $\mathscr{L}(\mathscr{H})$ be the set of bounded linear operators on $\mathscr{H}$. Let Ω be a open connected subset of complex plane $\mathbb{C}$. A class of Cowen–Douglas operator with index one: $B_1(\Omega)$ is defined as follows [6]: $B_1(\Omega) =:$ $\{T \in \mathscr{L}(\mathscr{H})$:

(i) $\Omega \subset \sigma(T) =: \{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible,
(ii) $\bigvee_{\lambda\in\Omega} \mathrm{Ker}(T-\lambda) = \mathscr{H}$,
(iii) $\mathrm{Ran}(T-\lambda) = \mathscr{H}$,
(iv) $\dim \mathrm{Ker}(T-\lambda) = 1, \forall\lambda \in \Omega\}$.

For any operator $T \in B_1(\Omega)$, it is shown that we can find a holomorphic family of eigenvectors $\{e(\lambda), \lambda \in \Omega\}$ such that $Te(\lambda) = \lambda e(\lambda), \forall\lambda \in \Omega$. A holomorphic curve with one dimension is a map from $\mathscr{H}$ to Grassmann manifold $\mathrm{Gr}(n, \mathscr{H})$ defined as $F(\lambda) =: \bigvee\{e(\lambda)\}$ for $\lambda \in \Omega$. We call two linear bounded operators T and S are unitarily equivalent if and only if there exists a unitary operator $U \in \mathscr{L}(\mathscr{H})$ such that $T = USU^*$, denoted by $T \sim_u S$. For two holomorphic curves F and G defined on Ω, if there exists a unitary operator $U \in \mathscr{L}(\mathscr{H})$ such that $F(\lambda) = UG(\lambda), \forall\lambda \in \Omega$, then we call them unitarily equivalent, denoted by $F \sim_u G$.

In [6], it is shown that unitary equivalence of operator T can be deduced to the same problem of holomorphic curve F associated to it. Following Cowen and Doulgas [6], a curvature function for $T \in B_1(\Omega)$ can be defined as

$$K_T = -\frac{\partial}{\partial\bar{\lambda}}\left(h^{-1}\frac{\partial h}{\partial\lambda}\right), \quad \text{for all } \lambda \in \Omega,$$

where $h(\lambda) = ||e(\lambda)||^2, \forall\lambda \in \Omega$. And a remarkable result is also proved in [6]: For $T, S \in B_1(\Omega)$, $T \sim_u S$ if and only if $K_T = K_S$ on Ω. Subsequently, the curvature function turns into an important object of the research of Cowen–Douglas operators. Many other mathematicians too have done a lot of work around the curvature [2–5, 11–13, 19–22, 25]. On the other hand, by using the K_0-group, a few others [7–10] worked on the problems of similarity classification of Cowen–Douglas operators and some holomorphic curves.

In [1], Apostol and Martin discussed the unitary equivalence problem of Cowen–Douglas operators in a C^*-algebraic setting. Let $\mathcal{U}$ be a unital C^*-algebra, then $p \in \mathcal{U}$ is called a projection in $\mathcal{U}$ whenever $p^2 = p = p^*$, and $\mathcal{P}(\mathcal{U})$ denotes the set of all projections in $\mathcal{U}$ which is called the Grassmann manifold of $\mathcal{U}$. Let $\Omega \subseteq \mathbb{C}$ be a connected open set. If $P : \Omega \to \mathcal{P}(\mathcal{U})$ is an infinite differential $\mathcal{U}$-valued map with $\bar{\partial}PP = 0$, then it is called an extended holomorphic curve on $\mathcal{P}(\mathcal{U})$ (see more details in [15]).

Let $P, Q : \Omega \to \mathcal{P}(\mathcal{U})$ be two extended holomorphic curves. We call P and Q are unitarily equivalent (denoted by $P \sim_u Q$) if there exists a unitary $U \in \mathcal{U}$ such that $P(\lambda) = UQ(\lambda)U^*, \forall\lambda \in \Omega$. Martin and Salinas did a series work of holomorphic curves on extended flag manifolds and Grassmann manifolds [15–18, 24].

In [15], Martin and Salinas proved as follows.

Lemma 1.1 (*Theorem* 4.5 *of* [15]). *Suppose* P,Q *be two extended holomorphic curves in Class* $\mathcal{A}_k(\Omega, \mathcal{U})$ *if* $\mathcal{U}$ *is inner* (*see Definition* 1.5) , *then the following are equivalent*:

(1) $P \sim_u Q$;
(2) *for each* $\lambda \in \Omega$, *there exists a unitary* v *such that*

$$v\bar{\partial}^J P(\lambda)\partial^I P(\lambda)v^* = \bar{\partial}^J Q(\lambda)\partial^I Q(\lambda), \quad \forall J, I \leq k.$$

Condition (2) is also said to be that P and Q have order of contact k at λ. In this note, we introduce a class of extended holomorphic curves including Bott projection on $\mathbb{C}^2$ and holomorphic curves in $\mathscr{L}(\mathscr{H})$, etc. With the same form of curvature of the Cowen–Douglas operator, we define the curvature of these extended holomorphic curves and give a unitarily equivalent theorem. The following is our main theorem:

Theorem 2.3. *Let* $P, Q \in \mathcal{P}_1(\Omega, \mathcal{U}) \cap \mathcal{A}_1(\Omega, \mathcal{U})$. *For each* $\lambda \in \Omega$, *if there exists a unitary* $v_\lambda \in \mathcal{U}$ *such that* $P(\lambda) = v_\lambda Q(\lambda)v_\lambda^*$, *then* $P \sim_u Q$ *if and only if* $K_P(\lambda) = K_Q(\lambda), \forall\lambda \in \Omega$.

We will first introduce some notations and results, and all the notations are adopted from [1], [7] and [15].

1.2. Let $\mathcal{U}$ be a C^*-algebra, and $P : \Omega \to \mathcal{P}(\mathcal{U})$ be an infinite differential map. Then P is holomorphic if and only if

$$\bar{\partial}P(\lambda)P(\lambda) = 0, \quad \forall\lambda \in \Omega, \tag{1.1}$$

where we use symbol $\bar{\partial}^J\partial^I$ to denote $\frac{\partial^{J+I}}{\partial^J\bar{\lambda}\partial^I\lambda}, \forall J, I \in \mathbb{Z}^+$. We assume that $\partial^I P = P$, $\bar{\partial}^J P = P$, $\bar{\partial}^J\partial^I P = P$, when $J = I = 0$.

1.3. Let $C^\infty(\Omega, \mathcal{U})$ denote the $*$-algebra of all $\mathcal{U}$-valued infinitely differentiable functions defined on Ω. Then we have that

$$(\partial^I(A))^* = \bar{\partial}^I(A), \quad \forall A \in C^\infty(\Omega, \mathcal{U}).$$

Let $\mathcal{U}$ be a unital C^*-algebra, and $P : \Omega \to \mathcal{P}(\mathcal{H})$ be an extended holomorphic curve. Assume that $S \subseteq \mathcal{U}$ is a fixed subset containing the unit of $\mathcal{U}$. For each $\lambda \in \Omega$ and every $\alpha \in \mathbb{Z}_+ \cup \{\infty\}$, set

$$\mathcal{B}_\lambda^\alpha = \left\{\bar{\partial}^J P(\lambda) y^* x \partial^I P(\lambda) : I, J \in \mathbb{Z}_+, I, J \leq \alpha, x, y \in S\right\}.$$

Let $\mathcal{U}_\lambda^\alpha$ be the closure of $*$-subalgebra of $\mathcal{U}$ generated by $\mathcal{B}_\lambda^\alpha$ with property

$$\mathcal{U}_\lambda^0 \subseteq \mathcal{U}_\lambda^1 \subseteq \cdots \subseteq \mathcal{U}_\lambda^\infty.$$

DEFINITION 1.4 [15]

Let $k \geq 1$ be an integer. If the following conditions are satisfied, then (P, S) is said to be in the class $\mathcal{A}_k(\Omega, \mathcal{U})$:

(1) $\mathcal{U}_\lambda^\infty$ is a finite-dimensional C^*-algebra for each $\lambda \in \Omega$.
(2) If k_λ denotes the cardinal of any maximal collection of mutually orthogonal minimal projections in $\mathcal{U}_\lambda^\infty$, then $k_\lambda \leq k$.
(3) If $a \in \mathcal{U}$ and $aP(\lambda) = 0$ for every $\lambda \in \Omega$, then $a = 0$.

In particular, when $S = \{\mathbf{1}\}$, (P, S) is equal to P.

DEFINITION 1.5 [15]

We say $\mathfrak{S} \subset \mathcal{U}$ is a separating subset of $\mathcal{U}$ if $\{a \in \mathcal{U} : as = 0, s \in \mathfrak{S}\} = \{0\}$. Assume $\mathfrak{S}$, $\mathfrak{T}$ are two separating subsets of $\mathcal{U}$, and $\theta : \mathfrak{S} \to \mathfrak{T}$ is a given bijection. We say θ is inner (semi-inner), if there exists a unitary $u \in \mathcal{U}$ (a unitary $v \in \mathcal{U}$) such that

$$usu^* = \theta(s), s \in \mathfrak{S}, \quad (\text{or } \; vt^*sv^* = \theta(t)\theta(s), s, t \in \mathfrak{S}).$$

$\mathcal{U}$ is said to be inner if each semi-inner bijection between two separating subsets of $\mathcal{U}$ is inner. In the following, we always assume that $\mathcal{U}$ is inner.

1.6. Let $\operatorname{diag}\{x_1, x_2, \ldots, x_n\}$ denote a diagonal matrix with the diagonal entries: $x_1, x_2, \ldots, x_n$.

2. Curvature and unitary equivalence of extended holomorphic curve

Let B be a C^*-algebra. A Hilbert B-module $l^2(\mathbb{N}, B)$ is defined as

$$l^2(\mathbb{N}, B) =: \left\{(a_i)_{i\in\mathbb{N}} : a_i \in B, \forall i \in \mathbb{N}, \quad \text{and} \quad \sum_{i\in\mathbb{N}} ||a_i||^2 < \infty\right\}.$$

We denote the set of all linear bounded operators on $l^2(\mathbb{N}, B)$ by $\mathscr{L}(l^2(\mathbb{N}, B))$. Then $\mathscr{L}(l^2(\mathbb{N}, B))$ is a C^*-algebra.

Let $\alpha^* = (a_i^*)_{i\in\mathbb{N}} \in l^2(\mathbb{N}, B)$ and $\alpha = (a_i)_{i\in\mathbb{N}}^T$ be the conjugate transpose of α^*. Let the symbol '$\cdot$' denote the multiplication of a matrix. We will first introduce the following two notations:

(1) $\alpha \cdot \alpha^* := (a_i a_j^*)_{i,j\in\mathbb{N}}$ which can be seen as an element in $\mathscr{L}(l^2(\mathbb{N}, B))$;
(2) $\alpha^* \cdot \alpha := \sum_{i=1}^{\infty} \alpha_i^* \alpha_i \in B$.

DEFINITION 2.1

Let Ω be a connected open subset of $\mathbb{C}$ and B be a unital C^*-algebra. For $\mathcal{U} = \mathscr{L}(l^2(\mathbb{N}, B))$, and let $\mathcal{P}_1(\Omega, \mathcal{U})$ denote the extended holomorphic curve P which satisfies:

(1) $P(\lambda) = (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1}\alpha(\lambda) \cdot \alpha^*(\lambda), \forall\lambda \in \Omega$, where $\alpha : \Omega \to l^2(\mathbb{N}, B)$ is a holomorphic function and α^* is the conjugate transpose of α.
(2) $\alpha^*(\lambda) \cdot \alpha(\lambda) \in Z(B), \forall\lambda \in \Omega$, where $Z(B) := \{x \in B : xb = bx, \forall b \in B\}$ is the centre of B.

DEFINITION 2.2

Let $P \in \mathcal{P}_1(\Omega, \mathcal{U})$. Considering $l^2(\mathbb{N}, B))$ is a Hilbert C^*-module, denote $h(\lambda) = \langle\alpha(\lambda), \alpha(\lambda)\rangle = \alpha^*(\lambda) \cdot \alpha(\lambda)$. A curvature function of P is defined as

$$K_P = -\frac{\partial}{\partial\bar{\lambda}}\left(h^{-1}\frac{\partial h}{\partial\lambda}\right), \qquad \text{for all } \lambda \in \Omega.$$

This curvature is the same inform as the curvature of the Cowen–Douglas operator.

Theorem 2.3. *Let $P, Q \in \mathcal{P}_1(\Omega, \mathcal{U}) \cap \mathcal{A}_1(\Omega, \mathcal{U})$. For each $\lambda \in \Omega$, if there exists a unitary $v_\lambda \in \mathcal{U}$ such that $P(\lambda) = v_\lambda Q(\lambda) v_\lambda^*$, then $P \sim_u Q$ if and only if $K_P(\lambda) = K_Q(\lambda), \forall\lambda \in \Omega$.*

Proof. Since $P(\lambda) = (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1}\alpha(\lambda) \cdot \alpha^*(\lambda), \forall\lambda \in \Omega$ and $\bar{\partial}\alpha = 0, \partial\alpha^* = 0$, we have

$$\begin{aligned}\bar{\partial}P &= \bar{\partial}((\alpha^* \cdot \alpha)^{-1})\alpha \cdot \alpha^* + (\alpha^* \cdot \alpha)^{-1}\bar{\partial}(\alpha \cdot \alpha^*)\\ &= \bar{\partial}((\alpha \cdot \alpha^*)^{-1})\alpha \cdot \alpha^* + (\alpha^* \cdot \alpha)^{-1}(\alpha \cdot \bar{\partial}\alpha^*)\\ &= \bar{\partial}h^{-1}\alpha \cdot \alpha^* + h^{-1}\alpha \cdot \bar{\partial}\alpha^*\end{aligned}$$

and

$$\partial P = \partial h^{-1}\alpha \cdot \alpha^* + h^{-1}\partial\alpha \cdot \alpha^*.$$

By $\bar{\partial}\alpha = 0, \partial\alpha^* = 0$, it follows that

$$\partial h = \partial\alpha \cdot \alpha^*, \bar{\partial}h = \alpha \cdot \bar{\partial}\alpha^*$$

and

$$\begin{aligned}\partial\bar{\partial}h &= \partial(\bar{\partial}\alpha^* \cdot \alpha + \alpha^* \cdot \bar{\partial}\alpha)\\ &= \partial\bar{\partial}\alpha^* \cdot \alpha + \bar{\partial}\alpha^* \cdot \partial\alpha\\ &= \bar{\partial}\alpha^* \cdot \partial\alpha.\end{aligned}$$

By condition (2) in Definition 2.1, $h := \alpha \cdot \alpha^* \subseteq Z(B)$. So h commutes with each element in B. Since $\alpha \in l^2(\mathbb{N}, B)$, h and h^{-1} also commute with α and α^*. Notice that condition (2) always means that $\mathrm{Span}\{\bar{\partial}^J \alpha^* \cdot \partial^I \alpha,\ J, I \in \mathbb{Z}^+ \cup \{0\}\} \subseteq Z(B)$. Then for any $J, I \in \mathbb{Z}^+ \cup \{0\}$, it follows that $\bar{\partial}^J \alpha^* \cdot \partial^I \alpha$ commutes with α and α^*.

From these observations, we have that

$$\begin{aligned}
\bar{\partial} P \partial P &= (\bar{\partial} h^{-1}(\alpha \cdot \alpha^*) + h^{-1}(\alpha \cdot \bar{\partial}\alpha^*))(\partial h^{-1}(\alpha \cdot \alpha^*) + h^{-1}(\partial\alpha \cdot \alpha^*)) \\
&= \bar{\partial} h^{-1} \partial h^{-1}(\alpha \cdot \alpha^* \cdot \alpha \cdot \alpha^*) + h^{-1}(\alpha \cdot \bar{\partial}\alpha^*) h^{-1}(\partial\alpha \cdot \alpha^*) \\
&\quad + \bar{\partial} h^{-1} h^{-1}(\alpha \cdot \alpha^* \cdot \partial\alpha \cdot \alpha^*) + h^{-1} \partial h^{-1}(\alpha \cdot \bar{\partial}\alpha^* \cdot \alpha \cdot \alpha^*) \\
&= \bar{\partial} h^{-1} \partial h^{-1} h(\alpha \cdot \alpha^*) + h^{-2}(\alpha \cdot \bar{\partial}\alpha^* \cdot \partial\alpha \cdot \alpha^*) \\
&\quad + \bar{\partial} h^{-1} h^{-1}(\alpha \cdot \alpha^* \cdot \partial\alpha \cdot \alpha^*) + h^{-1} \partial h^{-1}(\alpha \cdot \bar{\partial}\alpha^* \cdot \alpha \cdot \alpha^*) \\
&= [\bar{\partial} h^{-1} \partial h^{-1} h^2 + h^{-1} \partial\bar{\partial} h + \bar{\partial} h^{-1} \partial h + \partial h^{-1} \cdot \bar{\partial} h] P \\
&= [-h^{-1} \bar{\partial} h h^{-1} \partial h + h^{-1} \partial\bar{\partial} h] P \\
&= -K_P P.
\end{aligned}$$

Similarly, $\bar{\partial} Q \partial Q = -K_Q Q$. Chose $S = \{\mathbf{1}\}$, and from Lemma 1.1, we know that P and Q have order of contact one at each λ if and only if P and Q are unitarily equivalent.

Since the formulas

$$\bar{\partial} P P = P \partial P = 0,\ \bar{\partial} Q Q = Q \partial Q = 0$$

and

$$\bar{\partial} P \partial P = -K_P P,\ \bar{\partial} Q \partial Q = -K_Q Q$$

hold, it follows that if $K_P = K_Q$ holds, then P and Q satisfy condition (2) in Lemma 1.1, and P and Q are unitarily equivalent. So we finish the proof of the 'if part'.

If $P = uQu^*$, then $\bar{\partial} P \partial P = u \bar{\partial} Q \partial Q u^*$. By condition (2) of 3.1, it follows that $K_P, K_Q \in Z(B)$. Thus,

$$\begin{aligned}
\bar{\partial} P \partial P &= -K_P P = -K_P u Q u^* = -u K_P Q u^* \\
&= -u K_Q Q u^* = -K_Q u Q u^* \\
&= u \bar{\partial} Q \partial Q u^*.
\end{aligned}$$

That means $K_P(\lambda) Q(\lambda) = K_Q(\lambda) Q(\lambda), \forall \lambda \in \Omega$. By condition (3) of Definition 1.4, we have $K_P = K_Q$ on Ω. This completes the proof of Theorem 2.3.

Let $\alpha : \Omega \to l^2(\mathbb{N}, B)$ be a holomorphic function. In Definition 2.1, the C^∞ map $P : \Omega \to l^2(\mathbb{N}, B)$ is defined as $P := (\alpha^* \cdot \alpha)^{-1} \alpha \cdot \alpha^*$.

By the proof of Theorem 2.3, we can see that P is an extended holomorphic curve. In fact, by the following calculation, we can show that P satisfies equation (1.1) since we have that

$$\begin{aligned}
\bar{\partial} P P &= \bar{\partial}((\alpha^* \alpha)^{-1} \alpha \cdot \alpha^*)((\alpha^* \alpha)^{-1} \alpha \cdot \alpha^*) \\
&= (\bar{\partial} h^{-1} \alpha \cdot \alpha^* + h^{-1} \alpha \cdot \bar{\partial} \alpha^*) h^{-1} \alpha \cdot \alpha^* \\
&= (\bar{\partial} h^{-1} + (h^{-1})^2 \bar{\partial} h) \alpha \cdot \alpha^* \\
&= h^{-1}(\bar{\partial} h^{-1} + (h^{-1})^2 \bar{\partial} h) \alpha \cdot \alpha^* \\
&= h^{-1}(\bar{\partial}(h^{-1} h)) \alpha \cdot \alpha^* \\
&= 0.
\end{aligned}$$

By using the same construction in Definition 2.1, we can get a series of extended holomorphic curves which can be classified by using curvature. And the finite dimension case of our extended holomorphic curve contains the Bott projection (Example 2.4) and the infinite case contains the classical holomorphic curves in Grassmann manifold (Example 2.5).

Example 2.4 (*Bott projection*). Let $\mathcal{U}$ be $M_2(\mathbb{C})$ and $\Omega \subseteq \mathbb{C}$ be a connected open set, and let $P : \Omega \to M_2(\mathbb{C})$ be defined by

$$P(\lambda) = \frac{1}{1+|\lambda|^2}\begin{pmatrix} 1 & \bar{\lambda} \\ \lambda & |\lambda|^2 \end{pmatrix}, \quad \forall \lambda \in \Omega.$$

Then P is called Bott projection which is important in K-theory [12]. If we choose $\alpha(\lambda) := (1, \lambda)^T \in \mathbb{C}^2$ and $\alpha^*(\lambda) := (1, \bar{\lambda})$, then

$$P(\lambda) = (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1}\alpha(\lambda) \cdot \alpha^*(\lambda), \quad \forall \lambda \in \Omega,$$

and P is an extended holomorphic curve on Ω.

If we consider another extended holomorphic curve

$$Q(\lambda) = \frac{1}{1+|g(\lambda)|^2}\begin{pmatrix} 1 & \bar{g(\lambda)} \\ g(\lambda) & |g(\lambda)|^2 \end{pmatrix}, \quad \bar{\partial} g = 0,$$

by 3.3, we can see that the curvatures of P and Q are $\frac{-1}{1+|\lambda|^2}$ and $\frac{-|\partial g(\lambda)|^2}{1+|g(\lambda)|^2}$ respectively. In particular, if we choose $g(\lambda) = \lambda^2$, then the two curvatures can not be equal for $|\lambda| \neq \sqrt{2}-1$. By Theorem 2.3, P and Q are not unitarily equivalent. In fact, for any $U \in M_2(\mathbb{C})$ with $UP(\lambda) = Q(\lambda)U$, $\forall \lambda \in \mathbb{D}$, it can be shown that U is zero.

Example 2.5 (*Holomorphic curves*) Let $T \in B(\mathcal{H})$ be a Cowen–Douglas operator with index 1 and $\text{Ker}(T-\lambda) = \text{Span}\{e(\lambda)\}$, $\forall \lambda \in \Omega$ (see more details in [6] and [15]). For the kernel function e, let $\alpha(\lambda)$ be its coordinate function in l^2. $P(\lambda) = (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1}\alpha(\lambda) \cdot \alpha^*(\lambda)$ is just the matrix form for the projection from $\mathcal{H}$ on to $\text{Ker}(T-\lambda)$. For each $\lambda \in \Omega$, chose $\alpha(\lambda) = (a_0(\lambda), a_1(\lambda), \ldots, a_n(\lambda)\ldots)^T$, and we have that

$$\begin{aligned} P(\lambda) &= (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1}\alpha(\lambda) \cdot \alpha^*(\lambda) \\ &= \left((\overline{a_0(\lambda)}, \overline{a_1(\lambda)}, \ldots, \overline{a_n(\lambda)}\ldots) \begin{pmatrix} a_0(\lambda) \\ a_1(\lambda) \\ \vdots \\ a_n(\lambda) \\ \vdots \end{pmatrix} \right)^{-1} \begin{pmatrix} a_0(\lambda) \\ a_1(\lambda) \\ \vdots \\ a_n(\lambda) \\ \vdots \end{pmatrix} \\ &\quad \times (\overline{a_0(\lambda)}, \overline{a_1(\lambda)}, \ldots, \overline{a_n(\lambda)}\ldots) \\ &= \frac{1}{\sum\limits_{k=0}^{\infty} |a_k(\lambda)|^2} \begin{pmatrix} a_0(\lambda) \\ a_1(\lambda) \\ \vdots \\ a_n(\lambda) \\ \vdots \end{pmatrix} (\overline{a_0(\lambda)}, \overline{a_1(\lambda)}, \ldots, \overline{a_n(\lambda)}\ldots) \end{aligned}$$

$$= \frac{1}{\sum\limits_{k=0}^{\infty}|a_k(\lambda)|^2}\begin{pmatrix} |a_0(\lambda)|^2 & a_0(\lambda)\overline{a_1(\lambda)} & \cdots & a_0(\lambda)\overline{a_n(\lambda)} & \cdots \\ a_1(\lambda)\overline{a_0(\lambda)} & |a_1(\lambda)|^2 & \cdots & a_1(\lambda)\overline{a_n(\lambda)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n(\lambda)\overline{a_0(\lambda)} & a_n(\lambda)\overline{a_1(\lambda)} & \cdots & |a_n(\lambda)|^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_{\infty\times\infty}.$$

From the above discussion, for each operator in $B_1(\Omega)$, we can find a related extended holomorphic curve with the same form in Definition 2.1. Furthermore, they have the same unitary invariants. So we can deduce that the extended holomorphic curve class $\mathcal{P}_1(\Omega,\mathcal{U})$ is just the generation of operator class $B_1(\Omega)$.

3. Extended holomorphic curves and inductive limits

3.1. As is well known, it is a common to construct new C^*-algebras using the inductive limits of some inductive sequences of C^*-algebras which are familiar to us. In this chapter, we will focus on discussing the unitary classification problem of some special extended holomorphic curves associated with inductive limits. This kind of inductive limits was introduced in [7] and [14].

Let $\mathcal{U}$ be a unital C^*-algebra, and $\mathcal{U}_n := M_{k_n}(\mathcal{U})$, where $\{k_n\}_{n=1}^{\infty} \in \mathbb{N}$ and $l = \frac{k_m}{k_n} \in \mathbb{N}$, $\forall m \geq n$. Let $C(\Omega,\mathcal{U}_n)$ denote the $*$-algebra of all the $\mathcal{U}_n$-valued continuous functions defined on Ω. Let $\mathscr{A}$ be a inductive limit of $*$-algebras defined as follows:

$$C(\Omega,\mathcal{U}_1) \overset{\phi_{1,2}}{\to} C(\Omega,\mathcal{U}_2) \overset{\phi_{2,3}}{\to} \cdots \to C(\Omega,\mathcal{U}_n) \overset{\phi_{n,n+1}}{\to} C(\Omega,\mathcal{U}_{n+1}) \cdots \to \mathscr{A}.$$

Let $P \in \mathcal{A}_{\alpha_n}(\Omega,\mathcal{U}_n)$ be an extended holomorphic curve. Assume homomorphisms $\phi_{n,m} : C(\Omega,\mathcal{U}_n) \to C(\Omega,\mathcal{U}_m)$ satisfies the following three conditions:

(1) For any $m > n$, there exist a unitary $u_{n,m}$and holomorphic functions $\{x_i\}_{i=1}^{\infty}$ on Ω such that

$$\phi_{n,m}(P(\lambda)) = u_{n,m}(\lambda)\mathrm{diag}\{P(x_1(\lambda)), P(x_2(\lambda))\cdots P(x_l(\lambda))\}u_{n,m}^*(\lambda), \forall\lambda \in \Omega;$$

(2) There exists a connected open set $\Lambda \subset \bar{\Lambda} \subseteq \Omega$ such that $\mathrm{Ran}\, x_i \subseteq \Lambda$, $\forall i = 1, 2, \ldots, l$.
(3) There exist functions $h_i^{J,I} \in C^{\infty}(\Omega)$, $\forall i = 1, 2, \ldots, l$, $\forall 0 < J, I \leq \alpha_m$ such that

$$\begin{aligned}&\bar{\partial}^J\phi_{n,m}(P(\lambda))\partial^I\phi_{n,m}(P(\lambda))\\ &\quad = u_{n,m}(\lambda)\mathrm{diag}\left\{h_1^{J,I}(\lambda)P(x_1(\lambda)), \ldots, h_l^{J,I}(\lambda)P(x_l(\lambda))\right\}u_{n,m}^*(\lambda).\end{aligned}$$

Notice that $\phi_{n,m}(P)$ belongs to $\mathcal{A}_{\alpha_m}(\Omega,\mathcal{U}_m)$, where $\alpha_m = \frac{k_m}{k_n}\alpha_n$.

DEFINITION 3.2

The set $\{x_1(\lambda), x_2(\lambda), \ldots, x_l(\lambda)\}$ in 3.1 is called the spectrum of $\phi_{n,m}$ at λ, denoted by $SP(\phi_{n,m})_\lambda$. Let $\psi_{n,m}$ be another homomorphism in 4.1 and $SP(\psi_{n,m})_\lambda = \{\tilde{x}_1(\lambda), \tilde{x}_2(\lambda), \ldots, \tilde{x}_l(\lambda)\}$. And we define the distance of the two sets as follows:

$$\mathrm{dist}\{SP(\phi_{n,m})_\lambda, SP(\psi_{n,m})_\lambda\} = \max_i \min_\sigma\{|x_i(\lambda) - \tilde{x}_{\sigma(i)}(\lambda)|\},$$

where σ is a permutation of set $\{1, 2, \ldots, l\}$.

3.3 [23]. Let $\{\mathcal{U}_i\}_{i=1}^{\infty}$ be a family of C^*-algebras. We associate two new C^*-algebras $\prod_i^{\infty}\mathcal{U}_i$ and $\sum_i^{\infty}\mathcal{U}_i$ as follows. Let $\prod_i^{\infty}\mathcal{U}_i$ be the set of all functions $a:\mathbb{N}\to\cup_{i\in\mathbb{N}}\mathcal{U}_i$ for each $a(i)\in\mathcal{U}_i$, where $||a||=\mathrm{Sup}\{||a(i)||_{\mathcal{U}_i}:i\in\mathbb{N}\}$. Let $\sum_i^{\infty}\mathcal{U}_i$ be the closure of the subset

$$\mathcal{I}=\left\{a\in\prod_i^{\infty}\mathcal{U}_i : a(i)=0 \text{ for all but finite many } i\in\mathbb{N}\right\}.$$

Let $\mathcal{M}=\lim_{n\to\infty}(\mathcal{U}_n, i_{n,m})$ and $i_{n,m}:\mathcal{U}_n\to\mathcal{U}_m$ be the embedding homomorphisms. Then $\mathcal{M}=\prod_i^{\infty}\mathcal{U}_i/\sum_i^{\infty}\mathcal{U}_i$ and $i_{n,\infty}(a)=[\ldots,a,i_{n,n+1}(a),i_{n,n+2}(a),\ldots]$. For more details on inductive limits of C^*-algebras, see [14].

In the following theorem, we will consider the unitary equivalent problems of the extended holomorphic curves in inductive limits C^*-algebras. To give a classification of inductive limits C^*-algebras, we need to consider the spectrum of the homomorphisms between two C^*-algebras in the inductive sequences. In our case (with assumptions (1)–(3) in 3.1), we also can describe the unitary equivalence of two extended holomorphic curves in inductive limits of C^*-algebras by comparing the spectrums of the corresponding homomorphism sequences.

Theorem 3.4. *Let* $\mathcal{U}=\lim_{n\to\infty}(C(\Omega,\mathcal{U}_n),\phi_{n,m})$, $\tilde{\mathcal{U}}=\lim_{n\to\infty}(C(\Omega,\mathcal{U}_n),\psi_{n,m})$ *be inductive limits in* 3.1. *Let* $P\in\mathcal{A}_{\alpha_N}(\Omega,\mathcal{U}_N)$ *be an extended holomorphic curve in* 3.1, *where N is a given integer.*

The following statements are true if:

(1) $\lim_{m\to\infty}\max_{\lambda\in\Lambda}\mathrm{dist}\{SP(\phi_{n,m})_\lambda, SP(\psi_{n,m})_\lambda\}=0$;
(2) *For any* $m>N$ *and* $\lambda\in\Omega$,

$$h_i^{J,I}(\lambda)=\tilde{h}^{J,I}_{\sigma_0(i)}(\lambda)\ (in\ 3.1),$$

when $|x_i(\lambda)-\tilde{x}_{\sigma_0(i)}(\lambda)|=\min_\sigma\{|x_i(\lambda)-\tilde{x}_{\sigma(i)}(\lambda)|\}$, $\forall i\in\{1,2,\ldots,l\},\forall\lambda\in\Omega, 0<\forall J,I\le\alpha_m$.

Then there exists $M>0$ *such that* $\phi_{N,m}(P)\sim_u\psi_{N,m}(P)$ *for all* $m>M$. *Furthermore,* $\phi_{N,\infty}(P)\sim_u\psi_{N,\infty}(P)$ *in* $\mathcal{M}=\prod_i^{\infty}\mathcal{U}_i/\sum_i^{\infty}\mathcal{U}_i$.

Proof. For any $\lambda,\lambda'\in\Lambda$, there exists $\delta>0$ such that if $|\lambda-\lambda'|<\delta$, then $||P(\lambda)-P(\lambda')||<1$. Since $P(\lambda)$ and $P(\lambda')$ are projections, then $P(\lambda)\overset{u}{\sim}P(\lambda')$.

Suppose permutation σ_0 satisfies

$$|x_i(\lambda)-\tilde{x}_{\sigma_0(i)}(\lambda)|=\min_\sigma\{|x_i(\lambda)-\tilde{x}_{\sigma(i)}(\lambda)|\}.$$

Then

$$h_i^{J,I}(\lambda)=\tilde{h}^{J,I}_{\sigma_0(i)}(\lambda),\forall i=1,2,\ldots,l,\quad\forall\lambda\in\Omega.$$

By condition (1), there exists $M>0$ such that for any $m\ge M$,

$$|x_i(\lambda)-\tilde{x}_{\sigma_0(i)}(\lambda)|\le\mathrm{dist}(SP(\phi_{N,m})_\lambda, SP(\psi_{N,m})_\lambda)\le\delta,\quad\forall\lambda\in\Omega.$$

So when $m \geq M$,

$$P(x_i(\lambda)) \overset{u}{\sim} P(\tilde{x}_{\sigma_0(i)}(\lambda)), \quad \forall \lambda \in \Omega, \forall 0 < i \leq l.$$

Notice that for any $\lambda \in \Lambda$ and $0 \leq J, I \leq \alpha_m$,

$$\bar{\partial}^J \phi_{N,m}(P(\lambda)) \partial^I \phi_{N,m}(P(\lambda)) = u_{N,m}(\lambda) \begin{pmatrix} h_1^{JI}(\lambda) P(x_1(\lambda)) & & \\ & \ddots & \\ & & h_l^{JI}(\lambda) P(x_l(\lambda)) \end{pmatrix} u_{N,m}^*(\lambda). \tag{3.1}$$

Obviously, (3.1) also holds for $\psi_{N,m}$. That is, for any $\lambda \in \Lambda$ and $0 \leq J, I \leq \alpha_m$,

$$\bar{\partial}^J \phi_{N,m}(P(\lambda)) \partial^I \psi_{N,m}(P(\lambda)) = \tilde{u}_{N,m}(\lambda) \begin{pmatrix} \tilde{h}_1^{JI}(\lambda) P(\tilde{x}_1(\lambda)) & & \\ & \ddots & \\ & & \tilde{h}_l^{JI}(\lambda) P(\tilde{x}_l(\lambda)) \end{pmatrix} \tilde{u}_{N,m}^*(\lambda).$$

By (3.1), for any $\lambda \in \Omega$, we have

$$\bar{\partial}^J \phi_{N,m}(P(\lambda)) \partial^I \phi_{N,m}(P(\lambda)) \overset{u}{\sim} \bar{\partial}^J \psi_{N,m}(P(\lambda)) \partial^I \psi_{N,m}(P(\lambda)), \quad \forall m \geq M.$$

Since $\phi_{N,m}(P)$ and $\psi_{N,m}(P) \in \mathcal{A}_{\alpha_m}(\Lambda, \mathcal{U}_m)$ and $\mathcal{U}_m$ is an inner algebra, by Lemma 1.6, we have $\phi_{n,m}(P) \overset{u}{\sim} \psi_{n,m}(P)$, $\forall m \geq M$.

Suppose $\phi_{n,m}(P(\lambda)) = U_m \psi_{n,m}(P(\lambda)) U_m^*$, where U_m does not depend on λ. Set

$$U = i_{M,\infty}(U_M) = [U_M, \ldots, U_m, U_{m+1}, \ldots] \in \mathcal{M} = \prod_i^\infty \mathcal{U}_i \Big/ \sum_i^\infty \mathcal{U}_i.$$

Notice that $\mathcal{U}, \tilde{\mathcal{U}} \subset \mathcal{M}$, and we have

$$\begin{aligned} \phi_{N,\infty}(P(\lambda)) &= [\phi_{N,M}(P(\lambda)), \phi_{N,M+1}(P(\lambda)), \ldots] \\ U^* &= U[\psi_{N,M}(P(\lambda)), \psi_{N,M+1}(P(\lambda)), \ldots] \\ &= U \psi_{N,\infty}(P(\lambda)) U^*, \quad \forall \lambda \in \Omega. \end{aligned}$$

□

Remark 3.5. For conditions (1)–(3) given for the extended holomorphic curves mentioned in 3.1, conditions (1) and (2) are always used in the classification theory in C^*-algebras. In the last part of this note, we would like to point out that condition (3) in 3.1 can also be satisfied for some special extended holomorphic curves.

In fact, if we let $P : \Omega \to \mathcal{P}(\mathcal{U})$ be an extended holomorphic curve with the following properties:

(a) $P(\lambda) = f(\lambda) p(\lambda) p^*(\lambda)$, $\forall \lambda \in \Omega$, where $f \in C^\infty(\Omega)$, $p \in C^\infty(\Omega, \mathcal{U})$;
(b) $\partial P(\lambda) = f(\lambda) p^*(\lambda)$, $\bar{\partial} P(\lambda) = f(\lambda) p(\lambda)$, $\forall \lambda \in \Omega$;
(c) $\dfrac{\partial}{\partial \lambda} p^*(\lambda) = \dfrac{\partial}{\partial \bar{\lambda}} p(\lambda) = 0$, $\forall \lambda \in \Omega$.

Then, by conditions (b) and (c), we have

$$\partial^I P(\lambda) = \frac{\partial^I}{\partial \lambda} f(\lambda) p^*(\lambda), \bar{\partial}^J P(\lambda) = \frac{\partial^J}{\partial \bar{\lambda}} f(\lambda) p(\lambda)$$

and

$$\bar{\partial}^J P(\lambda) \partial^I P(\lambda) = \frac{\frac{\partial^I}{\partial \lambda} f(\lambda) \frac{\partial^J}{\partial \bar{\lambda}} f(\lambda)}{f(\lambda)} P(\lambda) = f^{J,I}(\lambda) P(\lambda).$$

A calculation shows that such a kind of extended holomorphic curves with properties: (a), (b) and (c) will satisfy condition (3) in 3.1. And we can also check that the Bott projection on to $\mathbb{C}^2$ satisfies conditions (a), (b) and (c).

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